A Sensitive of Maximum Generally Weighted Moving Average Control Chart For AR(p) Processes

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***Abstract*— The maximum generally weighted moving average (MaxGWMA) control chart efficient of monitoring in both mean or variation of process. The objective of this paper is related to compare the performance of MaxGWMA chart and maximum exponentially weighted moving average (MaxEWMA) chart in the change detection of mean change for autocorrelated process. The observations are presented as second-order autoregressive process (AR(2)) and third-order autoregressive process (AR(3)). The performance of control chart is determined in term of average run length (ARL). The results show that MaxGWMA chart is more sensitive than MaxEWMA chart for all magnitude shift, when $\omega = 1.1$ (α is adjust parameter).**

***Index Terms*— Max EWMA, Max GWMA, Autocorrelated, Average Run Length**

I. INTRODUCTION

Control charts are usually used tools to improve the quality of product and service. The first control chart was proposed by Shewhart in 1931, namely Shewhart chart. This chart is good perform for detecting large change of process. Alternative control charts used for monitoring small change of process, namely exponentially weighted moving average (EWMA) and cumulative sum chart (CUSUM) chart. Most control charts are not able to detect mean and variance in the same chart. Hawkin [1] proposed CUSUM chart for detecting mean and variance, but it's complicate in practice. Some literatures have been improved control charts for monitoring mean and variance in the single chart. Cheng and Cheng [2] proposed Max chart for monitoring mean and variance when process observations are normal distribution. Xie [3] first introduced concept of MaxEWMA control chart, then Chen et al. [4] extended Xie's research. The Max EWMA chart was combined the two EWMA charts into one chart such that the chart can simultaneously monitor the process mean and variability. Cheng and Thaga [5] proposed MaxCUSUM chart for autocorrelated processes. They found that this chart is more sensitive than EWMA chart for detecting small and moderate shifts of mean and variance. Cheng and Thaga [6] were investigate that MaxCUSUM chart was better than MaxEWMA and Maxchart for detecting mean and standard deviation. Sheu et al. [7] proposed MaxGWMA control chart, they found that its able to detect the change in mean and variance. In addition, MaxGWMA chart is more sensitive than MaxEWMA chart.

The performance of control is usually measured by average run length (ARL). The ARL is average number of point plotted in control process denoted by ARL_0. On the other hand, ARL of out of control is average number of point plotted until process out of control when process shift denoted ARL_1. Ideally, an acceptable ARL of an in-control process should be large enough to detect a small change in parameter distribution. Normally, SPC technique is designed for normal distribution or independent and identically distribution (i.i.d) observation.

However, there are some event does not follow assumption. For example, the process is skewness distribution see Teh et al. [8] and Phanyem [9]. Moreover, some processes are serially dependent which occur in chemical process see Petcharat [10-11] , Areepong and Sukparungsee [12] and Sunthornwa et al. [13]. In this paper, we concern the sensitivity of Max GWMA and Max EWMA for monitoring change point in mean of autocorrelated process with exponential white noise.

II. CHARACTERISTICS OF CONTROL CHARTS

A. *Maximum Exponentially Weighted Moving Average Control Chart : (MaxEWMA)*

The concept of MaxEWMA control chart was introduced by Xie's research, then Chen et al. extended Xie's research called MaxEWMA control chart. This chart is able to monitoring mean and variance in the single chart. In this paper, we consider simple change point detection of process observations. Suppose X_t be the observations of autoregressive process with exponential white noise denoted by AR(p), which can be is define as

$$X_t = \mu + \phi_1 X_{t-1} + \phi_2 X_{t-2} + ... + \phi_p X_{t-p} + \varepsilon_t \,, \qquad (1)$$

where autoregressive process coefficient $|\phi_i| < 1$, for $\forall i = 1,2,..,p$ and $\varepsilon_t \sim Exp(\alpha)$. The initial value $\varepsilon_0 = 1$. It assumed the initial value of AR(p) process equal 1. We also consider chart under assumption that $\varepsilon_1, \varepsilon_2, ...,$ are independent random variable with distribution function $F(x, \alpha)$, the parameter $\alpha = \alpha_0$ before change point (in

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control) and the parameter $\alpha = \alpha_1$ after change point (out of control), where $\alpha_1 = \alpha_0 + \delta\alpha_0$. Let $X_{ij}, i = 1,2,3,...$ and $j = 1,2,3,...,n_i$, where i is index of subgroup. We defined two statistics that proposed by Chen et al (2001) as follow

$$U_i = \frac{\bar{X}_i - \mu}{\sigma / \sqrt{n_i}}$$

and $$V_i = \Phi^{-1}\left\{F\left[\frac{(n_i - 1)S_i^2}{\sigma^2}, n_i - 1\right]\right\}, \qquad (2)$$

where $\bar{X}_i$ and S_i^2 are sample mean and sample variance of i^{th} sample. Let $\Phi^{-1}(.)$ is inverse standard normal distribution and $F[\pi, \upsilon]$ is χ^2 distribution with degree of freedom υ. Two statistics of mean and variance defined from two statistics U_i and V_i defined as

$$W_i = (1-\lambda)W_{i-1} + \lambda U_i, \quad i = 1,2,...$$
$$Y_i = (1-\lambda)Y_{i-1} + \lambda V_i, \quad i = 1,2,... \quad , \qquad (3)$$

where λ is smoothing parameter, $0 < \lambda \le 1$, the initial value of EWMA statistics are $W_0 = 0$ and $Y_0 = 0$. $W_i \sim N(0, \sigma_{W_i}^2)$ and $Y_i \sim N(0, \sigma_{Y_i}^2)$ are independent.

The MaxEWMA statistic defined as follow :

$$ME_i = \max\{|W_i|, |Y_i|\}, \; i = 1,2,3,... \qquad (4)$$

Hence, ME_i is defined nonnegative, thus the Upper control limit for MaxEWMA is

$$UCL_{ME} = E(ME_i) + L_{ME}\sqrt{Var(ME_i)}, \qquad (5)$$

where $E(ME_i)$ and $Var(ME_i)$ are mean and variance of MaxEWMA statistic and L_{ME} is width of control limit when process is in control. Such that, the $ME_i > UCL_{ME}$ is out of control state.

B. Maximum Generally Weighted Moving Average Control Chart: (MaxGWMA)

Generally Weighted Moving Average (GWMA) is a moving average of past data in which data point is assign the weight. Hence, these weights decrease from the present data to remote data, the GWMA reflects important information on recent processes. The GWMA control chart concept was first introduce by Sheu and Lin [7] The MaxGWMA control chart was proposed by Sheu et al. [14], this chart was developed from MaxEWMA and GWMA control chart.

According to the sequence of independent sample the sequence of independent samples, let M represent the number of samples until the first occurrence of event A since the previous occurrence of event A since the previous occurrence of even A. Since

$$\sum_{m=1}^{\infty} P(M = m) = P(M = 1) + \mathrm{P(M = 2)} + ... + P(M = t) + P(M > t) = 1 \qquad (6)$$

Suppose $P(M = 1), \mathrm{P(M = 2)}, .., P(M = t)$ be the weights of a current sample, the previous sample,…, and the most out of data sample , respectively. Hence, $P(M > t)$ is weighted with the target value of the process. Two GWMA statistics of mean and variance defined from two statistics G_i and H_i as follow

$$H_i = P(M=1)U_i + P(M=2)U_{i-1} + ... + P(M=i)U_1 + P(M>i)H_0, i = 1,2,...$$
$$K_i = P(M=1)V_i + P(M=2)V_{i-1} + ... + P(M=i)V_1 + P(M>i)K_0, \; i = 1,2,..., \qquad (7)$$

where $i = 1,2,3,...$, the initial value of GWMA statistics are $H_0 = 0$ and $K_0 = 0$. Since, U_i and V_i are independent, such that H_i and K_i are also independent. We have $H_i \sim N(0, \sigma_{H_i}^2)$ and $K_i \sim N(0, \sigma_{K_i}^2)$, where $\sigma_{H_i}^2 = \sigma_{K_i}^2$ $= Q_i = \sum_{j=1}^{i}[P(M = j)]^2$.

It's easy to computation, $P(M > i) = q^{i^{\omega}}$ is chosen; then

$P(M = i) = P(M > i-1) - P(M > i) = q^{(i-1)^{\omega}} q^{i^{\omega}}$,

where $i = 1,2,3,...$, q is constant: $0 \le q \le 1$ and ω is adjustment parameter chosen by practitioner: $\omega > 0$.

The MaxGWMA statistic defined as follow:

$$MG_i = \max\{|G_i|, |H_i|\} \;, i = 1,2,3,... \qquad (8)$$

Hence, MG_i is defined nonnegative, thus the Upper control limit for MaxGWMA is

$$UCL_{MG} = E(MG_i) + L_{MG}\sqrt{Var(MG_i)}, \qquad (9)$$

where $E(MG_i)$ and $Var(MG_i)$ are mean and variance of MaxEWMA statistic and L_{ME} is width of control limit when process is in control. Such that, the $MG_i > UCL_{MG}$ is out of control state.

In addition, The GWMA statistic in the paper is correspond as Robert [15] when $\omega = 1$ and $q = 1 - \lambda$. Otherwise, EWMA control chart is a special case of GWMA control chart. Such that, MaxEWMA chart is special case of MaxGWMA chart when $\omega = 1$ and $q = 1 - \lambda$.

III. Evaluation and Comparison of Control Chart

In this section, we evaluated ARL of MaxEWMA and MaxGWMA control chart when the model of process observation is AR(p) with exponential white noise by using Mote Carlo simulation technique. In control process, we set the exponential parameter $\alpha = \alpha_0 = 1$ and process is out of control $\alpha_1 = \alpha_0 + \delta\alpha_0$ where δ is shift size; $\delta = 0.05, 0.2,$

0.4, 0.6, 0.8,1.0,1.5,2.0 and 3.0, respectively. The ARL in control is ARL_0 =370. Sheu and Lin [7] noted that large value of q and ω value ($0.5<\omega<1$) increase the sensitivity of GWMA chart, so we design values for parameter q (q =0.9, 0.95,0.99) and expand adjust parameter ω (ω = 0.8,1.1) $\lambda = 0.01$, 0.05 and 0.10 for MaxGWMA control chart to achieve the optimal value. Table I to table III show comparison of ARL_0 and ARL_1 between MaxGWMA and MaxEWMA charts for AR(2) process. table IV to table VI show comparison of ARL_0 and ARL_1 between MaxGWMA and MaxEWMA charts for AR(3) process. The results found that, MaxGWMA chart is more sensitive than MaxEWMA chart for all magnitudes of shifts when ω =1.1.

IV. Conclusion

In this paper, we evaluate ARL of MaxGWMA and MaxEWMA chart for AR(p) process. The results show that MaxEWMA chart is more sensitive than MaxGWMA chart for all magnitude shifts, when adjust parameter of MaxGWMA lass than 1 $(\omega<1)$. In addition, MaxGWMA chart is more sensitive than MaxEWMA chart for all magnitude shifts when adjust parameter equal 1.1.

TABLE I
Comparison of ARLs values for AR(2) process from MaxGWMA and MaxEWMA for $f_1 = 0.2$, $f_2 = 0.3$ with $q = 0.9$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 4.435$	$w = 1.1$, $h_u = 4.477$	$w = 1$, $h_u = 4.43556$
0	368.64	368.83	370.56
0.05	52.56	**18.03**	24.11
0.20	22.44	**9.40**	11.83
0.40	12.89	**6.07**	7.48
0.60	8.82	**4.70**	5.37
0.80	6.50	**3.92**	4.14
1.00	5.08	**3.00**	3.59
1.50	3.04	**2.00**	2.13
2.00	2.01	**1.79**	1.99
3.00	1.00	**1.00**	1.00

TABLE II
Comparison of ARLs values for AR(2) process from MaxGWMA and MaxEWMA for $f_1 = 0.2$, $f_2 = 0.3$ with $q = 0.95$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 4.3865$	$w = 1.1$, $h_u = 4.4503$	$w = 1$, $h_u = 4.4436$
0	370.208	369.83	370.564
0.05	120.597	**34.47**	48.333
0.20	54.325	**18.29**	24.71
0.40	31.601	**12.22**	15.795
0.60	21.562	**9.25**	11.706
0.80	16.472	**7.48**	9.186
1.00	13.063	**6.07**	7.778
1.50	8.288	**4.39**	5.093
2.00	5.992	**3.23**	4.001
3.00	3.678	**2.23**	2.865

TABLE III
Comparison of ARLs values for AR(2) process from MaxGWMA and MaxEWMA for $f_1 = 0.2$, $f_2 = 0.3$ with $q = 0.99$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 3.005$	$w = 1.1$, $h_u = 4.39$	$w = 1$, $h_u = 4.3$
0	370.90	368.83	370.56
0.05	120.60	**34.47**	48.33
0.20	54.33	**18.29**	24.71
0.40	31.60	**12.22**	15.80
0.60	21.56	**9.25**	11.71
0.80	16.47	**7.48**	9.19
1.00	13.06	**6.07**	7.78
1.50	8.29	**4.39**	5.09
2.00	5.99	**1.79**	4.00
3.00	3.68	**1.00**	2.87

TABLE IV
Comparison of ARLs values for AR(3) process from MaxGWMA and MaxEWMA for $f_1 = 0.4$, $f_2 = 0.2$, $f_3 = 0.1$ with $q = 0.9$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 10.28$	$w = 1.1$, $h_u = 10.335$	$w = 1$, $h_u = 10.344$
0	370.74	367.76	373.47
0.05	120.41	**20.12**	27.59
0.20	60.28	**11.15**	14.4
0.40	16.98	**9.24**	9.54
0.60	11.92	**7.02**	7.04
0.80	9.17	**4.97**	5.83
1.00	7.23	**4.00**	4.85
1.50	5.41	**3.00**	3.02
2.00	3.18	**2.00**	2.24
3.00	2.00	**1.14**	1.83

TABLE V
Comparison of ARLs values for AR(3) process from MaxGWMA and MaxEWMA for $f_1 = 0.4$, $f_2 = 0.2$, $f_3 = 0.1$ with $q = 0.95$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 6.95$	$w = 1.1$, $h_u = 10.206$	$w = 1$, $h_u = 9.98$
0	368.84	369.37	369.82
0.05	140.86	**38.13**	54.84
0.20	68.28	**21.56**	29.74
0.40	41.58	**15.02**	19.77
0.60	29.71	**11.68**	14.97
0.80	22.85	**9.66**	12.04
1.00	18.52	**8.02**	10.06
1.50	12.06	**6.00**	7.04
2.00	8.97	**4.92**	5.55
3.00	5.39	**3.00**	3.95

TABLE VI
COMPARISON OF ARLS VALUES FOR AR(3) PROCESS FROM MAXGWMA AND MAXEWMA FOR $f_1 = 0.4$, $f_2 = 0.2$, $f_3 = 0.1$ WITH $q = 0.99$

	MaxGWMA		MaxEWMA
shift	$w = 0.8$, $h_u = 4.3865$	$w = 1.1$, $h_u = 4.4503$	$w = 1$, $h_u = 4.436$
0	368.36	368.50	368.38
0.05	316.98	**161.28**	240.96
0.20	221.91	**95.09**	142.91
0.40	156.61	**66.48**	97.64
0.60	119.63	**52.13**	75.12
0.80	95.73	**43.13**	61.34
1.00	79.47	**39.91**	51.73
1.50	54.78	**27.36**	37.37
2.00	41.1	**21.97**	29.19
3.00	26.7	**15.8**	20.23

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