A Series Representation for Multidimensional Rayleigh Distributions

Martin Wiegand, Saralees Nadarajah

***Abstract*—The Rayleigh distribution is of paramount importance in signal processing and many other areas, yet an expression for random variables of arbitrary dimensions has remained elusive. In this note, we generalise the results of Beard and Tekinay [1] for quadrivariate random variables to cases of unconstrained order and provide a simple algorithm for evaluation. The assumptions of cross-correlation between in-phase and quadrature, as well as non-singularity of the covariance matrix are retained throughout our computations.**

***Index Terms*—Complex Gaussian distribution, Multivariate Rayleigh distribution**

I. Introduction

CORRELATED Rayleigh random variables arise in signal processing and many other areas: correlated Rayleigh fading channels, correlated Rayleigh scattering spectroscopy, correlated Rayleigh envelopes, correlated Rayleigh co-channel interferers, correlated Rayleigh clutters and correlated Rayleigh targets; to mention just a few.

For correlated Rayleigh random variables Rice [2] and Miller [3] obtained probability distribution representations for the bivariate and trivariate cases. Their method of expressing the distribution via an underlying Gaussian distribution has still been utilised in recent publications and will be essential to our approach as well.

Based on the previous progressions, Beard and Tekinay [1] have derived a series representation for a quadrivariate Rayleigh distribution around a Bessel function expansion. We believe that the dimensional restriction can be relaxed, while the original assumption of non-singularity of the covariance matrix is maintained. In Section II, we circumnavigate the denomination problem of increasingly many Bessel function sum terms, as noted by Beard and Tekinay [1], with a Cauchy sum. The resulting integrals can then be more easily evaluated via the complex exponential notation of the cosine function. We also provide a pseudocode for the algorithm. Some practical applications of the pseudocode including runtimes are given in Section III. Some final remarks on what has been done are given in Section IV. Throughout, only the formulas cited in the text are numbered.

II. Multivariate Rayleigh distribution

We begin by introducing the $2n$ dimensional random variable $Z = \{z_{I_1}, z_{Q_1}, \ldots, z_{I_1}, z_{Q_1}\}$, where I_i represents the in-phase and Q_i the quadrature part of a signal. The joint distribution is assumed to be a $2n$-variate Gaussian ($\mu = 0$, $\sigma^2 = \zeta$), with the distribution function given below (as in Beard and Tekinay [1] or Rice [2]):

$$f\left(z_{I_1}, z_{Q_1}, \ldots, z_{I_1}, z_{Q_1}\right) = \frac{1}{(2\pi)^n |K|^{1/2}} \exp\left(-\frac{Z^T K^{-1} Z}{2}\right), \tag{1}$$

where

$$K = \zeta \begin{bmatrix} 1 & 0 & \rho_1 & \cdots & & \cdots & \rho_{n-1} & 0 \\ 0 & 1 & 0 & \rho_1 & \cdots & & \cdots & \rho_{n-1} \\ \rho_1 & 0 & 1 & & & & & \vdots \\ \vdots & \rho_1 & & \ddots & & & & \\ & \vdots & & & & & & \vdots \\ & & & & & \ddots & & \rho_1 \\ & & & & & & 1 & 0 \\ & & & & \cdots & \rho_1 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{2n} \tag{2}$$

denotes the covariance matrix for $2n$ dimensions with values $-1 \leq \rho_i \leq 1$. The in-phase and quadrature parts are presumed to be uncorrelated, for example, $\mathbb{E}\left[z_{I_i} z_{Q_i}\right] = 0$ without loss of generality. A more general result for correlated random variables can be obtained analogously.

We transform the cartesian coordinates of the input vector $z \in \mathbb{R}^{2n}$ into polar coordinates:

$$z = \begin{bmatrix} r_1 \cos(\theta_1) \\ r_1 \sin(\theta_1) \\ \vdots \\ \\ \vdots \\ r_n \cos(\theta_n) \\ r_n \sin(\theta_n) \end{bmatrix}.$$

The determinant of the Jacobian for the transformation $|J| = r_1 \cdots r_n$ can hence be written as a factor outside the exponential function.

To further expand the matrix vector product, we determine a general expression for the inverse matrix. We employ Cramer's rule utilising the cofactor matrix C such that

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$K^{-1} = \frac{1}{|K|}C^T$ holds:

$$C = \begin{bmatrix} c_0 & 0 & c_1 & \cdots & & \cdots & c_{n-1} & 0 \\ 0 & c_0 & 0 & c_1 & \cdots & & \cdots & c_{n-1} \\ c_1 & 0 & c_0 & & & & & \vdots \\ \vdots & c_1 & & \ddots & & & & \\ & \vdots & & & & & & \vdots \\ & & & & & \ddots & & c_1 \\ & & & & & & c_0 & 0 \\ & & & & \cdots & c_1 & 0 & c_0 \end{bmatrix}. \quad (3)$$

We note that the cofactor matrix has to retain the original shape of K, which aids in the evaluation of the exponent.

We introduce (2) and (3) into the $2n$-variate Gaussian distribution in (1), and apply common trigonometric identities [4] onto the resulting sine and cosine product terms. This yields the following result for $r = (r_1, \dots, r_n)$ and $\theta = (\theta_1, \dots, \theta_n)$:

$$\begin{aligned} f(r,\theta) &= \frac{|J|}{(2\pi)^n |K|^{1/2}} \exp\left(-\frac{1}{2|K|}\left(\sum_{i=1}^n r_i^2 c_0 \right.\right. \\ &\quad \left.\left. + 2\sum_{(i,k,l\in P^n)} c_i r_k r_l \cos(\theta_k - \theta_l)\right)\right) \\ &= \underbrace{\frac{|J|}{(2\pi)^n |K|^{1/2}} \exp\left(-\frac{1}{2|K|}\sum_{i=1}^n r_i^2 c_0\right)}_{=\gamma_{n,K}} \\ &\quad \prod_{(i,k,l\in P^n)} \exp\left(-\frac{1}{|K|} c_i r_k r_l \cos(\theta_k - \theta_l)\right). \end{aligned}$$

Here, $P^n = \Big\{(i,k,l) \in \mathbb{N}^3 |\ 1 \le l < k \le n,\ i = 1, \dots, n-1,\ |k-l| = i\Big\}$ is the set of all feasible coefficient combinations of c_i and r_i as they arise from the vector matrix product. We relabel the variables according to the counting scheme of algorithm 1, and dispose of the set P^n in favour of a summation for $t = 1, \dots, n(n-1)/2 = p = |P^n|$.

First we substitute the theta variables by $x_{l-1} - x_{k-1} = \theta_k - \theta_l$ [1], where $x_0 = 0$ needs to be introduced for consistency.

```
Result: Retain new variables x̄_t and coefficients a_t
Renaming t=1
for l = 1,...,n do
    for k = l+1,...,n do
        x̄_t = x_{l-1} − x_{k-1};
        a_t = −(1/|K|) c_{|k−l|} r_k r_l;
        t++;
    end
end
```

Algorithm 1: Renaming algorithm.

This naturally leaves us with a compacter version of the Gaussian distribution as we can see below:

$$f(r_1,\theta_1,\dots,r_n,\theta_n) = \gamma_{n,K}\prod_{t=1}^p \exp\left(a_t \cos(\overline{x}_t)\right). \quad (4)$$

Next we consider n-variate Rayleigh distribution by generating the marginal probability distribution. Integration over the angle component of the polar coordinates reduces the function in (4) to an n-dimensional one, solely dependent on the radius component:

$$f(r_1,\dots,r_n) = \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^p \exp\left(a_t\cos(\overline{x}_t)\right) dx_1\cdots dx_n.$$

The composition of exponential and cosine functions proves to be difficult to integrate analytically, we hence use the following Bessel function expansions [1] [4] to reduce the problem to a trigonometric integral:

$$\exp[a\cos(x)] = I_0(a) + 2\sum_{j=1}^\infty I_j(a)\cos(jx),$$
$$\exp[-a\cos(x)] = I_0(a) + 2\sum_{j=1}^\infty (-1)^j I_j(a)\cos(jx).$$

Additionally, we rename the appearing coefficients, to combine both cases, therefore making the series representation more lucid to us and to reduce the exceptions:

$$b_{t,j_t} = \begin{cases} I_0(|a_t|), & \text{if } j_t = 0, \\ 2(-1)^{j_t \mathbb{I}_{(a_t<0)}} I_{j_t}(|a_t|), & \text{if } j_t > 0. \end{cases} \quad (5)$$

With the series expansion and revised notation in place, we can move on to the arithmetic:

$$\begin{aligned} f(r) &= \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\left[\prod_{t=1}^p I_0(a_t) \right. \\ &\quad \left. + \sum_{j_t=1}^\infty I_{j_t}(a_t)\cos(j_t\overline{x}_t)\right] dx_1\cdots dx_n \\ &\overset{(5)}{=} \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^p\sum_{j_t=0}^\infty b_{t,j_t}\cos(j_t\overline{x}_t)\,dx_1\cdots dx_n \\ &= \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{j_1=0}^\infty\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}} b_{1,j_p}\cos(j_p\overline{x}_1) \\ &\quad b_{2,j_{p-1}-j_p}\cdots b_{p,j_1-j_2}\cos((j_1-j_2)\overline{x}_p)\,dx_1\cdots dx_n \\ &\overset{j_{p+1}=0}{=} \gamma_{n,K}\int_0^{2\pi}\cdots\int_0^{2\pi}\sum_{j_1=0}^\infty\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^p b_{t,j_{p-t+1}-j_{p-t+2}} \\ &\quad \cos((j_{p-t+1}-j_{p-t+2})\overline{x}_t)\,dx_1\cdots dx_n \\ &= \gamma_{n,K}\sum_{j_1=0}^\infty\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^p b_{t,j_{p-t+1}-j_{p-t+2}} \\ &\quad \int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^p\cos((j_{p-t+1}-j_{p-t+2})\overline{x}_t)\,dx_1\cdots dx_n. \end{aligned}$$

Expression (II) was derived by introducing a Cauchy product into the equation [5], enabling us to exchange product and sum of the series expansion, which in turn allows us then to further simplify the integral (convergence holds as a result of the expansion).

We then insert Euler representation of the cosine $\cos(z) = \frac{1}{2}[\exp(iz) + \exp(-iz)]$ into (II) to alleviate the integral evaluation. This yields the following result with the now

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modified index $j_t^* = j_{p-t+1} - j_{p-t+2}$ and we can determine the complex integral:

$$\begin{aligned} f(r_1,\ldots,r_n) &= \gamma_{n,K}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p} b_{t,j_t^*} \\ &\quad \int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p}\cos\left(j_t^*\overline{x}_t\right)dx_1\cdots dx_n \\ &= \gamma_{n,K}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}} \\ &\quad \prod_{t=1}^{p} b_{t,j_t^*}\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{t=1}^{p} \\ &\quad \frac{\left(\exp\left(ij_t^*\overline{x}_t\right)+\exp\left(-ij_t^*\overline{x}_t\right)\right)}{2}dx_1\cdots dx_n \\ &= \frac{\gamma_{n,K}}{2^p}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p} b_{t,j_t^*}\int_0^{2\pi}\cdots\int_0^{2\pi} \\ &\quad \sum_{\boldsymbol{\rho}\in\{-1,1\}^p}\prod_{t=1}^{p}\exp\left(ij_t^*\rho_t\overline{x}_t\right)dx_1\cdots dx_n \\ &= \frac{\gamma_{n,K}}{2^p}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p} b_{t,j_t^*}\sum_{\boldsymbol{\rho}\in\{-1,1\}^p} \\ &\quad \int_0^{2\pi}\cdots\int_0^{2\pi}\exp\left(i\sum_{t=1}^{p} j_t^*\rho_t\overline{x}_t\right)dx_1\cdots dx_n \end{aligned} \tag{6}$$

where $\boldsymbol{\rho} = (\rho_1,\ldots,\rho_p)$.

As all the factors $j_t^*\rho_t \in \mathbb{Z}$, it is clear that the integral in (6) is exactly non-zero ($(2\pi)^p$ in fact), when the exponent itself is zero [6]. This is precisely the case when the sum over the coefficients corresponding to their x_i are zero, for all x_i. To further clarify this, we denote $\alpha_t = j_t^*\rho_t$ and design two upper triangular matrices with $\overline{x}_t$, $t = 1,\ldots,p$ and their corresponding coefficients as their respective elements:

$$X = \begin{bmatrix} x_1 & x_1 - x_2 & \cdots & & x_{n-2} - x_{n-1} \\ x_2 & x_1 - x_3 & & & 0 \\ x_3 & x_1 - x_4 & & & \vdots \\ x_4 & \vdots & & & \vdots \\ \vdots & & & & \\ & \vdots & & & \\ \vdots & x_1 - x_{n-1} & 0 & & \vdots \\ x_{n-1} & 0 & & & 0 \end{bmatrix},$$

$$A = \begin{bmatrix} \alpha_1 & \alpha_n & \cdots & \cdots & \alpha_{p-2} & \alpha_p \\ \alpha_2 & \alpha_{n+1} & & & \alpha_{p-1} & 0 \\ \alpha_3 & \vdots & & & 0 & \vdots \\ \vdots & & & & \vdots & \\ & \vdots & & & & \\ \vdots & \alpha_{2n-2} & 0 & & & \vdots \\ \alpha_{n-1} & 0 & & & \cdots & 0 \end{bmatrix}.$$

The matrix entries for A have been generated as $A_{o,p} = \alpha_{\sum_{i=0}^{p-1}(n-i-1)} + o$ (filling the columns with progressively fewer elements). We can obtain the sum over all coefficients for one x_i with their respective signs by adding the n-th column to the n-th entry in the first column, and then subtracting the minor counter diagonal's negative entries. We denote the resulting i-th sum as follows:

$$\Sigma_{x_i} = A_{i,1} + \sum_{w=1}^{n-i} A_{w,i} - \sum_{w=1}^{i-1} A_{i-w,1+w}.$$

We can now very simply write the integral as a product of Kronecker Deltas, dependant on the previously defined coefficient sums. With this last step we are thus finished:

$$\begin{aligned} f(r_1,\ldots,r_n) = \pi^p\gamma_{n,K}\sum_{j_1=0}^{\infty}\sum_{j_2=0}^{j_1}\cdots\sum_{j_p=0}^{j_{p-1}}\prod_{t=1}^{p} \\ b_{t,j_t^*}\sum_{\boldsymbol{\rho}\in\{-1,1\}^p}\prod_{w=1}^{n}\delta_{\{\Sigma_{x_w}=0\}}. \end{aligned} \tag{7}$$

Equation (7) enables us to formulate a simple algorithm to evaluate the Rayleigh distribution for a given vector $r_1,\ldots,r_n$ in arbitrary dimensions. We provide a pseudocode below to illustrate the basic principles on how the series expansion can be evaluated.

Result: n-variate Rayleigh distribution for $r_1,\ldots,r_n$
temp=0;
for $j_1 = 0,\ldots,\infty$ **do**
 $\cdots$
 for $j_p = 0,\ldots,j_{p-1}$ **do**
  $c = \prod_{t=1}^{p} b_{t,j_{p-t+1}-j_{p-t+2}}$;
  for $\boldsymbol{\rho} \in \{-1,1\}^p$ **do**
   /* Set up the coefficient matrix */
   for $o = 1,\ldots,n-1$ **do**
    for $p = 1,\ldots,n-o$ **do**
     $t = \sum_{i=0}^{p-1}(n-i-1) + o$;
     $A_{o,p} = (j_{p-t+1} - j_{p-t+2})\,\rho_t$;
    end
   end
   /* Construct coefficient sums */
   for $i = 1,\ldots,n-1$ **do**
    $S_i = A_{i,1} + \sum_{w=1}^{n-i} A_{w,i} - \sum_{w=1}^{i-1} A_{i-w,1+w}$
   end
   /* Determine integral and add non-zero terms to the return value */
   if $S_i == 0$ *for all* i **then**
    temp = temp + c;
   end
  end
 end
end
return $\pi^p\gamma_{n,K}$ temp

Algorithm 2: Evaluation algorithm with notation as before and a_t as introduced in Algorithm 1.

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III. Applications

Outage Probabilites

In 'Infinite Series Representations of the Trivariate and Quadrivariate Rayleigh Distribution and their Applications' Y. Chen and C. Tellambura [7] discuss the application of multivariate Rayleigh distributions to determine the outage probability of three and four branch selection combining in correlated Rayleigh fading.

The outage probability is defined below, where γ is an output threshold and $\gamma_1, \ldots, \gamma_n$ are the respective average outputs.

$$\mathbb{P}_{out} = F_R\left(\sqrt{\frac{\gamma\rho_1}{\gamma_1}}, \ldots, \sqrt{\frac{\gamma\rho_n}{\gamma_n}}\right) \quad (8)$$

F_R is of Rayleigh distribution, with a covariance matrix K filled with the exemplary random values $\rho_0 \sim 0.5, \rho_1 \sim -0.12, \rho_2 \sim -0.09$ as covariances (values which we will recycle for the simulation testing in section 4 later on). Consider $\left(\sqrt{\gamma\rho_1/\gamma_1}, \ldots, \sqrt{\gamma\rho_n/\gamma_n}\right) = (1,1,1)$, such that the outage probability is visualised by the area underneath our density plot. Alternatively we can write the input as $\sqrt{\gamma}\bar{\gamma}$ where $\bar{\gamma} = \left(\sqrt{\rho_1/\gamma_1}, \ldots, \sqrt{\rho_n/\gamma_n}\right)$.

Note that the direction vectors are scaled,hence the cutoff points for the value $\sqrt{\gamma}$ are $(1,1,1), (1, \frac{1}{2}, \frac{1}{2})$ and $(1.\frac{1}{2}, \frac{1}{2})$ for the different directions.

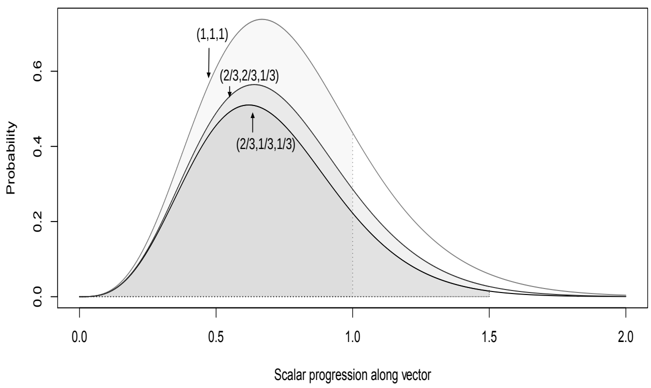

Fig. 1: Three dim. Rayleigh density, with the area underneath representing the outage probability

A multivariate Integral over the Rayleigh Distribution then reveals the cumulative distribution function, e.g. the outage probability as described in (8). We have formulated the probabilities as such below.

$$\mathbb{P}_{out}(\gamma) = \int_0^{\sqrt{\frac{\gamma\rho_1}{\gamma_n}}} \cdots \int_0^{\sqrt{\frac{\gamma\rho_1}{\gamma_1}}} f_R(t_1, \ldots, t_n) dt_1 \ldots dt_n \quad (9)$$

We visualise these results for different configurations of ρ_i and γ_i in figure 2. The x axis holds the given threshold γ, more specifically $\sqrt{\gamma}$ which we treat as a scalar factor to the different vectors of $\bar{\gamma}$.

As the dimension of 3 was chosen arbitrarily, and can be done for higher orders simultaneously, we have therefore found a mode of computation of outage probabilites for arbitrary branch SC correlated Rayleigh fading.

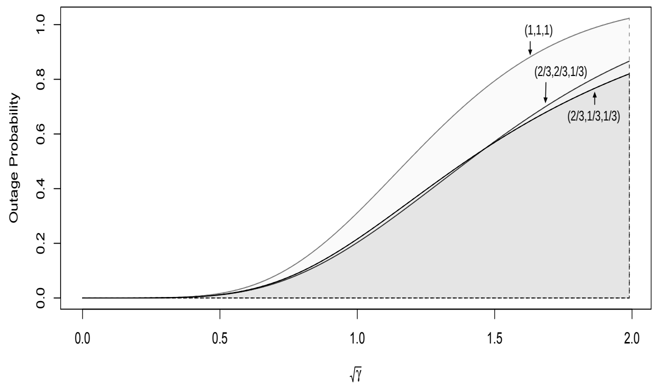

Fig. 2: Three dim. Rayleigh density, with CDF for the Outage Probability marked underneath

AMC Level Change Probabilites

The second application is based on a stochastic channel model expanded upon by Beard and Tekinay [1]. Here the authors intend to compute the probability for a channel to change adaptive modulation and coding levels (AMC) from one level to another. The introduction of an additional correlation variable (time - frequency) mandated a quadrivariate Rayleigh distribution representation.
Without the dimensional restriction, we can now evaluate arbitrary Rayleigh distributions, necessitated by models which use more than fourcorrelated input variables. In figure 3 (see Appendix) we show an example of a six dimensional Rayleigh distribution, which can be used for stochastic channel modelling with two or more additional correlated model variables (for example material constant or transmission type properties).
The correlation coefficients are $\rho_0 = 2.5, \rho_1 = 0.3, \rho_2 = -0.1, \rho_3 = 0.1, \rho_4, = -0.15\rho_5 = 0.2$ In this example. As mentioned before, two variables can be chosen to model Ω_t and A_t. We believe that these examples have emphazised the possible potential uses opened up by the newly gained representation.

Complexity

We construct an upper limit for the evaluation effort of algorithm 2, by setting the creation of matrix A and evaluation of the coefficients at $(n-1)(\frac{n}{2}-1)$ and $(n-1)n$ operations, respectively. The inner loop over the coefficients ρ runs 2^p iterations, while the outer loop has an upper bound of N^p iterations.

This gives a total upper limit of $\mathcal{O}(p^N 2^p n^2)$ computational operations. For four dimensions this gives us a maximum computation time increase by factor eight per order, which fig.2 shows we remain below. However this does lessen the practical application of the series expansion, as the additional contribution of the Bessel function addends are negligible. Virtually without loss of accuracy we can fix the maximum term order to naught, eliminating the need for a coefficient matrix (the approximation consists only of the $b_{t,0} = I_0(a_t)$ factors now, along with the lead coefficient $\pi^p\gamma_{n,K}$). This equates to a maximum computational effort of $\Theta(p) = \Theta(n^2)$. We have therefore used order zero approximations exclusively for the following sections.

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	Runtime		Contribution		
Highest order	Average	Total	Min	Average	Max
0	0.048	0.48	-	-	-
1	0.316	3.16	0	0	0
2	1.258	12.58	0	0	0
3	3.874	38.74	1.21E-15	1.28E-08	3.48E-08
4	10.007	100.07	0	9.64E-11	3.32E-10
5	22.865	228.65	0	2.22E-13	9.74E-13
6	47.234	472.34	0	5.55E-17	2.78E-16
7	95.969	959.69	0	2.78E-18	2.78E-17
8	171.666	1716.66	0	0	0
9	297.695	2976.95	0	0	0

Fig. 3: Runtimes and summand contribution for $n = 10$ evaluation points.

Unfortunately we could not realise the pseudocode given by Tekinay and Beard [1] as a functioning program. The Bessel expansion used does implicitly assume only positive values, since no distinction is made with respect to the argument sign. This means that Bessel functions of negative arguments are undefined in [1]. Furthermore we believe that some of the terms looped over in the method Beard and Tekinay put forth are not correctly specified. For example in algorithm 1 line 4 introduces a triple sum, with addends not containing the sum variables. This makes the loop over these terms somewhat pointless.

IV. Simulation

It is known that Rayleigh distributed random variables can be simulated through normally distributed RVs. We are therefore able to simulate a Rayleigh distributed random sample (similarly to the ansatz of our approximation) with $X \sim \mathcal{N}^n(0, S)$ and $Y \sim \mathcal{N}^n(0, S)$ where $S \in \mathbb{R}^{n \times n}$. The covariance matrix is equivalent to the Matrix K defined in section 2, with the zero value diagonals removed.

We generate a random sample of normal RVs and convert them to Rayleigh RVs. The resulting samples can then be used to form empirical density functions via an n-dimensional grid of cubes, containing different amounts of the sample. We have assumed the three dimensional case to lower the computational effort (which grows to the power of the dimension) from the channel modeling example in section 3. We generated 10^9 samples for each vector for optimal results, e.g. 3 Billion three dimensional samples altogether, with an adaptive grid mesh ranging between 0.01 and 0.001, depending on the local sample count.

Additionally we applied a kernel density estimator to the epdf to compensate for random peaks within the sample (dotted red). The plots below show the results along different vector directions, giving the approximation (solid black) along with both density estimation and epdf cell values (grey bars), as well as the interpolated absolute deviation between epdf and expression (7) (dotted blue).

We can clearly see that the shape of the approximation holds up, and the probabilities of the empirical, kernel density estimation and series approximation show only marginal deviations from one another. This gives strong indication for the correctness of the approximation.

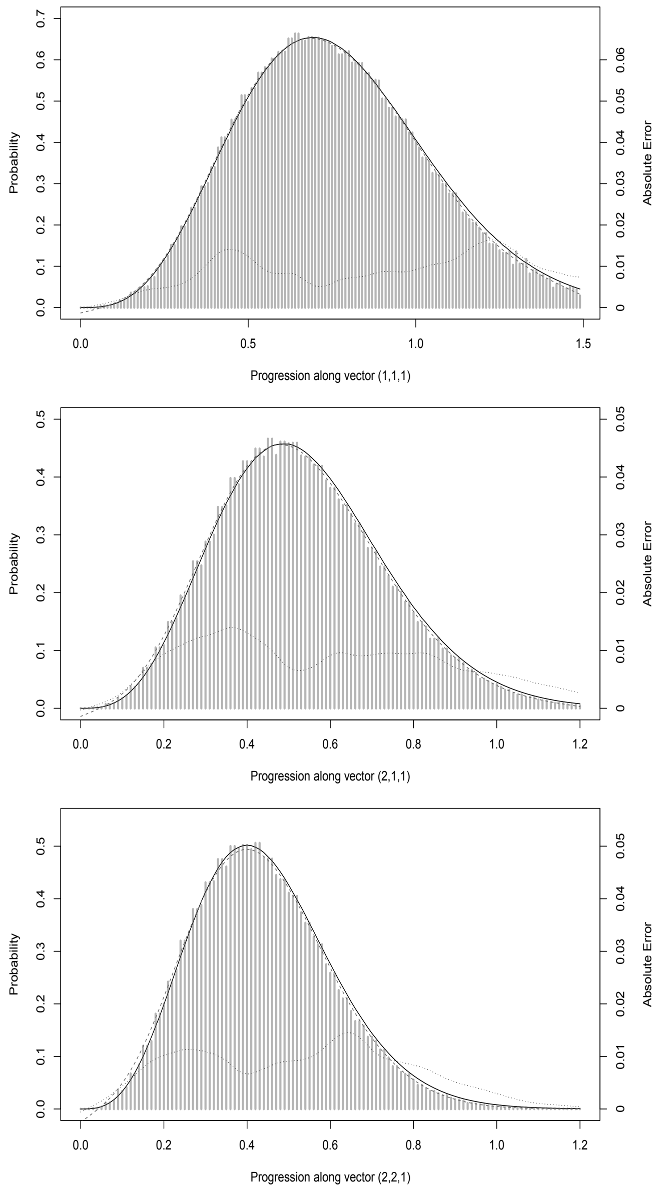

Fig. 4: Histogramm of simulated data with kernel estimator and approximation of order naught.

V. Conclusions

We have considered computing Rayleigh distributions with arbitrary dimensions. By revising the algorithm to determine the non-zero series contributions of the complex integral addends, we were able to compute the series' terms more efficiently. With expression (7) we have therefore devised a new, more general series representation for now arbitrary dimensions.

Besides the covariance matrix assumptions about uncorrelated in-phase and quadrature parts [1] no further restrictions have to be considered. The provided pseudocode gives a blueprint on how to implement computations for practical applications.

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APPENDIX

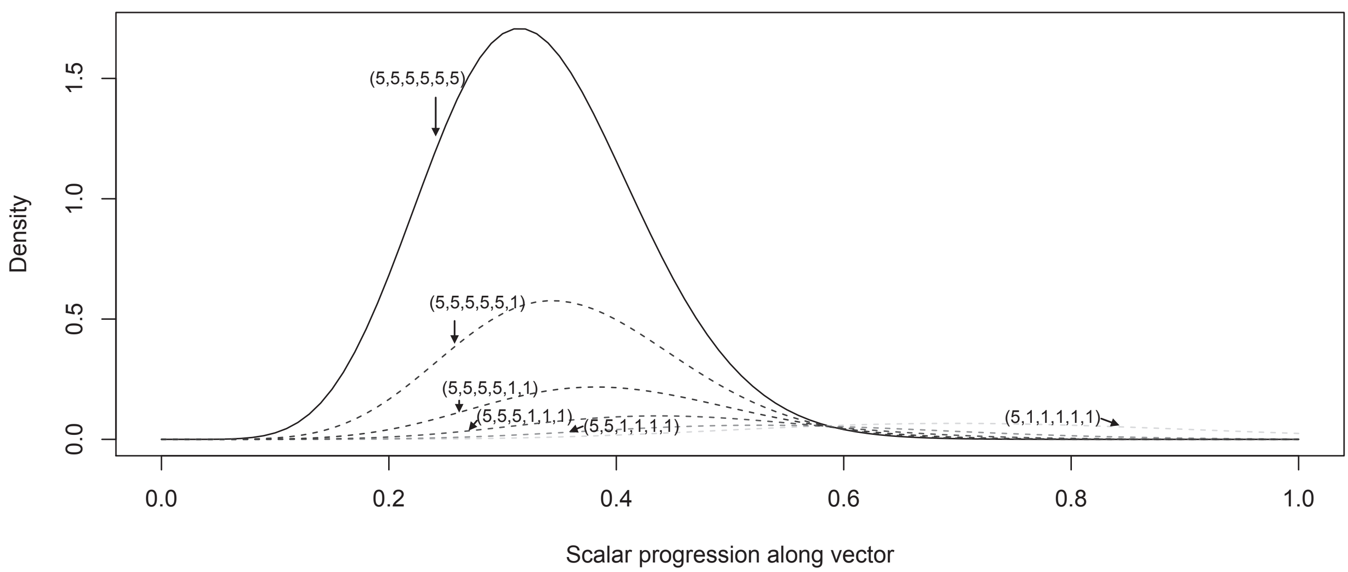

Fig. 5: 6 dim. Rayleigh Density along 6 different directional vectors.

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