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Exact Solutions and Numerical Comparison of Methods for Solving Fractional-Order Differential Systems

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***Abstract*—In this paper, we apply the Laplace-Adomian-Pade method (LAPM), which is based on the Laplace-Adomian decomposition method (LADM), and the Adams-Bashforth-Moulton type predictor-corrector scheme to solve a fractional-order model of the glucose-insulin homeostasis in rats for analytical and numerical solutions, respectively. Moreover, the exact solutions of this fractional-order model, which are solved using the Laplace transform, are employed to numerically and graphically compare with the results obtained using the two methods. The LAPM and the predictor-corrector scheme can also be applied simply and efficiently to other fractional-order differential systems arising in engineering problems.**

***Index Terms*—Laplace-Adomian decomposition method, Padé approximation, Adams-Bashforth-Moulton predictor-corrector scheme, Caputo fractional-order derivative, Exact solution**

I. INTRODUCTION

VARIOUS phenomena occurring in the applied sciences [1]–[3] and engineering [4], [5] have recently been modeled by fractional-order differential equations (FDEs). Several methods are now being used to analytically solve FDEs, for example, the Adomian decomposition method (ADM) [6], the Laplace-Adomian decomposition method (LADM) [7], the Duan-Rach modified ADM [8], the homotopy analysis method (HAM) [9], and the multistep generalized differential transform method (MSGDTM) [10]. Numerical solutions of FDEs can also be obtained via many approaches such as the Adams-Bashforth-Moulton type predictor-corrector scheme or PECE (Predict, Evaluate, Correct and Evaluate) method [11], the Galerkin finite element method [12], the Legendre wavelets method [4] and the spectral collocation method [13].

In the present paper, we study the system of fractional order differential equations in (1). This system is a generalization of an integer-order system proposed by Lombarte et al. [14] as a model for glucose-insulin homeostasis in healthy rats.

$$\begin{aligned} {}_{C}D_a^\alpha i(t) &= c_1 g(t) - c_6^\alpha i(t), \\ {}_{C}D_a^\alpha g(t) &= -c_4(i(t)-c_5) - c_2 i(t) + c_0 d(t) - c_3, \quad (1) \\ {}_{C}D_a^\alpha d(t) &= -c_7^\alpha d(t), \end{aligned}$$

with initial conditions

$$i(0) = i^0,\ g(0) = g^0,\ d(0) = d^0. \quad (2)$$

In (1), ${}_{C}D_a^\alpha$ is the Caputo fractional derivative operator of order $\alpha \in (0,1]$ starting from $t = a$. The state variables are the blood insulin concentration $i(t)$, the blood glucose concentration $g(t)$ and the amount of glucose in the intestine $d(t)$. The constants c_i, $i = 1, 2, ..., 7$ are non-negative model parameters.

In this paper, we solve Eq. (1) by three methods and compare the solutions. The first method is an analytical method based on the Laplace-Adomian-Padé method (LAPM) [1], [15]. The second method is a numerical method based on the Adams-Bashforth Moulton predictor-corrector method (PECE) [11]. The third method uses Laplace transforms to find an exact solution of the system.

The paper is organized as follows. In section 2, preliminary definitions and properties are given. In section 3, a description of the methods used in our work are briefly given. In section 4, the three methods for solving the system (1) and the solutions obtained are given. Finally, section 5 includes a discussion of the results and the conclusions.

II. PRELIMINARY DEFINITIONS AND PROPERTIES

In this section, we provide definitions of fractional-order operators such as the Riemann-Liouville fractional integral and the Caputo fractional derivative. The definition of the Mittag-Leffler functions and their important properties are briefly given.

A function $f(t)$ $(t > 0)$ is said to be in the space C_α $(\alpha \in \mathbb{R})$ if it can be expressed as $f(t) = t^p g(t)$ for some $p > \alpha$, where $g(t)$ is continuous in $[0, \infty)$. The function is also said to be in the space C_α^m if $f^{(m)} \in C_\alpha$, $m \in \mathbb{N}$ (for further details see [16]).

Definition 2.1: [16]. The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$ with $a \geq 0$ is defined as

$${}_{RL}J_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \quad (3)$$

where $\Gamma(\cdot)$ is the gamma function.

Definition 2.2: [16]. For a positive real number α, the Caputo fractional derivative of order α with $a \geq 0$ is defined

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in terms of the Riemann-Liouville fractional integral, i.e., ${}_CD_a^\alpha f(t) = {}_{RL}J_a^{m-\alpha} f^{(m)}(t)$, or it can be expressed as

$$ {}_CD_a^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \tag{4} $$

where $m-1<\alpha<m$, $t \geq a$ and $f \in C_{-1}^m,\ m \in \mathbb{N}$.
An important property of the Riemann-Liouville fractional integral and the Caputo fractional derivative of the same order γ can be written as [16]

$$ {}_{RL}J_a^\alpha\ {}_CD_a^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}, \tag{5} $$

where $m-1<\alpha<m,\ f \in C_\alpha^m$ for $m \in \mathbb{N}$ and $\alpha \geq -1$.

The Laplace transforms of a fractional derivative in the Caputo sense and of some types of the Mittag-Leffler functions [16] are as follows.

Lemma 2.1: [16] The Laplace transform of the Caputo fractional derivative of order $m-1<\alpha<m$ is

$$ \mathscr{L}\{{}_CD_a^\alpha f(t)\} = s^\alpha F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \tag{6} $$

where $F(s) = \mathscr{L}\{f(t)\}$.

Definition 2.3: [16] Given $\alpha,\ \beta > 0$, and $z \in \mathbb{C}$. The one parameter Mittag-Leffler function E_α is defined as

$$ E_\alpha(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(j\alpha+1)}, \tag{7} $$

and the Mittag-Leffler function with two parameters is defined as

$$ E_{\alpha,\beta}(z) = \sum_{j=0}^\infty \frac{z^j}{\Gamma(j\alpha+\beta)}. \tag{8} $$

Lemma 2.2: [17] The Laplace transforms for several Mittag-Leffler functions are given by

$$ \mathscr{L}\{E_\alpha(-\lambda t^\alpha)\} = \frac{s^{\alpha-1}}{s^\alpha+\lambda}, \tag{9} $$

$$ \mathscr{L}\{t^{\beta-1}E_{\alpha,\beta}(-\lambda t^\alpha)\} = \frac{s^{\alpha-\beta}}{s^\alpha+\lambda}, \tag{10} $$

provided that $s > |\lambda|^{1/\alpha}$, where λ is a constant parameter.

III. Algorithms of the Methods

In this section, we will present the methods and the algorithms for the LADM and the PECE methods that we use to analytically and numerically solve the fractional-order system (1).

A. The Laplace Adomian Decomposition Method

The Laplace Adomian Decomposition Method (LADM) [18], [19] for solving FDEs or a system of FDEs is as follows. Consider the following fractional-order initial value problem:

$$ {}_CD_a^\alpha u(t) + R(u) + N(u) = g(t), \tag{11} $$

where $m-1<\alpha<m$, $m \in \mathbb{N}$ and the solution $u(t)$ satisfies some given initial conditions. In Eq. (11), ${}_CD_a^\alpha$ denotes the Caputo fractional derivative of order α with respect to t, R and N are linear and nonlinear operators of u, respectively, and g is a source term.

Taking the Laplace transform of both sides of Eq. (11) and then applying the formula (2.1) to the resulting equation, we obtain

$$ \mathscr{L}\{{}_CD_a^\alpha u(t)\} + \mathscr{L}\{R(u)\} + \mathscr{L}\{N(u)\} = \mathscr{L}\{g(t)\}, $$

$$ \mathscr{L}\{u(t)\} = \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1}u^{(k)}(0) + \frac{1}{s^\alpha}\mathscr{L}\{g(t)\} - \frac{1}{s^\alpha}\mathscr{L}\{R(u)\} - \frac{1}{s^\alpha}\mathscr{L}\{N(u)\}. \tag{12} $$

In the LADM, we define the solution $u(t)$ as an infinite series

$$ u(t) = \sum_{i=0}^\infty u_i(t), \tag{13} $$

and represent the nonlinear term N by an infinite series of Adomian polynomials

$$ N(u) = \sum_{i=0}^\infty A_i, \tag{14} $$

where the A_i polynomials can be determined by the following formula

$$ A_i = \frac{1}{i!}\frac{d^i}{d\lambda^i} N\left(\sum_{k=0}^\infty \lambda^k u_k\right)\Bigg|_{\lambda=0},\ i \geq 0. \tag{15} $$

Substituting (13) and (14) into (12), we get

$$ \mathscr{L}\left\{\sum_{i=0}^\infty u_i(t)\right\} = \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1}u^{(k)}(0) + \frac{1}{s^\alpha}\mathscr{L}\{g(t)\} - \frac{1}{s^\alpha}\mathscr{L}\left\{R\left(\sum_{i=0}^\infty u_i(t)\right)\right\} - \frac{1}{s^\alpha}\mathscr{L}\left\{\sum_{i=0}^\infty A_i\right\}. \tag{16} $$

Then we have the Adomian recursion scheme as follows

$$ \begin{aligned} \mathscr{L}\{u_0\} &= \frac{1}{s^\alpha}\sum_{k=0}^{m-1} s^{\alpha-k-1}u^{(k)}(0) + \frac{1}{s^\alpha}\mathscr{L}\{g(t)\}, \\ \mathscr{L}\{u_{n+1}\} &= -\frac{1}{s^\alpha}\mathscr{L}\{R(u_n(t))\} - \frac{1}{s^\alpha}\mathscr{L}\{A_n\},\ n \geq 0. \end{aligned} \tag{17} $$

Applying the inverse Laplace transform to Eq. (17), we can evaluate the solution components u_n $(n \geq 0)$. Then the n-term approximation of the solution is

$$ \varphi_n(t) = \sum_{i=0}^{n-1} u_i(t), \tag{18} $$

which in the $\lim_{n\to\infty}$ yields the exact solution of Eq. (11) as

$$ u(t) = \lim_{n\to\infty} \varphi_n(t). \tag{19} $$

Sometimes the exact solution $u(t)$ in Eq. (19) may be written in a closed form.

If the exact solution $u(t)$ in Eq. (19) can be written as a power series in which an independent variable t is raised to fractional powers and the radius of convergence of the series is quite small, then the solution might not be valid for the entire domain of interest. Therefore, a technique of analytical continuation to obtain a solution valid in the domain of interest is required. The Padé approximant method constructs a rational function in t as an approximation for

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a slowly converging or diverging power series in t. It is one of the well-known convergence acceleration techniques, which can be applied to an n-term polynomial approximation $\phi_n(t)$. We denote the $[m/m]$ diagonal Padé approximant of $\phi_n(t)$ in t by $[m/m]\{\phi_n(t)\}$, i.e., $\text{Padé}_{[m/m]}\{\phi_n(t)\} = [m/m]\{\phi_n(t)\}$, where $m = (n-1)/2$ if $n = 3, 5, 7, \ldots$, and $m = n/2$ if $n = 4, 6, 8, \ldots$. However, if each variable t in the n-term approximation $\phi_n(t)$ has a fractional power, then we must change such fractions to new integer powers using a transformation before applying the Padé approximants. The LADM improved by the Padé approximants is called the Laplace-Adomian-Padé method (LAPM).

B. Adams-Bashforth-Moulton predictor-corrector scheme

Currently, the Adams-Bashforth-Moulton type predictor-corrector scheme or the PECE [11] method is extensively employed to numerically solve FDEs. In our work, we will use this method to obtain approximate numerical solutions for system (1). The relevant formulas of the method are as follows.

Consider the fractional-order initial value problem (FIVP)

$$\begin{aligned} {}_C D_a^\alpha u(t) &= f(t, u(t)), \quad 0 \le t \le T, \\ u^{(k)}(0) &= u_0^{(k)}, k = 0, 1, \ldots m-1, \quad \alpha \in (m-1, m), \end{aligned} \tag{20}$$

where f is a nonlinear function and m is a positive integer. The FIVP (20) can be transformed to the following Volterra integral equation

$$u(t) = \sum_{k=0}^{m-1} u_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau. \tag{21}$$

In order to approximate the integral in (21), we discretize the entire time T as the uniform grid $\{t_n = nh : n = 0, 1, \ldots N\}$ for some integer N and the step size $h := T/N$. Let $u_h(t_n)$ denotes the approximation to $u(t_n)$. Suppose that we have already calculated approximations $u_h(t_j), j = 1, 2, \ldots, n$, then the approximation $u_h(t_{n+1})$ of the FIVP (20) can be computed using the PECE method as follows:

$$\begin{aligned} u_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, u_h^P(t_{n+1})) + \\ \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^{n} a_{j,n+1} f(t_j, u_h(t_j)), \end{aligned} \tag{22}$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1. \end{cases} \tag{23}$$

The initial approximation $u_h^P(t_{n+1})$ in Eq. (22) is called a predictor and is given by

$$u_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{n} b_{j,n+1} f(t_j, u_h(t_j)), \tag{24}$$

where

$$b_{j,n+1} = \frac{h^\alpha}{\alpha}((n+1-j)^\alpha - (n-j)^\alpha). \tag{25}$$

IV. Main results

In this section, we demonstrate the use of the LAPM and the PECE as described above to solve the FIVP in Eqs. (1)-(2). However, we will first obtain the exact solution of the fractional-order model of glucose-insulin homeostasis in healthy rats using the Laplace transform method.

Taking the Laplace transform of system (1), we have

$$\begin{aligned} &\mathscr{L}\{{}_C D_a^\alpha i(t)\} = \mathscr{L}\{c_1 g(t) - c_6^\alpha i(t)\}, \\ &\mathscr{L}\{{}_C D_a^\alpha g(t)\} = \mathscr{L}\{-c_4(i(t) - c_5) - c_2 i(t) + c_0 d(t) - c_3\}, \\ &\mathscr{L}\{{}_C D_a^\alpha d(t)\} = \mathscr{L}\{-c_7^\alpha d(t)\}. \end{aligned} \tag{26}$$

Substituting the initial conditions (2) into (26), we obtain

$$\begin{aligned} &s^\alpha I(s) - s^{\alpha-1} i^0 = -c_6^\alpha I(s) + c_1 G(s), \\ &s^\alpha G(s) - s^{\alpha-1} g^0 = -(c_2 + c_4) I(s) + c_0 D(s) + \frac{c_4 c_5 - c_3}{s}, \\ &s^\alpha D(s) - s^{\alpha-1} d^0 = -c_7^\alpha D(s), \end{aligned} \tag{27}$$

where $I(s) = \mathscr{L}\{i(t)\}$, $G(s) = \mathscr{L}\{g(t)\}$, and $D(s) = \mathscr{L}\{d(t)\}$.

Algebraically manipulating the resulting system, we obtain the following linear system for the variables $I(s)$, $G(s)$, and $D(s)$:

$$\begin{bmatrix} s^\alpha + c_6^\alpha & -c_1 & 0 \\ c_2 + c_4 & s^\alpha & -c_0 \\ 0 & 0 & s^\alpha + c_7^\alpha \end{bmatrix} \begin{bmatrix} I(s) \\ G(s) \\ D(s) \end{bmatrix} = \begin{bmatrix} s^{\alpha-1} i^0 \\ \frac{s^\alpha g^0 + c_4 c_5 - c_3}{s} \\ s^{\alpha-1} d^0 \end{bmatrix}. \tag{28}$$

From the last equation in system (28), we can easily obtain $D(s)$ as follows

$$D(s) = \frac{s^{\alpha-1} d^0}{s^\alpha + c_7^\alpha}. \tag{29}$$

Taking the inverse Laplace transform of (29) and then using the formula (9), we obtain

$$d(t) = d^0 E_\alpha(-c_7^\alpha t^\alpha). \tag{30}$$

Applying Cramer's rule to system (28) to obtain the remaining two variables, we find

$$I(s) = \frac{s^{2\alpha} i^0 + c_1(s^\alpha + c_7^\alpha)(g^0 s^\alpha + \mu_2) + c_1 c_0 d^0 s^\alpha}{s(s^\alpha + c_7^\alpha)(s^\alpha - \beta_1)(s^\alpha - \beta_2)}, \tag{31}$$

$$\begin{aligned} G(s) = &\frac{1}{s(s^{2\alpha} + c_6^\alpha s^\alpha + c_1(c_2 + c_4))} \Big[c_0 d^0 s^\alpha (s^\alpha + c_6^\alpha) \\ &+ (s^\alpha + c_6^\alpha)(s^\alpha g^0 + c_4 c_5 - c_3)(s^\alpha + c_7^\alpha) \\ &- s(s^\alpha + c_7^\alpha)(c_2 + c_4 s^{\alpha-1} i^0) \Big] \end{aligned} \tag{32}$$

Separating the solutions in Eqs. (31)-(32) into partial fractions, we have

$$\begin{aligned} I(s) = &\frac{i^0}{s} + \frac{\phi_1}{\beta_1 - \beta_2}\left(\frac{s^{-1}}{s^\alpha - \beta_1}\right) - \frac{\phi_2}{\beta_1 - \beta_2}\left(\frac{s^{-1}}{s^\alpha - \beta_2}\right) \\ &- \phi_3 \left(\frac{s^{-1}}{s^\alpha + c_7^\alpha}\right), \end{aligned} \tag{33}$$

$$\begin{aligned} G(s) = &\frac{g^0}{s} + \frac{\omega_1}{\beta_1 - \beta_2}\left(\frac{s^{-1}}{s^\alpha - \beta_1}\right) - \frac{\omega_2}{\beta_1 - \beta_2}\left(\frac{s^{-1}}{s^\alpha - \beta_2}\right) \\ &- \omega_3 \left(\frac{s^{-1}}{s^\alpha + c_7^\alpha}\right), \end{aligned} \tag{34}$$

where

$$\phi_1 = \beta_1^2 i^0 + c_1(g^0\beta_1 + \mu_2) + \frac{c_1\, c_0 d^0 \beta_1}{\beta_1 + c_7^\alpha}, \quad (35)$$

$$\phi_2 = \beta_2^2 i^0 + c_1(g^0\beta_2 + \mu_2) + \frac{c_1\, c_0 d^0 \beta_2}{\beta_2 + c_7^\alpha}, \quad (36)$$

$$\phi_3 = \frac{c_1\, c_0 d^0 c_7^\alpha}{(\beta_1 + c_7^\alpha)(\beta_2 + c_7^\alpha)}, \quad (37)$$

$$\beta_1 = \frac{-c_6^\alpha + \sqrt{(c_6^\alpha)^2 - 4\mu_1 c_1}}{2}, \quad (38)$$

$$\beta_2 = \frac{-c_6^\alpha - \sqrt{(c_6^\alpha)^2 - 4\mu_1 c_1}}{2}, \quad (39)$$

$$\omega_1 = g^0\beta_1^2 + c_6^\alpha\mu_2 + \beta_1\mu_2 + c_6^\alpha g^0 + \frac{(\beta_1 + c_6^\alpha)c_0\beta_1 d^0}{\beta_1 + c_7^\alpha} - \mu_1\beta_1 i^0, \quad (40)$$

$$\omega_2 = g^0\beta_2^2 + c_6^\alpha\mu_2 + \beta_2\mu_2 + c_6^\alpha g^0 + \frac{(\beta_2 + c_6^\alpha)c_0\beta_1 d^0}{\beta_2 + c_7^\alpha} - \mu_1\beta_2 i^0, \quad (41)$$

$$\omega_3 = \frac{(c_6^\alpha - c_7^\alpha)c_0 c_7^\alpha d^0}{(\beta_1 + c_7^\alpha)(\beta_2 + c_7^\alpha)}, \quad (42)$$

$$\mu_1 = c_2 + c_4, \quad \text{and} \quad \mu_2 = c_4c_5 - c_3. \quad (43)$$

Taking the inverse Laplace transforms for $I(s)$ and $G(s)$ in Eqs. (33) and (34), respectively, and then using the formula (10), we finally obtain the solutions in terms of the Mittag-Leffler functions as follows:

$$i(t) = i^0 + \frac{\phi_1}{\beta_1 - \beta_2}E_{\alpha,\alpha+1}(\beta_1 t^\alpha) - \frac{\phi_2}{\beta_1 - \beta_2}E_{\alpha,\alpha+1}(\beta_2 t^\alpha) - \phi_3 E_{\alpha,\alpha+1}(-c_7^\alpha t^\alpha), \quad (44)$$

$$g(t) = g^0 + \frac{\omega_1}{\beta_1 - \beta_2}E_{\alpha,\alpha+1}(\beta_1 t^\alpha) - \frac{\omega_2}{\beta_1 - \beta_2}E_{\alpha,\alpha+1}(\beta_2 t^\alpha) - \omega_3 E_{\alpha,\alpha+1}(-c_7^\alpha t^\alpha). \quad (45)$$

Therefore, Eqs. (30), (44) and (45) are the exact solutions of the FIVP (1)-(2). In particular, for the special case of integer order $\alpha = 1$, the Mittag-Leffler functions can be reduced to exponential functions [20]. Thus, for $\alpha = 1$, the exact solutions (30), (44), and (45) reduce to the following:

$$\begin{aligned} i(t) &= i^0 + \frac{\phi_1}{\beta_1 - \beta_2}\left(\frac{e^{\beta_1 t} - 1}{\beta_1 t}\right) - \frac{\phi_2}{\beta_1 - \beta_2}\left(\frac{e^{\beta_2 t} - 1}{\beta_2 t}\right) - \phi_3\left(\frac{e^{-c_7 t} - 1}{-c_7 t}\right), \\ g(t) &= g^0 + \frac{\omega_1}{\beta_1 - \beta_2}\left(\frac{e^{\beta_1 t} - 1}{\beta_1 t}\right) - \frac{\omega_2}{\beta_1 - \beta_2}\left(\frac{e^{\beta_2 t} - 1}{\beta_2 t}\right), - \omega_3\left(\frac{e^{-c_7 t} - 1}{-c_7 t}\right) \\ d(t) &= d^0 e^{-c_7 t}. \end{aligned} \quad (46)$$

A. The Laplace-Adomian-Padé method

The present section is devoted to the use of the Laplace-Adomian-Padé method (LAPM) to obtain an analytical solution for the FIVP in Eqs. (1)-(2). We begin by using the Laplace Adomian Decomposition method (LADM) to obtain a series solution of the FIVP. We then show that the radius of convergence of the series is very small and that the series diverges over a large region of the domain of interest. Thus, the LAPM is required to obtain a solution which can be used over the whole domain by replacing the series solution by a Padé approximant, i.e., by a rational function. The LAPM for solving the FIVP in Eqs. (1)-(2) is as follows.

We begin the LAPM from Eq. (27). After some straightforward algebraic manipulation and taking the inverse Laplace transforms, we obtain the following implicit formulas for the solutions

$$\begin{aligned} i(t) &= i^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{c_1 g(t) - c_6^\alpha i(t)\right\}\right\}, \\ g(t) &= g^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\{-c_4(i(t) - c_5) - c_2 i(t) + c_0 d(t) - c_3\}\right\}, \\ d(t) &= d^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\{-c_7^\alpha d(t)\}\right\}. \end{aligned} \quad (47)$$

We can then expand the implicit formulas $i(t)$, $g(t)$, $d(t)$ into infinite series by iteration. The infinite series for the solutions are then given by

$$i(t) = \sum_{k=0}^{\infty} i_k(t), \quad g(t) = \sum_{k=0}^{\infty} g_k(t), \quad d(t) = \sum_{k=0}^{\infty} d_k(t). \quad (48)$$

Fortunately, the model does not have any nonlinear terms, so we do not need to replace them by the Adomian polynomials. Substituting Eq. (48) into Eq. (47) for $i(t)$, $g(t)$, $d(t)$, we obtain

$$\begin{aligned} \sum_{k=0}^{\infty} i_k(t) &= i^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{c_1\sum_{k=0}^{\infty} g_k(t) - c_6^\alpha \sum_{k=0}^{\infty} i_k(t)\right\}\right\}, \\ \sum_{k=0}^{\infty} g_k(t) &= g^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-c_4\left(\sum_{k=0}^{\infty} i_k(t) - c_5\right) - c_2\sum_{k=0}^{\infty} i_k(t) + c_0\sum_{k=0}^{\infty} d_k(t) - c_3\right\}\right\}, \\ \sum_{k=0}^{\infty} d_k(t) &= d^0 + \mathscr{L}^{-1}\left\{\frac{1}{s^\alpha}\mathscr{L}\left\{-c_7^\alpha\sum_{k=0}^{\infty} d_k(t)\right\}\right\}. \end{aligned} \quad (49)$$

Matching the two sides of Eq. (49), we can determine the solution components from the following recursion scheme.

$$\begin{aligned} \mathscr{L}\{i^0(t)\} &= \frac{i^0}{s}, \\ \mathscr{L}\{g^0(t)\} &= \frac{g^0}{s} + \frac{c_4c_5 - c_3}{s^{\alpha+1}}, \\ \mathscr{L}\{d^0(t)\} &= \frac{d^0}{s}, \\ &\vdots \\ \mathscr{L}\{i_{n+1}(t)\} &= \frac{c_1}{s^\alpha}\mathscr{L}\{g_n(t)\} - \frac{c_6^\alpha}{s^\alpha}\mathscr{L}\{i_n(t)\}, \\ \mathscr{L}\{g_{n+1}(t)\} &= -\frac{c_2 + c_4}{s^\alpha}\mathscr{L}\{i_n(t)\} + \frac{c_0}{s^\alpha}\mathscr{L}\{d_n(t)\}, \\ \mathscr{L}\{d_{n+1}(t)\} &= \frac{-c_7^\alpha}{s^\alpha}\mathscr{L}\{d_n(t)\}, \qquad n = 0, 1, 2, 3.... \end{aligned} \quad (50)$$

Iteratively taking the inverse Laplace transform of the recursion scheme in Eq. (50), we obtain

$$
\begin{aligned}
i_0(t) &= i^0, \quad g_0(t) = g^0 + t^\alpha \left(-\frac{c_3}{\Gamma(\alpha+1)} + \frac{c_5 c_4}{\Gamma(\alpha+1)} \right), \\
d_0(t) &= d^0, \\
i_1(t) &= t^\alpha \left(\frac{g^0 c_1}{\Gamma(\alpha+1)} - \frac{i^0 c_6^\alpha}{\Gamma(\alpha+1)} \right) \\
&\quad + t^{2\alpha} \left(-\frac{c_1\, c_3}{\Gamma(2\alpha+1)} + \frac{c_5 c_1\, c_4}{\Gamma(2\alpha+1)} \right), \\
g_1(t) &= t^\alpha \left(\frac{d^0 c_0}{\Gamma(\alpha+1)} - \frac{i^0 c_2}{\Gamma(\alpha+1)} - \frac{i^0 c_4}{\Gamma(\alpha+1)} \right), \\
d_1(t) &= -\frac{d^0 t^\alpha c_7^\alpha}{\Gamma(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
i_2(t) &= t^{2\alpha} \left(\frac{d^0 c_0\, c_1}{\Gamma(2\alpha+1)} - \frac{g^0 c_1 c_6^\alpha}{\Gamma(2\alpha+1)} - \frac{i^0 c_1\, c_2}{\Gamma(2\alpha+1)} - \frac{i^0 c_1\, c_4}{\Gamma(2\alpha+1)} \right) \\
&\quad + t^{2\alpha} \frac{i^0 c_6^{2\alpha}}{\Gamma(2\alpha+1)} + t^{3\alpha} \left(\frac{c_1\, c_3 c_6^\alpha}{\Gamma(3\alpha+1)} - \frac{c_5 c_1\, c_4 c_6^\alpha}{\Gamma(3\alpha+1)} \right), \\
g_2(t) &= t^{2\alpha} \left(-\frac{d^0 c_0 c_7^\alpha}{\Gamma(2\alpha+1)} - \frac{g^0 c_1\, c_2}{\Gamma(2\alpha+1)} - \frac{g^0 c_1\, c_4}{\Gamma(2\alpha+1)} + \frac{i^0 c_2 c_6^\alpha}{\Gamma(2\alpha+1)} \right) \\
&\quad + t^{2\alpha} \frac{i^0 c_4 c_6^\alpha}{\Gamma(2\alpha+1)} \\
&\quad + t^{3\alpha} \left(\frac{c_1\, c_3\, c_4}{\Gamma(3\alpha+1)} + \frac{c_1\, c_2\, c_3}{\Gamma(3\alpha+1)} - \frac{c_5 c_1\, c_4}{\Gamma(3\alpha+1)} - \frac{c_5 c_1\, c_2\, c_4}{\Gamma(3\alpha+1)} \right), \\
d_2(t) &= \frac{d^0 t^{2\alpha} c_7^{2\alpha}}{\Gamma(2\alpha+1)}.
\end{aligned}
$$

Since the expressions of the solution components are quite long for $n \geq 3$, only the first two solution components are expressed as above. The n-term approximations of the solution $i(t)$, $g(t)$, and $d(t)$ are defined as

$$
\mathscr{I}_n(t) = \sum_{k=0}^{n-1} i_k(t), \ \mathscr{G}_n(t) = \sum_{k=0}^{n-1} g_k(t), \ \mathscr{D}_n(t) = \sum_{k=0}^{n-1} d_k(t), \quad (51)
$$

respectively. The initial conditions and parameter values described in [14], which are employed in our simulations, are as follows.

$$
\begin{aligned}
&i^0 = 100, \ g^0 = 150, \ d^0 = 50, \ c_0 = 0.1, \ c_1 = 0.7, \\
&c_2 = 0.0003, \ c_4 = 0.05, \ c_6 = 0.25, \ c_7 = 0.14, \ c_5 = 250.
\end{aligned}
\quad (52)
$$

Using the symbolic algebra package MATHEMATICA, the LADM approximating solutions of the FIVP (1)-(2) for the special case $\alpha = 1$ with the given initial conditions and parameter values in Eq. (52) are

$$
\begin{aligned}
\mathscr{I}_{20}(t) &= 100 + 65t - 10.56t^2 ... + 3.549 \times 10^{-28} t^{20}, \\
\mathscr{G}_{20}(t) &= 150. + 6.97t - 1.672t^2 ... + 6.632 \times 10^{-29} t^{20}, \\
\mathscr{D}_{20}(t) &= 50 - 7.5t + 0.5625t^2 ... - 9.119 \times 10^{-32} t^{19}.
\end{aligned}
\quad (53)
$$

The corresponding solutions using the LAPM are

$$
\begin{aligned}
\text{Padé}_{[10/10]} \{\mathscr{I}_{20}(t)\} &= \frac{100 + 119297t ... + 2.917 \times 10^{-10} t^{10}}{1 + 1192.32t ... + 4.487 \times 10^{-13} t^{10}}, \\
\text{Padé}_{[10/10]} \{\mathscr{G}_{20}(t)\} &= \frac{150 + 187164t ... + 9.83 \times 10^{-11} t^{10}}{1 + 1247.71t ... + 4.31 \times 10^{-13} t^{10}}, \\
\text{Padé}_{[10/10]} \{\mathscr{D}_{20}(t)\} &= \frac{50 + 18228.8t ... - 3.97 \times 10^{-14} t^{10}}{1 + 364.725t ... + 7.95 \times 10^{-16} t^{10}}.
\end{aligned}
\quad (54)
$$

The corresponding exact solutions in (46) obtained using Laplace transforms for $\alpha = 1$, the approximate numerical solutions using the RK4 method, the LADM, and the LAPM are compared in Fig. 1. It is obvious from Fig. 1 that the solutions for $i(t)$, $g(t)$ obtained using the LADM are different from those obtained using the other methods when t is approximately close to $t = 25$ and that the LADM solution diverges for $t > 25$. This divergence is due to the fact that the infinite series solution for the LADM diverges for $t > 25$. Due to this divergence of the LADM compared with the LAPM it is clear that the LAPM will be a better method than the LADM for other values of α.

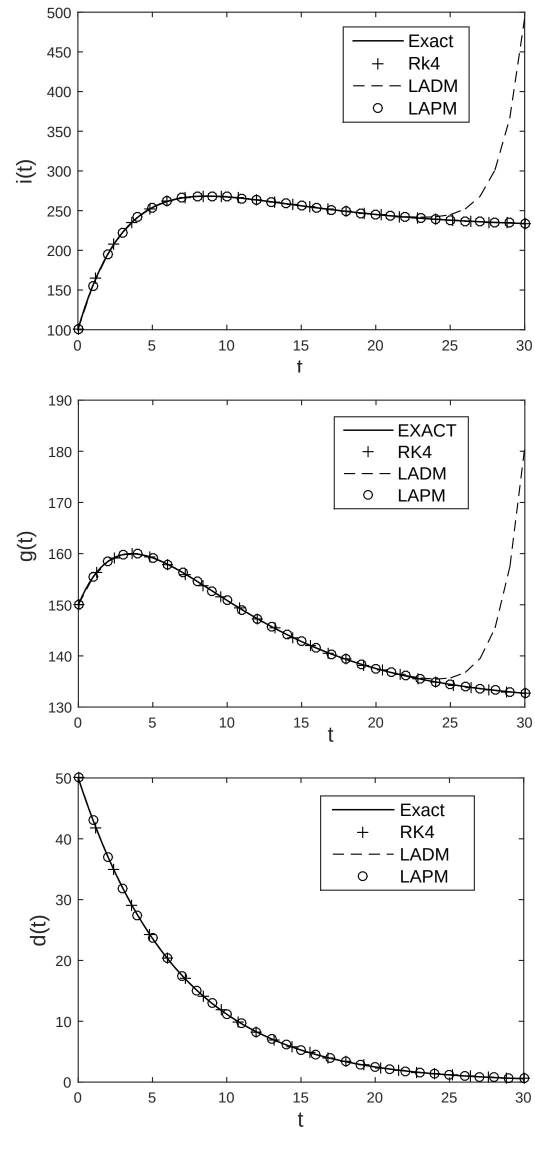

Fig. 1. Simulation comparisons of the solutions $i(t)$, $g(t)$, $d(t)$ for the FIVP (1)-(2) with $\alpha = 1$ using the exact solutions, the RK4 method, the LADM, and the LAPM. The simulation results $i(t)$, $g(t)$ obtained from the LADM are diverging for $t \geq 25$.

B. Adams-Bashforth-Moulton predictor-corrector scheme

Applying the Adams-Bashforth-Moulton predictor-corrector scheme in Eqs. (21)–(25) [11] to the FIVP (1)-(2), we discretize the time interval with points $\{t_n\}$ and obtain the formulas for $i_{h,n} = i_h(t_n)$, $g_{h,n} = g_h(t_n)$,

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$d_{h,n} = d_h(t_n)$ as follows:

$$\begin{aligned} i_{h,n+1} &= i^0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(c_1 g^P_{h,n+1} - c_6^\alpha i^P_{h,n+1}\right) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{1,j,n+1}\left(c_1 g_{h,j} - c_6^\alpha i_{h,j}\right), \end{aligned} \quad (55)$$

$$\begin{aligned} g_{h,n+1} &= g^0 + \frac{h^\alpha}{\Gamma(\alpha+2)}\left(-c_4(i^P_{h,n+1} - c_5)\right. \\ &- \left. c_2 i^P_{h,n+1} - c_3 + c_0 d^P_{h,n+1}\right) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{2,j,n+1}\left(-c_4(i_{h,j} - c_5)\right. \\ &- \left. c_2 i_{h,j} - c_3 + c_0 d_{h,j}\right), \end{aligned} \quad (56)$$

$$\begin{aligned} d_{h,n+1} &= d^0 + \frac{h^\alpha}{\Gamma(\alpha+2)}(-c_7^\alpha d^P_{h,n+1}) \\ &+ \frac{h^\alpha}{\Gamma(\alpha+2)}\sum_{j=0}^{n} a_{3,j,n+1}(-c_7^\alpha d_{h,j}), \end{aligned} \quad (57)$$

in which

$$i^P_{h,n+1} = i^0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{1,j,n+1}[c_1 g_{h,j} - c_6^\alpha i_{h,j}], \quad (58)$$

$$\begin{aligned} g^P_{h,n+1} &= g^0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{2,j,n+1}[-c_4(i_{h,j} - c_5) \\ &- c_2 i_{h,j} - c_3 + c_0 d_{h,j}], \end{aligned} \quad (59)$$

$$d^P_{h,n+1} = d^0 + \frac{1}{\Gamma(\alpha)}\sum_{j=0}^{n} b_{3,j,n+1}[-c_7^\alpha d_{h,j}], \quad (60)$$

where, for $l = 1, 2, 3$,

$$a_{l,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} & \\ -2(n-j+1)^{\alpha+1}, & \text{if } 1 \le j \le n, \\ 1, & \text{if } j = n+1, \end{cases} \quad (61)$$

$$b_{l,j,n+1} = \frac{h^\alpha}{\alpha}\left((n+1-j)^\alpha - (n-j)^\alpha\right) \quad 0 \le j \le n. \quad (62)$$

We will use the discretized formulas (55)-(62) to obtain the numerical solutions for the FIVP (1)-(2) in the next section.

C. Simulation results

In this section, we will show the simulation results of the FIVP (1)-(2) with $\alpha = \frac{3}{4}, \frac{1}{2}$. obtained using the formulas for the exact solutions in Eqs. (30), (44), and (45) and the function mlf for the Mittag Leffler functions, which is implemented by [21]. The approximating solutions of the problem generated by the PECE method, the LAPM, and the LADM are also described. In addition, the absolute errors of the numerical results obtained by the LAPM and the PECE method compared to those obtained using the exact solution formulas are shown.

The following simulation results are for $\alpha = \frac{3}{4}$. Applying the LADM to the problem via the recursion scheme (50), the 20-term approximations of the solutions are demonstrated as

$$\begin{aligned} \mathscr{I}_{20}(t) &= 100 + 59.52t^{3/4} ... + 1.216 \times 10^{-18} t^{15}, \\ \mathscr{G}_{20}(t) &= 150 + 7.584t^{3/4} ... + 1.460 \times 10^{-19} t^{15}, \\ \mathscr{D}_{20}(t) &= 50 - 13.113t^{3/4} ... - 5.328 \times 10^{-22} t^{57/4}. \end{aligned} \quad (63)$$

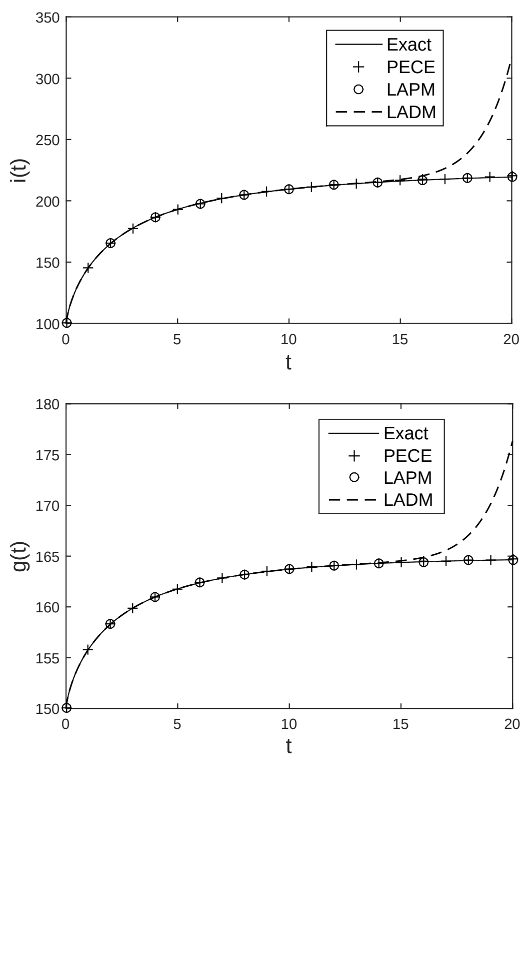
For simplicity, let $t^{\frac{1}{4}} = z$, then we have

$$\begin{aligned} \mathscr{I}_{20}(z) &= 100 + 59.52z^3 ... + 1.216 \times 10^{-18} z^{60}, \\ \mathscr{G}_{20}(z) &= 150 + 7.584z^3 ... + 1.460 \times 10^{-19} z^{60}, \\ \mathscr{D}_{20}(z) &= 50 - 13.113z^3 ... - 5.328 \times 10^{-22} z^{57}. \end{aligned} \quad (64)$$

Calculating the Padé$_{[10/10]}$ of the resulting solutions in Eq. (64) using the command 'PadeApproximant' in Mathematica and then recalling that $z = t^{\frac{1}{4}}$, the LAPM leads us to the following approximating solutions

$$\begin{aligned} \text{Padé}_{[10/10]}\mathscr{I}_{20}(t) &= \frac{100 - 2.54 \times 10^{-10} t^{\frac{1}{4}} ... + 7.67 \times 10^{-12} t^{\frac{5}{2}}}{1 - 2.54 \times 10^{-12} t^{\frac{1}{4}} ... + 4.08 \times 10^{-14} t^{\frac{5}{2}}}, \\ \text{Padé}_{[10/10]}\mathscr{G}_{20}(t) &= \frac{150 + 1.61 \times 10^{-8} t^{\frac{1}{4}} ... - 1.41 \times 10^{-9} t^{\frac{5}{2}}}{1 + 7.74 \times 10^{-11} t^{\frac{1}{4}} ... - 8.91 \times 10^{-12} t^{\frac{5}{2}}}, \\ \text{Padé}_{[10/10]}\mathscr{D}_{20}(t) &= \frac{50 + 2.7 \times 10^{-11} t^{\frac{1}{4}} ... + 2.33 \times 10^{-12} t^{\frac{5}{2}}}{1 + 5.38 \times 10^{-13} t^{\frac{1}{4}} ... + 1.45 \times 10^{-13} t^{\frac{5}{2}}}. \end{aligned} \quad (65)$$

The simulation results of the problem for $\alpha = \frac{3}{4}$ using all of the methods, i.e., the exact formulas (30), (44), and (45), the PECE method in Eqns. (55)-(62) with the step size $h = 10^{-3}$, the LAPM in Eq. (65), and the LADM in Eq. (63), are shown in Fig. 2. The solution curves $i(t), g(t)$ constructed by the LADM are diverging when $t \approx 15$ and the numerical simulations obtained by the PECE method and the LAPM are in very good agreement with the exact solutions. The absolute errors between numerical solutions, which are computed using the LAPM and the PECE method, and the exact solutions are shown in Table I. It can be numerically concluded from Table I that the PECE method achieves a higher degree of accuracy than the LAPM when t is larger.



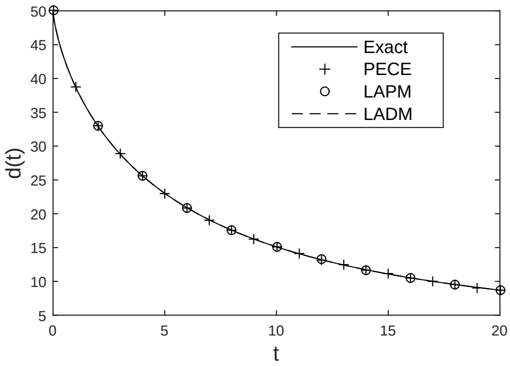


Fig. 2. Simulation comparisons of the solutions $i(t)$, $g(t)$, $d(t)$ for the FIVP (1)-(2) using the exact solutions, the PECE method, the LAPM, and the LADM for $\alpha = \frac{3}{4}$.

TABLE I
THE ABSOLUTE ERRORS OF DIFFERENT METHODS CAOMPARED WITH THE EXACT SOLUTIONS OF THE FIVP (1)-(2) WITH $\alpha = \frac{3}{4}$

$\alpha = \frac{3}{4}$	|Exact-LAPM|		
t	$\|\Delta i\|$	$\|\Delta g\|$	$\|\Delta d\|$
0	0	0	0
2	1.615E-04	9.970E-05	7.932E-06
4	4.017E-03	4.897E-04	8.802E-05
6	1.840E-02	1.869E-03	4.675E-04
8	4.890E-02	4.726E-03	1.478E-03
10	9.830E-02	9.256E-03	3.473E-03
12	1.670E-01	1.543E-02	6.769E-03
14	2.540E-01	2.301E-02	1.162E-02
16	3.575E-01	3.201E-02	1.822E-02
18	4.753E-01	4.196E-02	2.669E-02
20	6.051E-01	5.272E-02	3.711E-02

$\alpha = \frac{3}{4}$	|Exact-PECE|		
t	$\|\Delta i\|$	$\|\Delta g\|$	$\|\Delta d\|$
0	0	0	0
2	9.246E-06	1.215E-06	2.167E-06
4	2.769E-06	3.738E-07	1.082E-06
6	8.442E-07	1.164E-07	5.858E-07
8	1.727E-07	2.366E-08	3.295E-07
10	7.730E-08	1.230E-08	1.857E-07
12	1.607E-07	2.510E-08	1.027E-07
14	1.743E-07	2.786E-08	5.504E-08
16	1.706E-07	2.774E-08	2.435E-08
18	1.507E-07	2.517E-08	6.862E-09
20	1.234E-07	2.145E-08	5.653E-10

Next, we will simulate numerical results of the problem for $\alpha = \frac{1}{2}$ as follows. Applying the LADM to the problem via the recursion scheme (50), the 20-term approximations of the solutions are expressed as

$$\begin{aligned} \mathscr{I}_{20}(t) &= 100 + 47.115t^{1/2}... + 1.028 \times 10^{-10}t^{10}, \\ \mathscr{G}_{20}(t) &= 150 + 7.865t^{1/2}... + 9.059 \times 10^{-12}t^{10}, \\ \mathscr{D}_{20}(t) &= 50 - 21.851t^{1/2}... - 6.57 \times 10^{-13}t^{19/2}. \end{aligned} \tag{66}$$

For simplicity, let $t^{\frac{1}{2}} = z$, then we get

$$\begin{aligned} \mathscr{I}_{20}(z) &= 100 + 47.115z... + 1.028 \times 10^{-10}z^{20}, \\ \mathscr{G}_{20}(z) &= 150 + 7.865z... + 9.059 \times 10^{-12}z^{20}, \\ \mathscr{D}_{20}(z) &= 50 - 21.851z... - 6.57 \times 10^{-13}z^{19}. \end{aligned} \tag{67}$$

Similarly as above, we compute the Padé$_{[10/10]}$ of the obtained solutions in Eq. (67) and then substitute $z = t^{\frac{1}{2}}$ into the resulting equations. The LAPM eventually brings the following approximating rational solutions

$$\begin{aligned} \text{Padé}_{[10/10]}\mathscr{I}_{20}(t) &= \frac{100 + 1.08 \times 10^9 t^{\frac{1}{2}}... + 25.106 \times t^5}{1 + 1.08 \times 10^7 t^{\frac{1}{2}}... + 0.11t^5}, \\ \text{Padé}_{[10/10]}\mathscr{G}_{20}(t) &= \frac{150 - 1.855 \times 10^1 0 t^{\frac{1}{2}}... - 286.91 \times t^5}{1 - 1.237 \times 10^8 t^{\frac{1}{2}}... - 1.40t^5}, \\ \text{Padé}_{[10/10]}\mathscr{D}_{20}(t) &= \frac{50 - 1.80 \times 10^9 t^{\frac{1}{2}}... - 1.04 \times 10^{-3}t^5}{1 + 3.583 \times 10^7 t^{\frac{1}{2}}...0.285t^5}. \end{aligned} \tag{68}$$

The numerical simulations of the problem for $\alpha = \frac{1}{2}$ using all of the methods, i.e., the exact formulas (30), (44), and (45), the PECE method in Eqns. (55)-(62) with the step size $h = 10^{-3}$, the LAPM in Eq. (68), and the LADM in Eq. (66), are described in Fig. 3. The solution curves $i(t), g(t)$ obtained using the LADM are diverging when $t \approx 10$ and the numerical simulations obtained by the PECE method and the LAPM are in very good agreement with the exact solutions. The absolute errors between numerical solutions, which are computed using the LAPM and the PECE method, and the exact solutions are shown in Table II. It is not difficult to observe from Table II that the PECE method attains a better accuracy than the LAPM when t is far away from the initial point.

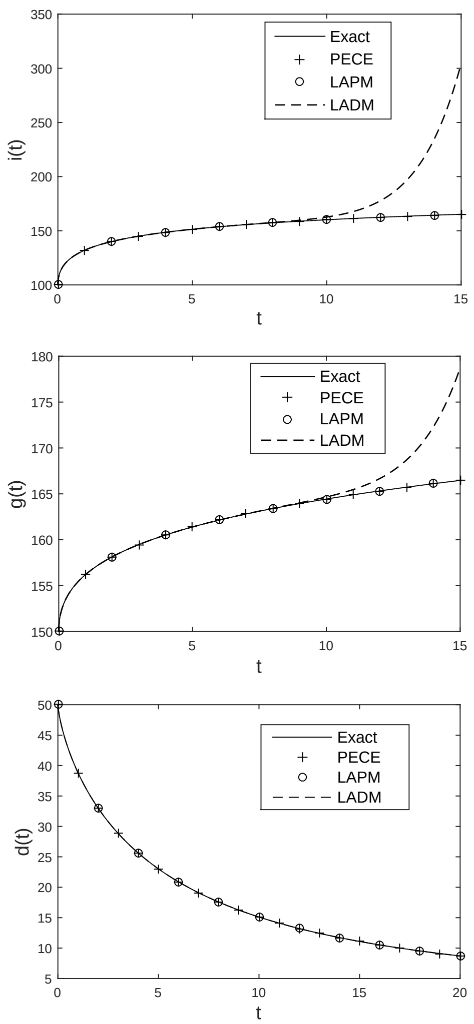


Fig. 3. Graphical comparisons of the solutions $i(t)$, $g(t)$, $d(t)$ for the FIVP (1)-(2) using the exact solutions, the PECE method, the LAPM, and the LADM for $\alpha = \frac{1}{2}$.

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TABLE II
THE ABSOLUTE ERRORS OF DIFFERENT METHODS CAOMPARED WITH THE EXACT SOLUTIONS OF THE FIVP (1)-(2) WITH $\alpha = \frac{1}{2}$

$\alpha = \frac{1}{2}$	|Exact-LAPM|		
t	$\|\Delta i\|$	$\|\Delta g\|$	$\|\Delta d\|$
0	0	0	0
1.5	2.274E-04	1.835E-04	6.387E-06
3	2.387E-04	9.357E-05	8.239E-06
4.5	2.294E-04	4.001E-05	8.574E-06
6	2.150E-04	2.979E-06	8.457E-06
7.5	1.996E-04	2.497E-05	8.197E-06
9	1.848E-04	4.735E-05	7.903E-06
10.5	1.708E-04	6.605E-05	7.615E-06
12	1.580E-04	8.216E-05	7.347E-06
13.5	1.462E-04	9.635E-05	7.104E-06
15	1.354E-04	1.091E-04	6.884E-06

$\alpha = \frac{1}{2}$	|Exact-PECE|		
t	$\|\Delta i\|$	$\|\Delta g\|$	$\|\Delta d\|$
0	0	0	0
1.5	3.001e-05	4.166E-06	1.731E-05
3	1.112e-05	1.914E-06	7.561E-06
4.5	5.602e-06	1.176E-06	4.260E-06
6	3.259e-06	8.288E-07	2.698E-06
7.5	2.055e-06	6.316E-07	1.829E-06
9	1.369e-06	5.066E-07	1.295E-06
10.5	9.496e-07	4.219E-07	9.473E-07
12	6.762e-07	3.606E-07	7.076E-07
13.5	4.837e-07	3.129E-07	5.328E-07
15	3.627e-07	2.787E-07	4.121E-07

V. CONCLUSION

In this article, we have successfully obtained the exact solutions of the fractional-order initial value problem (1)-(2) by using the Laplace transform. We have also obtained approximating analytical solutions of the problem via the LADM and the LAPM. The numerical solutions of the problem simulated via the PECE method have also been computed. The simulations of the solutions calculated by the above approaches have been compared for the integer order $\alpha = 1$ and the fractional orders $\alpha = \frac{3}{4}, \ \frac{1}{2}$. The comparisons have shown that the approximate solutions generated via the LAPM and the PECE are in very good agreement with the exact solutions of the problem, whereas the infinite series solutions generated by the LADM become divergent as the time increases.

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REFERENCES

[1] Y. Cho, I. Kim, and D. Sheen, "A fractional-order model for minmod millennium," *Mathematical Biosciences*, vol. 262, no. Supplement C, pp. 36 – 45, 2015.
[2] M. Asgari, "Numerical solution for solving a system of fractional integro-differential equations," *IAENG International Journal of Applied Mathematics*, vol. 45, no. 2, pp. 85–91, 2015.
[3] S. Zhang and Y.-Y. Zhou, "Multiwave solutions for the toda lattice equation by generalizing exp-function method," *IAENG International Journal of Applied Mathematics*, vol. 44, no. 4, pp. 177–182, 2014.
[4] H. Song, M. Yi, J. Huang, and Y. Pan, "Numerical solution of fractional partial differential equations by using legendre wavelets,," *Engineering Letter*, vol. 24, no. 3, pp. 358–364, 2016.
[5] K. A. Gepreel, T. A. Nofal, and N. S. Al-Sayali, "Exact solutions to the generalized hirota–satsuma kdv equations using the extended trial equation method," *Engineering Letters*, vol. 24, no. 3, pp. 274–283, 2016.
[6] K. Kataria and P. Vellaisamy, "Saigo space–time fractional poisson process via adomian decomposition method," *Statistics & Probability Letters*, 2017.
[7] F. Haq, K. Shah, G. ur Rahman, and M. Shahzad, "Numerical solution of fractional order smoking model via laplace adomian decomposition method," *Alexandria Engineering Journal*, 2017.
[8] S. Sirisubtawee and S. Kaewta, "New modified adomian decomposition recursion schemes for solving certain types of nonlinear fractional two-point boundary value problems," *International Journal of Mathematics and Mathematical Sciences*, vol. 2017, 2017.
[9] Q. Liu, J. Liu, and Y. Chen, "Asymptotic limit cycle of fractional van der pol oscillator by homotopy analysis method and memory-free principle," *Applied Mathematical Modelling*, vol. 40, no. 4, pp. 3211–3220, 2016.
[10] A. Freihat and S. Momani, "Application of multistep generalized differential transform method for the solutions of the fractional-order chua's system," *Discrete Dynamics in Nature and Society*, vol. 2012, 2012.
[11] K. Diethelm, N. J. Ford, and A. D. Freed, "A predictor-corrector approach for the numerical solution of fractional differential equations," *Nonlinear Dynamics*, vol. 29, no. 1, pp. 3–22, 2002.
[12] H. Wang, D. Yang, and S. Zhu, "A petrov–galerkin finite element method for variable-coefficient fractional diffusion equations," *Computer Methods in Applied Mechanics and Engineering*, vol. 290, pp. 45–56, 2015.
[13] H. Lu, P. W. Bates, W. Chen, and M. Zhang, "The spectral collocation method for efficiently solving pdes with fractional laplacian," *Advances in Computational Mathematics*, pp. 1–18, 2017.
[14] M. Lombarte, M. Lupo, G. Campetelli, M. Basualdo, and A. Rigalli, "Mathematical model of glucose–insulin homeostasis in healthy rats," *Mathematical biosciences*, vol. 245, no. 2, pp. 269–277, 2013.
[15] N. A. Khan, A. Ara, and N. A. Khan, "Fractional-order riccati differential equation: analytical approximation and numerical results," *Advances in Difference Equations*, vol. 2013, no. 1, p. 185, 2013.
[16] I. Podlubny, *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Academic press, 1998, vol. 198.
[17] R. Magin, X. Feng, and D. Baleanu, "Solving the fractional order bloch equation," *Concepts in Magnetic Resonance Part A*, vol. 34, no. 1, pp. 16–23, 2009.
[18] S. Momani and Z. Odibat, "Numerical comparison of methods for solving linear differential equations of fractional order," *Chaos, Solitons & Fractals*, vol. 31, no. 5, pp. 1248–1255, 2007.
[19] H. Ahmed, M. S. Bahgat, and M. Zaki, "Numerical approaches to system of fractional partial differential equations," *Journal of the Egyptian Mathematical Society*, vol. 25, no. 2, pp. 141–150, 2017.
[20] H. J. Haubold, A. M. Mathai, and R. K. Saxena, "Mittag-leffler functions and their applications," *Journal of Applied Mathematics*, vol. 2011, 2011.
[21] I. Podlubny, "Calculates the mittag-leffler function with desire accuracy (2005)."