Further-Step Optimization Towards Accuracy-Enhanced Digital Phase System

Tian-Bo Deng

***Abstract*—This paper shows that the phase system resulting from the quadratic-cone-programming (QconeP) feasibility-check (FC) can be further improved through adopting a further-step nonlinear optimization. Although the QconeP feasibility-check (QconeP-FC) design technique can get a sufficiently accurate phase system, the further-step nonlinear optimization can further reduce the peak errors of both frequency response (FR) and phase response. Clearly, this new approach includes two-step optimizations. The first one uses the QconeP-FC approach, and the second one employs a further-step nonlinear optimization. Incorporating the nonlinear optimization into the QconeP-FC approach yields an accuracy-enhanced digital phase system. An illustrative example is included to demonstrate the performance of this two-step-optimization design methodology.**

***Index Terms*—Signal processing, phase-system, feasibility-check (FC), nonlinear optimization, two-step optimization.**

I. Introduction

PHASE system is required for compensating for the phase distortion of a digital system. Such requirements can be found in a lot of digital signal processing (DSP) applications. Phase systems can be designed to be either FIR type [1]-[8] or IIR type [9], [10]. This paper utilizes the IIR-type allpass transfer function to approximate a given ideal phase (phase design specification). In [10], a quadratic-cone-programming (QconeP) feasibility-check (FC) approach is proposed, where the FC is performed. This approach is called QconeP-FC appraoch. By incorporating the bisection search (binary search) into the FC process, the search interval can be gradually reduced, and finally the interval can be reduced to a very small one that is below a preset small threshold. As a result, a sufficiently accurate phase system can be designed. This paper shows that the phase system resulting from the QconeP-FC design can be further enhanced through adopting a further-step nonlinear optimization. That is, this approach includes an FC process and a further-step nonlinear optimization. This two-step optimization technique can further reduce the design errors and thus yields an accuracy-enhanced phase system. An illustrative example is presented to verify the enhanced accuracy.

II. First-Step Optimization

Suppose that $\theta_d(\omega)$ is the given phase specification. Here, we want to use the following allpass phase system to approximate the given $\theta_d(\omega)$:

$$\begin{aligned} H(z) &= \frac{\beta_M + \beta_{M-1}z^{-1} + \beta_{M-2}z^{-2} + \cdots + z^{-M}}{1 + \beta_1 z^{-1} + \beta_2 z^{-2} + \cdots + \beta_M z^{-M}} \\ &= \frac{z^{-M}\beta(z^{-1})}{\beta(z)}. \end{aligned} \tag{1}$$

Here, $\beta(z)$ is the denominator of $H(z)$, and

$$\beta(z) = \sum_{i=0}^{M} \beta_i z^{-i}$$

with

$$\beta_0 = 1$$

is an Mth-order polynomial in z. Moreover, $\beta(z^{-1})$ is the mirror-image polynomial of $\beta(z)$, i.e.,

$$\beta(z^{-1}) = \sum_{i=0}^{M} \beta_i z^{i}.$$

The transfer function $H(z)$ contains unknown parameters β_i, $i = 1, 2, \cdots, M$. Finding the optimal coefficients β_i to best approximate the given $\theta_d(\omega)$ is the final goal of this research.

The frequency response (FR) is

$$H(\omega) = e^{-jM\omega} \cdot \frac{B^*(\omega)}{\beta(\omega)}. \tag{2}$$

The parameter $\omega \in [0, \pi]$ represents the angular frequency. The phase specification $\theta_d(\omega)$ corresponds to the desired FR

$$H_d(\omega) = e^{j\theta_d(\omega)}.$$

Thus, the FR error is

$$\begin{aligned} e_H(\omega) &= H(\omega) - H_d(\omega) \\ &= \frac{\Re(\omega) + j\Im(\omega)}{\beta(\omega)} \end{aligned} \tag{3}$$

where

$$\begin{aligned} \Re(\omega) &= -\sum_{i=0}^{M} \beta_i u_i(\omega) \\ \Im(\omega) &= -\sum_{i=0}^{M} \beta_i v_i(\omega) \end{aligned} \tag{4}$$

with $u_i(\omega)$ and $v_i(\omega)$ defined by

$$\begin{aligned} u_i(\omega) &= \cos[\theta_d(\omega) - i\omega] - \cos(M-i)\omega \\ v_i(\omega) &= \sin[\theta_d(\omega) - i\omega] + \sin(M-i)\omega. \end{aligned}$$

Substituting $\beta_0 = 1$ into (4) yields

$$\Re(\omega) = -u_0(\omega) - \sum_{i=1}^{M} \beta_i u_i(\omega)$$

$$\Im(\omega) = -v_0(\omega) - \sum_{i=1}^{M} \beta_i v_i(\omega).$$

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Hence, the FR error becomes

$$e_H(\omega) = \frac{\Re(\omega) + j\Im(\omega)}{\beta(\omega)}. \tag{5}$$

The problem here is to find the optimal values of the coefficients $\beta_1, \beta_2, \cdots, \beta_M$ that minimize the peak error of $e_H(\omega)$. That is, the problem is to minimize

$$\delta_H = \max\{|e_H(\omega)|, \omega \in [0, \pi]\}. \tag{6}$$

This problem can be described by

$$\begin{aligned} &\text{minimize} \quad \delta_H \\ &\text{subject to} \quad |e_H(\omega)| \le \delta_H \end{aligned} \tag{7}$$

with

$$|e_H(\omega)| = \frac{\sqrt{\Re^2(\omega) + \Im^2(\omega)}}{|\beta(\omega)|}.$$

However, the minimization problem in (7) is a difficult nonlinear problem. In [10], an approach using feasibility-check is formulated as follows.

Let

$$\phi = \frac{M\omega + \theta_d(\omega)}{2}.$$

The minimization (7) can be approximately formulated as

$$\begin{aligned} &\text{minimize} \quad \delta_H \\ &\text{subject to} \quad \sqrt{\Re^2(\omega) + \Im^2(\omega)} \le \delta_H \cdot \sum_{i=0}^{M} \beta_i \phi_i(\omega) \end{aligned} \tag{8}$$

with

$$\phi_i(\omega) = \cos(i\omega - \phi).$$

Here, it should be noted once again that the denominator $\beta(\omega)$ is approximated by the linear expression [8]

$$\begin{aligned} \beta(\omega) &\approx \sum_{i=0}^{M} \beta_i \phi_i(\omega) \\ &= \phi_0(\omega) + \sum_{i=1}^{M} \beta_i \phi_i(\omega) \end{aligned}$$

and

$$\phi_0(\omega) = \cos(\phi).$$

The constraint in (8) is equivalent to

$$\begin{bmatrix} \delta_H \cdot \sum_{i=0}^{M} \beta_i \phi_i(\omega) \\ \Re(\omega) \\ \Im(\omega) \end{bmatrix} \in \mathcal{K}_q \tag{9}$$

where $\mathcal{K}_q$ denotes the quadratic-cone (Qcone). That is, the constraint is a Qcone constraint. This constraint can be elaborated to the following matrix form.

Let

$$\boldsymbol{c} = \begin{bmatrix} \delta_H \cdot \phi_0(\omega) \\ -u_0(\omega) \\ -v_0(\omega) \end{bmatrix}$$

$$\boldsymbol{A}^T = \begin{bmatrix} -\delta_H \cdot \phi_1(\omega) & -\delta \cdot \phi_2(\omega) & \cdots & -\delta \cdot \phi_M(\omega) \\ u_1(\omega) & u_2(\omega) & \cdots & u_M(\omega) \\ v_1(\omega) & v_2(\omega) & \cdots & v_M(\omega) \end{bmatrix}$$

$$\boldsymbol{y} = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_M \end{bmatrix}^T.$$

The constraint can be expressed by using the matrix form

$$\boldsymbol{c} - \boldsymbol{A}^T \boldsymbol{y} \in \mathcal{K}_q.$$

Suppose that the minimum δ_H is located in $[\delta_\mathrm{L}, \delta_\mathrm{U}]$. A bisection search (binary search) method is proposed in [10] for finding the minimum solution. That is, for a fixed $\delta \in [\delta_\mathrm{L}, \delta_\mathrm{U}]$, we check the feasibility of the Qcone constraint and see if or not it is feasible. More specifically, either of the following two cases is identified:

$$\boldsymbol{c} - \boldsymbol{A}^T \boldsymbol{y} \in \mathcal{K}_q \tag{10}$$

$$\boldsymbol{c} - \boldsymbol{A}^T \boldsymbol{y} \notin \mathcal{K}_q. \tag{11}$$

The former means that the constraint is feasible, and the latter means that the constraint is infeasible.

According to the feasibility (10) or infeasibility (11), the search interval is further halved. The binary search is repeated until the interval is shrunk below a small threshold $\zeta = 10^{-7}$.

III. Second-Step Optimization

Our computer simulations have verified that the above approach using QconeP-FC truly leads to a considerably accurate approximation. However, it is possible to further enhance the accuracy of the phase-system by further minimizing the peak error

$$\delta_H = \max\{|e_H(\omega)|, \omega \in [0, \pi]\}.$$

This error definition is also given in (6). Here, the MATLAB minimizer *fminsearch* is utilized to minimize (6). As mentioned earlier, the reason why the nonlinear minimization is employed is because minimizing δ_H in (6) is a nonlinear minimization problem.

IV. Example

Let us consider

$$\theta_d(\omega) = \begin{cases} -12\omega, & \omega \in [0, \pi/2] \\ -8\omega - 2\pi, & \omega \in [\pi/2, \pi]. \end{cases} \tag{12}$$

This example is also used in [10]. The first-step (QconeP-FC) uses the same parameters set in [10], which are

System order : $M = 10$
Grid points for ω : $L = 1001$
Search interval : $[\delta_\mathrm{L}, \delta_\mathrm{U}] = [0, 0.098]$.

Table I lists the coefficient values from the first-step optimization using QconeP-FC, and Table II lists the coefficient values after the second-step optimization using the nonlinear optimization. To see the differences, Table III lists the coefficient differences ($\beta_i^{(0)} - \beta_i$) between Table I and Table II. Note that $\beta_i^{(0)}$ denote the initial coefficient values from the QconeP-FC design. The achieved coefficient improvements (differences) listed in Table III are also plotted in Fig. 1.

Fig. 2 shows the desired phase $\theta_d(\omega)$ and the designed phase $\theta(\omega)$ using the final coefficient values tabulated in Table II, and Fig. 3 shows the phase errors. Furthermore, Fig. 4 shows the FR-errors $|e_H(\omega)|$ in dB.

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TABLE I
COEFFICIENTS BEFORE NONLINEAR OPTIMIZATION

i	$\beta_i^{(0)}$
1	-1.274888072685988
2	0.812527182512196
3	-0.202109894303676
4	-0.071602578846389
5	0.034127015709933
6	0.030691754759400
7	0.004082010918542
8	-0.054728415426804
9	0.042823574342412
10	0.000124872383799

TABLE II
COEFFICIENTS AFTER NONLINEAR OPTIMIZATION

i	β_i
1	-1.274655747819261
2	0.812188676778089
3	-0.201949291608419
4	-0.071532758376946
5	0.034208262169398
6	0.030342730010751
7	0.004209501864216
8	-0.054324211815872
9	0.042406333299630
10	0.000158958913055

TABLE III
COEFFICIENT DIFFERENCES

i	$\beta_i/10^{-3}$
1	-0.232324866726419
2	0.338505734106986
3	-0.160602695257556
4	-0.069820469443690
5	-0.081246459464918
6	0.349024748649448
7	-0.127490945673766
8	-0.404203610932755
9	0.417241042781923
10	-0.034086529255724

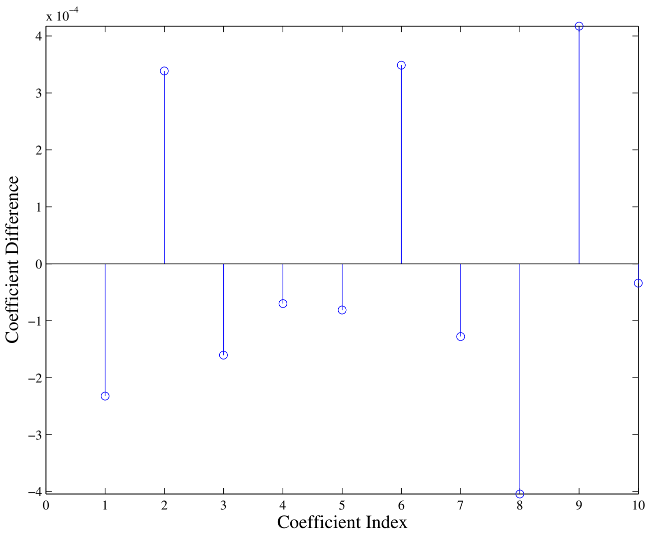

Fig. 1. Coefficient improvements (differences).

TABLE IV
MAXIMUM ERRORS

Method	δ_θ	δ_H (dB)
Rcone Design [9]	0.1090	-19.26
Feasibility-Check (FC) [10]	0.0638	-23.90
Two-Stage Optimizations	0.0629	-24.03

Table IV tabulates the maximum phase errors δ_θ and the maximum FR errors δ_H (dB) from the proposed two-step-optimization design, along those from the Rcone design in [9] and the feasibility-check (QconeP-FC) design [10]. It is obvious that the two-step optimization approach can further improve the phase-system accuracy.

The designed IIR-type phase system is recursive, sp its stability must be guaranteed. The stability can be checked by seeing if or not all the poles of $H(z)$ are inside the unit-circle (radius $= 1$). Fig. 5 shows all the poles of $H(z)$. Obviously, all the poles have radii smaller than one. Therefore, the phase system after the second-step optimization is stable.

V. CONCLUSION

This paper has presented a two-step design methodology that combines the QconeP-FC technique with a further-step optimization (nonlinear optimization) for further enhancing the final accuracy of the designed digital phase system. A comparative example has demonstrated that incorporating this further-step optimization (nonlinear optimization) is able to significantly improve the performance of the phase system. That is, the two-step optimization yields a much better phase system than the phase system from the single step (QconeP-FC approach) proposed in [10]. The computer simulations have clearly verified the accuracy enhancement by employing this two-step optimization methodology.

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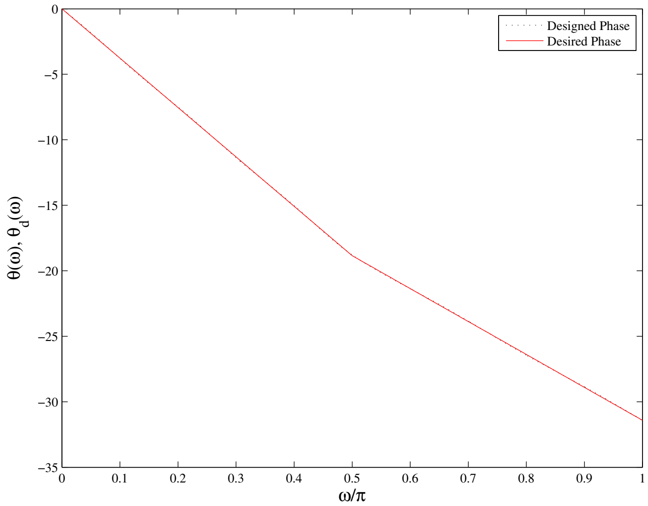

Fig. 2. Phases (desired and designed).

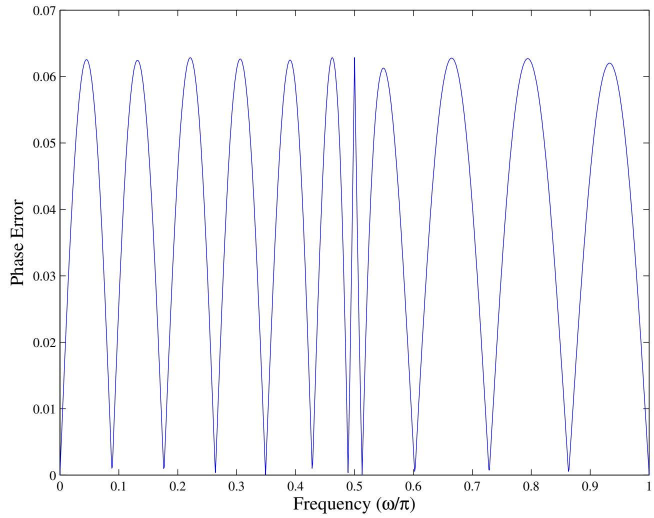

Fig. 3. Phase errors.

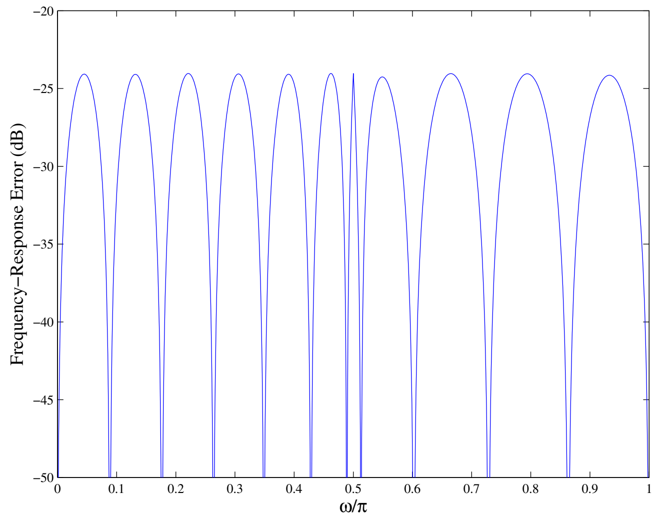

Fig. 4. FR errors in dB.

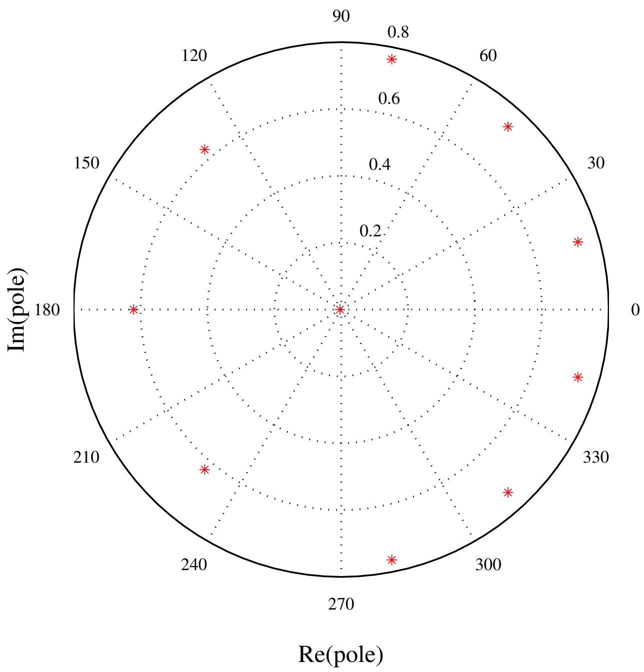

Fig. 5. Stability check using pole locations.

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