The He Homotopy Perturbation Method for Heat-like Equation with Variable Coefficients and Non Local Conditions

A. Cheniguel, H. Benaddi

Abstract— In this paper , one-dimensional, two-dimensional and three-dimensional heat-like equations subject to initial and non local boundary conditions are newly introduced and Homotopy Perturbation Method (HPM) is utilized for solving these new types of problems. The obtained results are highly accurate. Also HPM provides continuous solution in contrast to finite difference method, which only provides discrete approximations. It is found that this method is a powerful mathematical tool and can be applied to a large class of linear and nonlinear problems in different fields of science and technology

Index Terms— Homotopy perturbation method (HPM), Partial differential equations, Heat –like equations, non local boundary Conditions

I. INTRODUCTION

Recently, some promising analytic methods for solving initial boundary value problems have been proposed. Examples of these methods are series solution methods which include Adomian decomposition method [8], Homotopy Analysis Method [9], Variational iteration method [7] and Homotopy perturbation method [6]. These methods have received great interest for finding approximate and analytic solutions to partial differential equations. This interest was driven by the needs from applications both in industry and sciences. Theory and numerical methods for solving initial boundary value problems were investigated by many researchers see for instance [1-4] and the reference therein. The widely applied techniques are perturbation methods. He [6] has proposed a new perturbation technique coupled with the homotopy technique, which is called the homotopy perturbation method (HPM). In contrast to the traditional perturbation methods. a homotopy is constructed with an embedding parameter $p \in [0,1]$,which is considered as a small parameter. This method has been the subject of intense investigation during recent years and many researchers have used it in their works involving differential equations. He [5], applied HPM to solve initial boundary value problems governed by the nonlinear differential equations. The results show that this method is efficient and simple.

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A. Cheniguel is with Department of Mathematics, Faculty of Mathematics and Matter Sciences, Kasdi Merbah University, Ouargla, Algeria, 30000 (email: cheniguelahmed@yahoo.fr)
H. Benaddi is with Department of Mathematics, Faculty of Mathematics and Matter Sciences, Kasdi Merbah University, Ouargla, Algeria, 30000 (email: hadda.maths@gmail.com)

The main goal of this work is to apply the homotopy perturbation method (HPM) for solving heat-like equation with variable coefficients subject to non local boundary conditions in both one-dimensional, two-dimensional and three-dimensional cases.

II. ANALYSIS OF THE METHOD

To illustrate the basic ideas, let X and Y be two topological spaces. If f and g are continuous maps of X into Y, it is said that f is homotopic to g if there is continuous map $F: X \times [0,1] \longrightarrow Y$ such that $F(x,0) = f(x)$ and $F(x,1) = g(x)$ for each $x \in X$, then the map is called homotopy between f and g.

We consider the following nonlinear partial differential equation:

$$A(u) - f(r) = 0, in\ \Omega \tag{1}$$

Subject to the boundary conditions

$$B\left(u, \frac{\partial u}{\partial \eta}\right) = 0, on\ \Gamma \tag{2}$$

Where A is a general differential operator. f is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial \eta}$ denotes directional derivative in outward normal direction to Ω. The operator A, is generally divided into two parts, L and N, where L is linear, while N is nonlinear. Using A $=L$+N, eq. (1) can be rewritten as follows:

$$L(v) + N(v) - f(r) = 0 \tag{3}$$

By the homotopy technique, we construct a homotopy defined as

$$H(v,p): \Omega \times [0,1] \longrightarrow R \tag{4}$$

Which satisfies:

$$H(v,p) = (1-p)\big(L(v) - L(u_0)\big) + p\big(A(v) - f(r)\big), p \in [0,1], r \in \Omega \tag{5}$$

Or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p\big(N(v) - f(r)\big) = 0, p \in [0,1], r \in \Omega \tag{6}$$

Where $p \in [0,1]$ is an embedding parameter , u_0 is an initial approximation of equation (1), which satisfies the boundary conditions. It follows from equation (6) that:

$$H(v,0) = L(v) - L(u_0) = 0 \tag{7}$$

$$H(v,1) = A(v) - f(r) = 0 \tag{8}$$

The changing process of p from 0 to 1 monotonically is a trivial problem. $H(v,0) = L(v) - L(u_0) = 0$ is continuously transformed to the original problem

$$H(v,1) = A(v) - f(r) = 0. \tag{9}$$

In topology, this process is known as continuous deformation. $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. We use the embedding parameter p as a small

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parameter, and assume that the solution of equation (6) can be written as power series of p :

$v = p^0 v_0 + p v_1 + p^2 v_2 + p^3 v_3 + \cdots + p^n v_N + \cdots$ (10)

Setting $p = 1$ we obtain the approximate solution of equation (1) as:

$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$ (11)

The series of equation (11) is convergent for most of the cases, but the rate of the convergence depends on the nonlinear operator $N(v)$. He (1999) has suggested that the second derivative of $N(v)$ with respect to v should be small because the parameter may be relatively large i.e $p = 1$ and the norm of $L^{-1}(\frac{\partial N}{\partial u})$ must be smaller than one in order for the series to converge.

III. CONVERGENCE ANALYSIS

Lemma:

Suppose that L^{-1} exist then the exact solution satisfies

$u = L^{-1}\big(f(r)\big) - \sum_{i=0}^{\infty} L^{-1}\left(N(v_i)\right)$ (12)

Proof

Rewriting equation (6), in the following form

$L(v) = L(u_0) + p[f(r) - N(v) - L(u_0)]$ (13)

Applying the inverse operator L^{-1} to both sides of Eq.(13), we have

$v = u_0 + p[L^{-1}(f(r) - L^{-1}\big(N(v)\big) - u_0]$ (14)

We write Eq. (10) in compact form as follows

$v = \sum_{i=0}^{\infty} p^i v_i$ (15)

Substituting into the right hand side of Eq. (14) we get

$v = u_0 + p[L^{-1} f(r) - (L^{-1}(N\{\sum_{i=0}^{\infty} p^i v_i)\} - u_0]$

From Eq. (11) , we obtain the exact solution

$u = \lim_{p \to 1} v = L^{-1}\big(f(r)\big) - L^{-1}(N(\sum_{i=0}^{\infty} v_i)) =$

$= L^{-1}\big(f(r)\big) - \sum_{i=0}^{\infty} L^{-1}\left(N(v_i)\right)$

In order to study the convergence of the method , we present the sufficient condition of the convergence in the following .

Theorem 1.1

Supposing that X and Y are Banach spaces and N:X→Y is a contraction non linear mapping, that is

$\forall\, u, v \in X$:‖N(u)-N(v)‖<=L‖u-v‖, 0<L<1 (16)

Then, according to Banach's theorem N has a unique fixed point w, that is N(w)=w. Supposing that the sequence generated by homotopy perturbation method can be written as:

$V_n = N(V_{n-1}), V_{n-1} = \sum_{i=0}^{n-1} v_i$, n=1,2,3…. . (17)

And

$V_0 = v_0 \in B_r(v))$ where

$B_r(v) = \{u \in X, \|v - u\| < r\}$

then we have

(i)$V_n \in B_r(v)$

(ii)$\lim_{n\to\infty} V_n = v$.

Proof

(i)By induction, for n=1, we have

$$\|V_1 - v\| = \|N(V_0) - N(v)\| \le L\|v_0 - v\|$$

Assume that $\|V_{n-1} - v\| \le L^{n-1}\|v_0 - v\|$, then

$\|V_n - v\| \le \|N(V_{n-1}) - N(v)\| \le L\|V_{n-1} - v\| \le L^n\|v_0 - v\|$

Using (i), we obtain:

$\|V_n - v\| \le L^n\|v_0 - v\| \le L^n r$ <r $\Rightarrow V_n \in B_r(v)$.

(ii)Because $\|V_n - v\| \le L^n\|v_0 - v\|$ and $\lim_{n\to\infty} L^n = 0$, $\lim_{n\to\infty}\|V_n - v\| = 0$, that is $\lim_{n\to\infty} V_n = v$.

IV. EXAMPLES

A. *Example 1*

We consider the problem:

$\frac{\partial u}{\partial t} = x^6 + (1/30)x^2 \frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1, t > 0$ (18)

With the initial condition:

$u(x, 0) = 0,$

And the boundary conditions:

$u(0,t) = \int_0^1 u(x,t)dx + g_1 = (1/7)(e^t - 1), g_1 = 0$ (19)

For solving this problem, we construct the HPM as follows:

$H(v,p) = (1-p)\left(\frac{\partial v}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p(\frac{\partial v}{\partial t} - (\frac{1}{30})x^2 \frac{\partial^2 v}{\partial x^2} - x^6) = 0$ (20)

The components v_i of (11) are obtained as follows:

$\frac{\partial v_0}{\partial t} - \frac{\partial u_0}{\partial t} = 0, v_0 = u_0 = u(x,0) = 0)$ (21)

$\frac{\partial v_1}{\partial t} - (1/30)x^2 \frac{\partial^2 v_0}{\partial x^2} - x^6 = 0, v_1(x,0) = 0$ (22)

$\frac{\partial^2 v_0}{\partial x^2} = 0$

$\frac{\partial v_1}{\partial t} = x^6$

Hence

$v_1 = x^6 t$ (23)

$\frac{\partial v_2}{\partial t} - \frac{1}{30}\left(x^2 \frac{\partial^2 v_1}{\partial x^2}\right) = 0, v_2(x,0) = 0$

$\frac{\partial^2 v_1}{\partial x^2} = 30x^4 t$

$\frac{\partial v_2}{\partial t} = x^6 t$

Then, we obtain:

$v_2 = x^6 \frac{t^2}{2!}$ (24)

For the next component:

$\frac{\partial v_3}{\partial t} - \frac{1}{30}x^2 \frac{\partial^2 v_2}{\partial x^2} = 0, v_3(x,0) = 0$

$v_3 = x^6\ \frac{t^3}{3!}$ (25)

And so on, we obtain the approximate solution as follows:

$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots + v_n + \cdots$

And this leads to the following solution

$u(x,t) = x^6(e^t - 1)$ (26)

We can, immediately observe that this solution is exact.

B. *Example 2*

Consider the following two dimensional heat-like equation:

$\frac{\partial u}{\partial t} = x^6 y^6 + \frac{1}{60}(x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2}), 0 \le x \le 1, t > 0$ (27)

Subject to the initial condition

$u(x,y,0) = 0$ (28)

$$u(0,y,t) = \int_0^1 \int_0^1 u(x,y,t)dxdy + g_1 = \frac{1}{49}(e^t - 1), g_1 = 0$$

$$u(1,y,t) = \int_0^1 \int_0^1 u(x,y,t)dxdy + g_2 = \frac{1}{49}(e^t - 1) + \frac{t}{2}, g_2 = \frac{-t}{2}$$

$u(x,0,t)=\int_0^1\int_0^1 u(x,y,t)dxdy+g_3=\frac{1}{49}e^t, g_3=\frac{-1}{49}$

$$u(x,1,t)=\int_0^1\int_0^1 u(x,y,t)dxdy+g_4=\frac{1}{49}(e^t+3), g_4=\frac{-4}{49}$$

Solving the equation (27) with the initial condition (28), yields:

$\frac{\partial v_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, v_0=u_0=0$

$\frac{\partial v_1}{\partial t}-x^6y^6-\frac{1}{60}(x^2\frac{\partial^2 v_0}{\partial x^2}+y^2\frac{\partial^2 v_0}{\partial y^2})=0, v_1(x,y,0)=0$

$v_1=x^6y^6t$ (29)

$\frac{\partial v_2}{\partial t}-\frac{1}{60}\left(x^2\frac{\partial^2 v_1}{\partial x^2}+y^2\frac{\partial^2 v_1}{\partial y^2}\right)=0, v_2(x,y,0)=0$

$v_2=x^6y^6\frac{t^2}{2!}$

$\frac{\partial v_3}{\partial t}-\frac{1}{60}\left(x^2\frac{\partial^2 v_2}{\partial x^2}+y^2\frac{\partial^2 v_2}{\partial y^2}\right)=0$

$v_3=x^6y^6\frac{t^3}{3!}$ (30)

. .

Repeating the above process gives the remaining components as:

$v_n=x^ny^n\frac{t^n}{n!}$ (31)

Using equations in the above, we get:

$u(x,t)=x^6y^6\left(\left(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+..+\frac{t^n}{n!}+\cdots\right)-1\right)$

And finally the approximate solution is obtained as :

$u(x,y,t)=x^6y^6(e^t-1)$
(32)

C. *Example 3*

As a last example, consider the following problem:

$u_t=\boldsymbol{x}^6y^6z^6+\frac{1}{90}(x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2})$ (33)

$0\le x,y,z\le 1, t>0$

With the initial condition

$u(x,y,z,0)=0$ (34)

And the boundary conditions:

$u(0,y,z,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_1=\frac{1}{343}(e^t+t), g_1=\frac{-t-1}{343}$

$u(1,y,z,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_2=\frac{1}{343}(e^t-1), g_2=0$

$$u(x,0,z,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_3=\frac{1}{343}(e^t+3t), g_3=\frac{-3t-1}{343}$$

$$u(x,1,z,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_4=\frac{1}{343}(e^t-1), g_4=0$$

$$u(x,y,0,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_5=\frac{1}{343}e^t, g_5=\frac{-1}{343}$$

$$u(x,y,1,t)=\int_0^1\int_0^1\int_0^1 u(x,y,z,t)dxdydz+g_6=\frac{1}{343}(e^t-1)+t, g_6=-t$$

According to the HPM, we have:

$H(v,p)=(1-p)\left(\frac{\partial v_0}{\partial t}-\frac{\partial u_0}{\partial t}\right)+p\left(\frac{\partial v}{\partial t}-\frac{1}{90}(x^2\frac{\partial^2 v}{\partial x^2}+\right.$
y2∂2V∂y2+z2∂2V∂z2−f=0 (34)

Where $f=x^6y^6z^6$

By equating the terms with the identical powers of π , yields

$p^0: \frac{\partial v_0}{\partial t}-\frac{\partial u_0}{\partial t}=0, \frac{\partial v_0}{\partial t}=0, v_0=0$ (36)

$p^1: \frac{\partial v_1}{\partial t}-x^6y^6z^6-\frac{1}{90}(x^2\frac{\partial^2 v_0}{\partial x^2}+y^2\frac{\partial^2 v_0}{\partial y^2}+z^2\frac{\partial^2 v_0}{\partial z^2})=0, v_1(x,y,z,0)=0,$

$\frac{\partial v_1}{\partial t}=x^6y^6z^6$

Then:

$v_1=x^6y^6z^6t$ (37)

$p^2: \frac{\partial v_2}{\partial t}-\frac{1}{90}(x^2\frac{\partial^2 v_1}{\partial x^2}+y^2\frac{\partial^2 v_1}{\partial y^2}+z^2\frac{\partial^2 v_1}{\partial z^2})=0, v_2(x,y,z,0)=0,$

$\frac{\partial v_2}{\partial t}=x^6y^6z^6t$

$v_2=x^6y^6z^6\frac{t^2}{2!}$ (38)

$$p^3: \frac{\partial v_3}{\partial t}-\frac{1}{90}\left(x^2\frac{\partial^2 v_2}{\partial x^2}+y^2\frac{\partial^2 v_2}{\partial y^2}+z^2\frac{\partial^2 v_2}{\partial z^2}\right)=0$$

$$\frac{\partial v_3}{\partial t}=x^6y^6z^6\frac{t^2}{2!}$$

Hence

$v_3=x^6y^6z^6\frac{t^3}{3!}$ (39)

Continuing like-wise we get:

$$v_n=x^6y^6z^6\frac{t^n}{n!}$$

And so on we obtain the approximate solution as :

$u_{nhpm}=x^6y^6z^6[(1+\frac{t}{1!}+\frac{t^2}{2!}+\frac{t^3}{3!}+..+\frac{t^n}{n!}+\cdots)-1]$ (40)

From this result we deduce that the series solution converges to the exact one :

$u(x,t)=x^6y^6z^6(e^t-1)$ (41)

Table 1 Example 1

$h_x=0.1, h_t=0.1$, 3-Iterates

x_i	u_{ex}	u_{hpm}	$\lvert u_{ex}-u_{hpm}\rvert$
0.0	0.0	0.0	0.0
0.1	1.0517e-07	1.0517e-07	0.0
0.2	6.7309e-06	6.7307e-06	2.0000e-10
0.3	7.6670e-05	7.6666e-05	4.0000e-09
0.4	4.3078e-04	4.3076e-04	2.0000e-08
0.5	0.0016	0.0016	0.0
0.6	0.0049	0.0049	0.0
0.7	0.0124	0.0124	0.0
0.8	0.0276	0.0276	0.0
0.9	0.0559	0.0559	0.0
1.0	0.1052	0.105 2	0.0

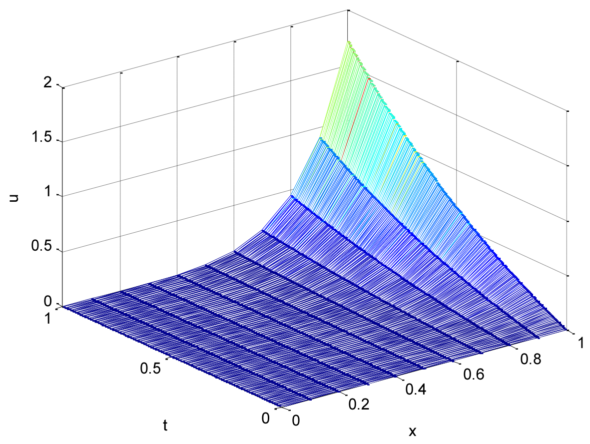

Fig.1.Series solution u_{hpm}

Table 2 Example 2

$h_x = 0.1, h_y = 0.1, h_t = 0.01$, 3-Iterates

x_i	y_i	u_{ex}	u_{hpm}	$\lvert u_{ex} - u_{hpm} \rvert$
0.0	0.0	0.0	0.0	0.0
0.1	0.1	1.0517e-13	1.0517e-13	0.0
0.2	0.2	4.3078e-10	4.3076e-10	2.0000e-14
0.3	0.3	5.5892e-08	5.5890e-08	2.0000e-12
0.4	0.4	1.7645e-06	1.7644e-06	1.0000e-10
0.5	0.5	2.5676e- 05	2.5675e-05	1.0000e-09
0.6	0.6	2.2893e-04	2.2892e-04	1.0000e-08
0.7	0.7	0.0015	0.0015	0.0
0.8	0.8	0.0072	0.0072	0.0
0.9	0.9	0.0297	0.0297	0.0
1.0	1.0	0.01052	0.01052	0.0

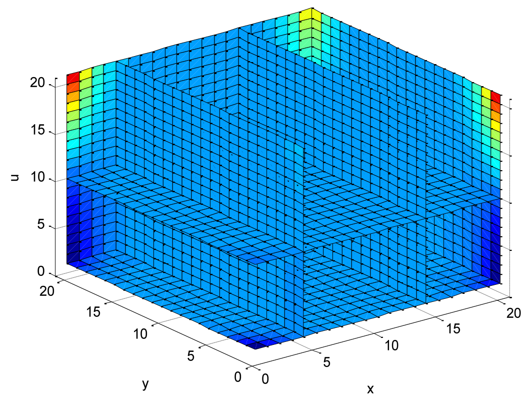

Fig. 2. Series Solution u_{hpm}

Table 3 Example 3

$h_x = h_y = h_z = 0.1, h_t = 0.1,\ 3 - Iterates$

x_i	y_j	z_k	u_{ex}	u_{hpm}	$\lvert u_{ex} - u_{2hpm} \rvert$
0.0	0. 0	0.0	0.0	0.0	0.0
0.1	0.1	0.1	1.0517e-19	1.0517e-19	0.0
0.2	0.2	0.2	2.7570e-14	2.7569e-14	1.0000e-18
0.3	0.3	0.3	4.0745e-11	4.0744e-11	1.0000e-15
0.4	0.4	04	7.2273e-09	7.2270e-09	3.0000e-13
0.5	0.5	0.5	4.0120e- 07	4.0118e-07	2.0000e-11
0.6	0.6	0.6	1.0681e-05	1.0681e-05	0.0
0.7	0.7	0.7	1.7126e-04	1.7125e-04	1.0000e-08
0.8	0.8	0.8	0.0019	0.0019	0.0
0.9	0.9	0.9	0.0158	0.0158	0.0
1.0	1.0	1.0	0.1052	0.1052	0.0

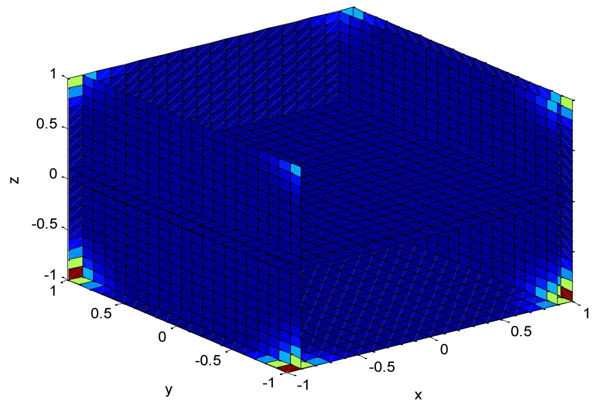

Fig.3. Series solution u_{hpm} at t=.1

V. CONCLUSION

In this work, new types of heat-Like equations with non local conditions are introduced and homotopy perturbation method is used for solving these kinds of problems. The method gives a series solution which converges rapidly. The speed of convergence means that few terms are required. The problems solved using the (HPM) gave satisfactory results in comparison to those recently obtained by other researchers using finite difference schemes. The case studies gave sufficiently good agreements with the exact solutions. The proposed iterative scheme finds the solution without any linearization, discretization, transformation or restrictive assumptions. The results demonstrate the stability and convergence of the method. Moreover, the method is easier to implement than the traditional techniques. It is worth mentioning that the homotopy perturbation method can be used for solving non linear problems without using Adomian's polynomials , this fact can be considered as clear advantage of this technique over the Adomian decomposition method. It can be shown from the obtained results that the technique is a powerful tool for solving a wide range of linear and nonlinear problems.

REFERENCES

[1] A. Cheniguel, Analytic Method for Solving Heat and Heat-Like Equation with Classical and Non Local Boundary Conditions, Chapter 7 in the Book"Transactions on Engineering Technologies, Springer Dordrecht, Heidelberg, 2015.

[2] A. Cheniguel, Numerical Method for the Heat Equation with dirichlet and Neumann Conditions. Lecture Notes in Engineering and Computer Science, pp:535-539.

[3] A. Cheniguel, On the Numerical Solution of Multi-Dimensional Diffusion Equation with Non Local Conditions, chapter 43 in the Book: Transactions on Engineering Technologies, Springer Dordrecht Heidelberg, 2014.

[4] A. Cheniguel, Numerical method for solving Wave Equation with non local boundary conditions, Lecture Notes in Engineering and Computer Science, pp:1190-1193.

[5] He. J.H.,A Coupling Method of Homotopy Technique for Non Linear Problems, Int. J. Non Linear Mech, 35:37-43, 2000.

[6] J. H. He,Homotopy Perturbation Technique, Computer Methods in Applied Mechanics and Engineering, Vol. 178, no. 3-4, pp:257-262, 1999.

[7] J.H.He., Variational Iteration Method for Delay Differential Equations. Commun. Non Linear Sci . Numer. Simul. Vol. 2, pp:235-236, 1997

[8] G. Adomian, Solving Frontier Problems of Physics:The Decomposition Method, Kluwer, Boston, 1994.

[9] S. J. Liao,Ph.D. Thesis, Shanghai Jiao Tong University, Shanghai, China, 1992.

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