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Minimizing the Initial Capital for the Discrete-time Surplus Process with Investment Control Under Alpha-regulation

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ABSTRACT - We consider the minimum initial capital problem of the insurance company with investment. The insurance company has to reserve sufficient initial capital to ensure that probability of ruin does not exceed the given quantity α. We prove the existence of the minimum initial capital. Then we give an example in approximating the minimum initial capital for Lognormal claim of an motor insurance company in Thailand and Laplace rate of return of gold price in London gold market by the bisection technique, and probability of ruin computed by simulation.

INDEX TERMS - initial capital, insurance, probability of ruin, investment

I. Introduction

Nowadays, insurance business is widely interested in financial investment. Insurance business must manage rules and covenant of Office of Insurance Commission. Insurance is a risk treatment option which involves risk transfer from an insured to an insurer with an insurance contract. Security of the insurer depends on capital reserves, claim severity and premium income.

In this research, we consider an insurance model that claims are day-to-day happen, i.e., the insolvent opportunity may be occurs in every day, for example, motor insurance. The claim severity (capital outflow) at time n is described as the positive random variable Y_n , and the premium of insurer (capital inflow) is assumed as a linear equation, $c(n) = nc_0$, where $c_0 > 0$ is so-called a premium rate for one unit of time of insurer. Therefore, the quantity

$$U_n(u, c_0) = u + nc_0 - \sum_{k=1}^{n} Y_k \, : \, n \in \mathbb{N}, \qquad (1)$$

is the insurer's balance (or surplus) at time n with the constant $U_0(u, c_0) = u \geq 0$ as the initial capital.

In 2006, Reference [1] considered a case of discrete time surplus process (1) and proposed another approach to deriving recursive and explicit formulas for the probability of ruin with exponential claim severity Y_n.

In 2013, Reference [3] generalized the recursive formula of the probability of insolvency, introduced the minimum initial problem, and controlled the probability of insolvency so that it was not greater than a given quantity.

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Reference [2] studied the model (1) for the minimum initial capital under α of the discrete-time risk process in the case of motor insurance which separated two claim severities and calculated the regression analysis.

In addition, we permit that the insurer can borrow money for investing in a financial market; called stocks, described by the price process $\{S_n : n \in \mathbb{N}\}$ where $S_n > 0$ is the price of one share at the time n . Therefore, the rate of return at time n is given by

$$R_n = \frac{S_n - S_{n-1}}{S_{n-1}} = \frac{S_n}{S_{n-1}} - 1,$$

and the sequence of random variables $\{R_n : n \in \mathbb{N}\}$ is called rate of return process. Let δ_{n-1} represent the amount of investment at time n. This means that the insurer holds $\frac{\delta_{n-1}}{S_{n-1}}$ shares during period n, so that the value of these shares at the time n is

$$\frac{\delta_{n-1}}{S_{n-1}} \cdot S_n = \delta_{n-1}(1 + R_n).$$

We write $U_n(u, c_0, \delta_n)$ instead of the surplus process, where u is the initial capital, c_0 is the premium rate, and δ_n is the value of investment during period n . Then, for each $n \in \mathbb{N}$, the surplus process in (1) is extended as follows:

$$U_n(u, c_0, \delta_n) = U_{n-1}(u, c_0, \delta_{n-1}) + c_0 - Y_n + \delta_{n-1} R_n.$$

This leads to the

$$U_n(u, c_0, \delta_n) = u + nc_0 - \sum_{k=1}^{n} Y_k + \sum_{k=1}^{n} \delta_{n-1} R_k. \qquad (2)$$

II. Behaviors of Ruin Probability

Throughout this paper, we assume that the claim size process $\{Y_n : n \in \mathbb{N}\}$ and the rate of return process $\{R_n : n \in \mathbb{N}\}$ are independent and identically distributed (i.i.d.). In addition, they are mutually independent, and c_0 can be calculated in terms of the expected value principle, as the following equation $c_0 = (1 + \theta) E[Y_1]$ where $\theta > 0$ is the safety loading of insurer. We fix the value of investment of each period with $\delta_n = \delta_0$. Then we obtain the considered model as follows:

$$U_n(u, c_0, \delta_n) = u + nc_0 - \sum_{k=1}^{n} Y_k + \sum_{k=1}^{n} \delta_0 R_k. \qquad (3)$$

Let $u \geq 0$ be an initial capital. For each $n \in \mathbb{N}$, we have the probability of survival at the times n as

$$\varphi_n(u,c_0,\delta_0) := P(U_k(u,c_0,\delta_0) \geq 0 \text{ for all } k = 1,2,3,\dots,n), \quad (4)$$

and the probability of ruin at one of the time $1,2,3,\dots,n$ is given by

$$\Phi_n(u,c_0,\delta_0) = 1 - \varphi_n(u,c_0,\delta_0). \quad (5)$$

Definition2.1. Let $\{U_n(u,c_0,\delta_n): n \in \mathbb{N}\}$ be a surplus process as mentioned in (3), $\alpha \in (0,1), N \in \mathbb{N}$, $c_0 > 0$, and $u \geq 0$, if $\Phi_n(u,c_0,\delta_0) \leq \alpha$, then u is called an acceptable initial capital corresponding to (α, N, c_0, δ). We denoted by MIC(α, N, c_0, δ) the minimum initial capital corresponding to (α, N, c_0, δ) if $\min_{u\geq 0}\{u: \Phi_n(u,c_0,\delta_0) \leq \alpha\}$ exists.

We define the total claim size process $\{\Lambda_n: n \in \mathbb{N}\}$ by

$$\Lambda_n \coloneqq Y_1 + Y_2 + Y_3 + \cdots + Y_n$$

and define the process of the total rate of return $\{\Gamma_n: n \in \mathbb{N}\}$ by

$$\Gamma_n \coloneqq R_1 + R_2 + R_3 + \cdots + R_n.$$

The probability of survival at the time N as mentioned in (4) can be expressed as

$$\begin{aligned}\varphi_N(u,c_0,\delta_0) &= P(\Lambda_k - \delta_0\Gamma_k \leq u + kc_0 \ \ for\ all\ k = 1,2,3,\dots,N) \\ &= P\left(\bigcap_{i=1}^{N}\{\omega: \Lambda_i(\omega) - \delta_0\Gamma_i(\omega) \leq u + ic_0\}\right).\end{aligned} \quad (6)$$

From (6), we have

$$\varphi_N(u,c_0,\delta_0) = E[\prod_{i=1}^{N} I_{(-\infty,0]}(\Lambda_i - \delta_0\Gamma_i - ic_0 - u)] \quad (7)$$

where I_A is a characteristic function on the set A. Then $I_{(-\infty,0]}(a-u)$ is increasing and right continuous in u. This implies that $\prod_{k=1}^{N} I_{(-\infty,0]}(a_k - u)$ is also increasing and right continuous in u. Therefore, $\varphi_N(u,c_0,\delta_0)$ is increasing and right continuous. Moreover, we can conclude that $\Phi_N(u,c_0,\delta_0)$ is decreasing and also right continuous.
Let $F_{Y_1-\delta_0 R_1(y)}$ be the distribution function of $Y_1 - \delta_0 R_1$, i.e.,

$$F_{Y_1-\delta_0 R_1(y)} = P(\ Y_1 - \delta_0 R_1 \leq y).$$

Since $\{Y_n: n \in \mathbb{N}\}$ and $\{R_n: n \in \mathbb{N}\}$ are i.i.d. and mutually independent, $\{Y_n - \delta_0 R_n: n \in \mathbb{N}\}$ is i.i.d. We conclude that $F_{Y_n-\delta_0 R_n}(y) = F_{Y_1-\delta_0 R_1}(y)$ for all $y \in \mathbb{R}$.

Theorem2.2. Let $c_0 > 0$ and $\delta_0 \in \mathbb{R}$ be given. For each $N \in \mathbb{N}$,

$$\lim_{u\to\infty}\varphi_N(u,c_0,\delta_0) = 1 \ and \ \lim_{u\to\infty}\Phi_N(u,c_0,\delta_0) = 0. \quad (8)$$

Proof. Consider

$$\begin{aligned}P\left(\bigcap_{i=1}^{N}\{\omega: Y_i(\omega) - \delta_0 R_i(\omega) \leq u + c_0\}\right) &= \prod_{i=1}^{N} P(Y_i - \delta_0 R_i \leq u + c_0) \\ &= \prod_{i=1}^{N} F_{Y_i-\delta_0 R_i}(u + c_0) \\ &= \left(F_{Y_1-\delta_0 R_1}(u+c_0)\right)^N.\end{aligned} \quad (9)$$

Obviously, we obtain the following properties

$$\bigcap_{i=1}^{N}\{\omega: Y_i(\omega) - \delta_0 R_i(\omega) \leq u + c_0\} \subset \bigcap_{i=1}^{N}\{\omega: \Lambda_i(\omega) - \delta_0\Gamma_i(\omega) \leq Nu + ic_0\}. \quad (10)$$

Using (9) and (10), we obtain

$$\left(F_{Y_1-\delta_0 R_1}(u+c_0)\right)^N \leq \varphi_N(Nu,c_0,\delta_0) \leq 1.$$

Since

$$\lim_{u\to\infty}\left(F_{Y_1-\delta_0 R_1}(u+c_0)\right)^N = 1,$$

we conclude that

$$\lim_{u\to\infty}\varphi_N(u,c_0,\delta_0) = 1,$$

and

$$\lim_{u\to\infty}\Phi_N(u,c_0,\delta_0) = 1 - \lim_{u\to\infty}\varphi_N(u,c_0,\delta_0) = 0.$$

Corollary2.3. Let $\alpha \in (0,1), \delta_0 \in \mathbb{R}$, $N \in \mathbb{N}$ and $c_0 > 0$ be given. Then there exists $\tilde{u} \geq 0$ such that u is an acceptable initial capital corresponding to $(\alpha, N, c_0, \delta_0)$ for all $u \geq \tilde{u}$.

III. Existence of Minimum Initial Capital

A quantity α is described as the most acceptable probability that the insurance company will become insolvent. As a result of Corollary 2.3, we obtain that $\{u \in [0,\infty): \Phi_N(u,c_0,\delta_0) \leq \alpha\}$ is a non-empty set. Therefore, we can always choose an initial capital which makes the value of finite-time probability of ruin not to exceed the quantity α. Since the set $\{u \in [0,\infty): \Phi_N(u,c_0,\delta_0) \leq \alpha\}$ is infinite, there are many acceptable initial capital corresponding to $(\alpha, N, c_0, \delta_0)$. In this section, we shall prove the existence of

$$\text{MIC}(\alpha, N, c_0, \delta_0) = \min\{u \in [0,\infty): \Phi_N(u,c_0,\delta_0) \leq \alpha\}. \quad (11)$$

Lemma2.4. [3] Let a, b and α be real numbers such that $a \leq b$. If f is decreasing and right continuous on $[a,b]$ and $\alpha \in [f(b), f(a)]$, then there exists $d \in [a,b]$ such that

$$d = \min\{x \in [a,b]: f(x) \leq \alpha\}. \quad (12)$$

Theorem2.5. Let $\alpha \in (0,1)$, $N \in \mathbb{N}$ and $c_0 > 0$. If $\Phi_N(u,c_0,\delta_0) > \alpha$, then theres exist $u^* \geq 0$ such that

$$u^* = \text{MIC}(\alpha, N, c_0, \delta_0).$$

Proof. By Corollary 2.3, there exists $\tilde{u} > 0$ such that $\Phi_N(\tilde{u},c_0,\delta_0) < \alpha$,i.e., $\alpha \in [\Phi_N(\tilde{u},c_0,\delta_0), \Phi_N(0,c_0,\delta_0)]$. Since $\Phi_N(u,c_0,\delta_0)$ is decreasing and right continuous, by Lemma 2.4 there exist $u^* \in [0,\tilde{u}]$ such that

$$u^* = \min\{u \in [0,\tilde{u}): \Phi_N(u,c_0,\delta_0) \leq \alpha\} = \min\{u \in [0,\infty): \Phi_N(u,c_0,\delta_0) \leq \alpha\}.$$

That is, $u^* = \text{MIC}(\alpha, N, c_0, \delta_0)$.

IV. Applications

We consider the data of the claim size of motor insurance company in Thailand. The data consist of the claim size (Y_n) shown in Fig 1. We assume that the claim size has lognormal distribution with the two parameters. The probability density function for lognormal distribution is given by

$$f(x;\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}}\exp\left(-\frac{(lnx-\mu)^2}{2\sigma^2}\right), \mu \in \mathbb{R}, \sigma > 0.$$

Here, μ is a location parameter and σ is scale parameter.

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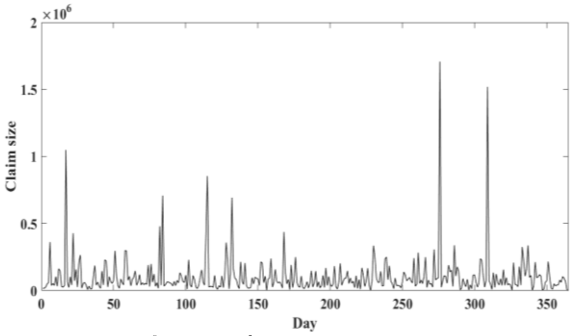

Fig 1. Claim size of motor insurance in one year

In addition, we consider the data of the rate of return in gold price from January 2006 to October 2014 provided by London gold market. The data consist of the gold price and the times shown in Fig 2. We assume that the rate of return has Laplace distribution with the two parameters. The probability density function for Laplace distribution is given by

$$f(x;\gamma,\lambda)=\frac{\lambda}{2}\exp(-\lambda|x-\gamma|)\,;\ \gamma>0,\lambda>0.$$

Here, γ is a location parameter and λ , which is sometimes referred to as the diversity, is a scale parameter.

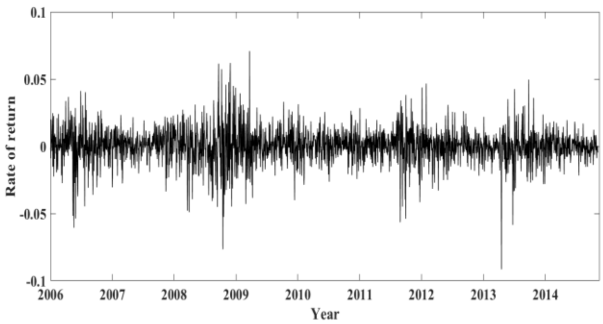

Fig 2. Rate of Return of gold price from January 2006 to October 2014 provided by London gold market

The claim size process $\{Y_n : n \in \mathbb{N}\}$ and rate of return process $\{R_n : n \in \mathbb{N}\}$ are defined as mentioned in (2) and assumed to be independent and identically distributed. Using the MLE method, we find that the estimated parameter vector of the claim size is $(\mu,\sigma) = (11.1, 0.9939)$ for lognormal distribution, and the estimated parameter of the rate of return is also found to be $(\gamma,\lambda) = (0.0005, 107.4)$ for Laplace distribution. The simulation result of finite-time probability of ruin is carried out with 100000 paths for the surplus process as mentioned in the surplus process (2). We assume that Y_1~Lognormal $(11.1, 0.9939)$ and R_1 ~Laplace$(0.0005, 107.4)$.

Next, we approximate the minimum initial capital, MIC$(\alpha, N, c_0, \delta_0)$, by applying the bisection technique in [3] for the discrete-time surplus process with claim size and rate of return as lognormal and Laplace distributions, respectively. We obtain minimum initial capital as shown in Table I.

Table I
Minimum initial capital **MIC**$(\boldsymbol{\alpha}, \boldsymbol{N}, \boldsymbol{c_0}, \boldsymbol{\delta_0})$ in the discrete-time surplus process with lognormal claim $(\boldsymbol{\theta = 0.1})$

$\boldsymbol{N}$	$\boldsymbol{\alpha = 0.1}$			$\boldsymbol{\alpha = 0.2}$		
	$p = 0.0$	$p = 0.1$	$p = 0.2$	$p = 0.0$	$p = 0.1$	$p = 0.2$
10	575867	522949	531683	327942	314261	311657
20	806427	782732	795483	506493	494922	481225
30	966287	948900	955355	627710	612507	606455
40	1094238	1062499	1083203	714844	704688	698201
50	1194109	1206250	1200115	773346	776076	767139
100	1570315	1544058	1553128	1031250	993677	1003436
200	1870712	1847217	1835937	1221758	1216821	1215625
300	2044298	2020472	2014222	1313347	1316672	1311867
400	2127431	2109268	2074990	1368896	1367781	1363530
500	2179230	2134375	2200049	1404877	1382811	1390020
1000	2246219	2203119	2205963	1444997	1427876	1430247

Table I shows the chosen model parameter combinations $\theta = 0.1$ and setting $\theta = 0.1, 0.2$ and 0.3 respectively. We set $p = 0.1$ and 0.2, where p is the proportion of initial capital with investment, we have $\delta_0 = pU_0$. The approximation of the minimum initial capital MIC$(\alpha, N, c_0, \delta_0)$ is done by the bisection technique, with $v_0 = 0, u_0 = 3000000, and$ $\Phi_N(u, c_0, \delta_0)$ is computed by simulation.

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