

On Numerical Solutions of Fractional-Order with a Delay of CSOH Model

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Abstract—Based on the current problem of squirrels within the coconut farms in SamutSongkhram, a province of Thailand, it encourages us to formulate the mathematical model to study the dynamics in this area. The system of fractional delayed differential equations is thus constructed for presenting the dynamics and interacting pattern of the number of coconut yields, squirrels, bran owls, and squirrel hunters in the coconut farm environment. The resulting model is shortly called the CSOH model. The well-known solver such as ODE45 and DDE23 are first used to numerically solve the first-order differential system of the PECE model for the approximate solutions. They are compared with those obtained using the Adams-Bashforth-Moulton type predictor-corrector scheme (PECE method). The comparisons among the methods demonstrate the consistency of the results. In addition, we apply the PECE method to the fractional delayed PECE model to investigate the dynamics of its numerical solutions.

Index Terms—SamutSongkhram, Thailand, Squirrel, Coconut, Bran owl, squirrel hunter, CSOH model, Adams-Bashforth-Moulton type predictor-corrector scheme, Caputo fractional-order derivative

I. INTRODUCTION

THE interaction of some species in a natural environment is successively studied in a numbers of contexts. However one well-known model that is employed to test an accuracy and efficiency of computation methods is LotkaVolterra model, also known as a predator-prey model, proposed in 1920. This model is adjusted to fit various assumptions under interest of scholars. Some adjusted models that are currently proposed, for instant, include models that represent interaction between two species under schooling behavior and a proportional harvest of both species[1], a model of two species with one of them can be divided to sub-classes that incorporate with a transition rate between sub-classes, competition within the classes, and the Holling type II response function [2], a model of an interaction of two species that include additional causes of mortality within the respond function [3]

Moreover, a model is extended to recognize the essence of other characteristics embed in the environment that can significantly effect on behaviors of the species within the

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model. The time delay, hence, becomes an important element that is employed for this purpose. The works that currently impose the time delay into models are, for example, the model of one class of prey and two classes of predator, immature and mature type, that assume the delay in the conversion time from immature to mature type which represented by $\frac{k_i b_i x(t-\tau) y_i(t-\tau)}{1+m_1 x(t-\tau)}$ where k_i is the consumption rate, b_i is a searching rate, and τ is a constant time delay notation[4], the two species model that utilize the function of $\frac{\theta a(1-c)(x(t-\tau)+\xi)}{1+\alpha\xi+a(1-c)hx(t-\tau)} - dy$ to represent the time delay in gestation where θ is a conversion rate, a is an attack rate, ξ is a quantity of additional food, c is a strength of habitat complexity, h is a handling time, and τ is the time delay for the gestation of predator [5], a model of two species with one of them can be divided to sub-classes, immature and mature classes, which identify the time delay in the function of $re^{d_1\tau}x_1(t-\tau)$ where x_1 is the immature prey and τ is time delay of immature class become mature class [6], and the two species model with the gestation delay in the logistic growth term that represent by $r - bx(t-\tau)x(t)$ [7].

Although computation results produced from a kind of integer order of differential equations like the models referred above are worth for studying, the marginal benefit of the fractional order of differential equation can draw many scholars attention to apply this technique for the purpose of model analysis. The main advantage of fractional derivative is that it provide an excellence tool for describing the memory that modify behavioral respond of species which make a crucial link between species [8]. This implies that the more accuracy of approximation should be produced by fractional technique. Examples of works related to this fractional derivative technique are a model of two types of species with a logistic growth of prey and Michaelis Menten type functional response [9], and a model of two species with a logistic growth of prey and Holling-type II functional response [10].

Of Course, all of these kinds of works, whether based on models of integer or fractional order, can provide valuable knowledge for inquiring solutions according to an assumption set by scholars and can do their best by closely approximation for solutions that benefit for further actions. It, in addition, also provides an idea of formulating the model and represent problems which are actually happen in community. This thus derive us the intention for searching the way to formulate such kind of a model.

In particular, the problem of a squirrel is recognized by the coconut farmer living in the SamutSongkhram, a province of Thailand, for long period of time since it is a major animal that destroys numbers of output produced in the farm while a farmer still don't have an effective way to prevent the output loss from it. Even though there are some strategies,

such as poisoning, catching, and hunting, that are previously applied to control numbers of squirrels within a farm, they seem less effective because of the limitation of capability of farmers to control squirrels and the limitation of support from related agent especially from a government. Therefore the problem of a squirrel is still embedded in this area. This becomes the point of interest of this work in that we will try to provide the foundation for actions to solve this problem. Hence it is worth, at this point, for starting to formulate the best-fit model to visualize of the squirrel phenomena within the coconut farm environment and to generate the basic idea for guesstimate some necessary actions for relieving this problem in the future. To meet the long term objective, we will begin our work by 1) setting the reasonable model for coconut yields, squirrels, bran owls, and squirrel hunters in the coconut farm environment, which shortly called CSOH model, and roughly identify an existence of its solution 2) confirming the validity of Predictor Corrector method that we will utilized for approximating the solution of fractional order based on accuracy and efficiency by comparing with numerical result from ODE45 and DDE23, and 3) applying fractional derivative technique, which will benefit for our work in the future, for approximate the solution of CSOH model by utilizing the PECE method. To complete this work, we will processed through 4 main sections which include the section 1 that will explain the logic behind the model construction, the section 2 and 3 that will provide the concept utilized for computation and verification of our model which include fractional derivative and numerical method, Adams-Bashforth-Moulton type predictor-corrector scheme, respectively, and section 4 that will present the computation result and, at the same time, discuss the conformity between ODE45, DDE23, and PECE method as well as the application of fractional derivative in CSOH model.

A. CSOH Model Formulation

The construction of model is based on the relationship between the coconut yield, squirrel, bran owl, and squirrel hunter. This model represents how the yield of coconuts can be varied by a squirrel density and how the action of the bran owl as well as a squirrel hunter can affect a density of squirrel. To present their relationship, we propose the following model:

$$\begin{aligned}
 {}_C D_0^\alpha C(t) &= \beta C(t) \left(1 - \frac{C(t)}{K}\right) - \gamma C(t)S(t) - \mu C(t), \\
 {}_C D_0^\alpha S(t) &= \eta S(t) \left(1 - \frac{S(t)}{\omega C(t) + Q}\right) + \lambda C(t - \tau)S(t) \\
 &\quad - \delta S(t)O(t) - \sigma S(t)H(t) - \rho S(t), \\
 {}_C D_0^\alpha O(t) &= \xi O(t) \left(1 - \frac{O(t)}{\kappa S(t) + B}\right) + \zeta S(t)O(t), \\
 {}_C D_0^\alpha H(t) &= cS(t) - dH(t),
 \end{aligned} \tag{1}$$

with the following history

$$[C(t), S(t), O(t), H(t)]^T = [C_0, D_0, O_0, H_0]^T, \quad -\tau \leq t \leq 0, \tag{2}$$

where ${}_C D_0^\alpha C(t)$ is the Caputo fractional derivative defined in the next section. The state variables and parameters in model (1) can be described as follows. $C(t)$ is a number of coconut yield at time t , $S(t)$, $O(t)$, and $H(t)$ are the

population of squirrel at time t , the population of bran owl at time t , and the population of squirrel hunter at time t respectively.

The basic assumptions of the model are that coconut yield (C), squirrel (S), and bran owl (O) are grown in the logistic way. For coconut yield, K is the coconut constant carrying capacity. However, as squirrel and bran owl are not rely on only one source of foods, so the carrying capacity of the squirrel can be combined between coconut and other sources of foods Q and the carrying capacity of bran owl depend on combination between number of squirrel and other sources B . For the rest of parameters, the constant β , η , and ξ denote the reproduction rate of coconut yields, squirrels, and bran owls, while c denote the enter rate of squirrel hunters, γ is the predation rate of the squirrels, μ is the harvesting rate of the coconut yields, ω is the coconut carrying capacity rate, λ is the conversion rate of the squirrels, δ is the predation rate of the bran owls, σ is the hunting rate of the squirrel hunters, ρ is the squirrel trapping rate, τ is the constant time delay, κ is the squirrel carrying capacity rate, ζ is the conversion rate of the bran owls, and d is the exit rate of the squirrel hunters.

II. PRELIMINARY

In this section, we will introduce some important concepts of fractional-order operators that will be utilized for approximating the solution of the fractional CSOH model. The definitions of the Riemann-Liouville fractional integral, the Caputo fractional derivative and the vital property are included as follows (see more details in [11]).

Definition 2.1: [11], [12] A function $f(t)$, ($t > 0$) is said to be in the space C_α ($\alpha \in \mathbb{R}$) if there exists a real number $p > \alpha$, such that $f(t) = t^p g(t)$ where $g(t) \in [0, \infty)$.

Definition 2.2: [11], [12] A function $f(t)$, ($t > 0$) is said to be in the space C_α^m , where $m \in \mathbb{N} \cup \{0\}$ if $f^{(m)} \in C_\alpha$.

Definition 2.3: [11], [12] The Riemann-Liouville fractional integral operator of order $\alpha > 0$ of a function $f \in C_\alpha$ with $a \geq 0$ is defined as

$${}_R L_a^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t - \tau)^{\alpha-1} f(\tau) d\tau, \quad t > a, \tag{3}$$

where $\Gamma(\cdot)$ is the gamma function. If $\alpha = 0$, then ${}_R L_a^\alpha f(t) = f(t)$

Definition 2.4: [11], [12] For a positive real number α , the Caputo fractional derivative of order α with $a \geq 0$ is defined in terms of the Riemann-Liouville fractional integral, i.e., ${}_C D_a^\alpha f(t) = {}_R L_a^{m-\alpha} f^{(m)}(t)$, or it can be expressed as

$${}_C D_a^\alpha f(t) = \frac{1}{\Gamma(m - \alpha)} \int_a^t \frac{f^{(m)}(\tau)}{(t - \tau)^{\alpha-m+1}} d\tau, \tag{4}$$

where $m - 1 < \alpha < m$, $t \geq a$ and $f \in C_{-1}^m$, $m \in \mathbb{N}$, and

$${}_C D_a^\alpha f(t) = f^{(m)} \quad \text{where } \alpha = m. \tag{5}$$

The important property of the Riemann-Liouville fractional integral and the Caputo fractional derivative of the same order α can be written as

$${}_R L_a^\alpha {}_C D_a^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^{(k)}(a) \frac{(t-a)^k}{k!}, \tag{6}$$

where $m - 1 < \alpha < m$, $f \in C_\alpha^m$ for $m \in \mathbb{N}$ and $\alpha \geq -1$.

III. METHOD OF SOLVING FRACTIONAL DELAY DIFFERENTIAL EQUATION

In this section, we will provide the description of the Adams-Bashforth-Moulton type predictor-corrector scheme or the PECE method for solving fractional-order differential equations with and without delay terms.

A. The predictor-corrector scheme for solving fractional-order differential equations

The PECE method [13] for numerically solving fractional-order differential equations is briefly described as follows. Consider the following initial value problem

$${}_C D_0^\alpha u(t) = f(t, u(t)), 0 \leq t \leq T, \quad (7)$$

$$u^{(k)}(0) = u_0^{(k)}, k = 0, 1, m-1, \alpha \in (m-1, m),$$

where f is a nonlinear function and m is a positive integer. The IVP (7) is equivalent to the Volterra integral equation

$$u(t) = \sum_{k=0}^{m-1} u_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, u(\tau)) d\tau. \quad (8)$$

To estimate the integral in (8), the entire interval $[0, T]$ of the independent variable t is discretized as the uniform grid $\{t_n = nh : n = 0, 1, \dots, N\}$ for some integer N with the step size $h := T/N$. Let $u_h(t_n)$ denote the approximation to $u(t_n)$. Suppose we have already approximated $u_h(t_j)$, $j = 1, 2, \dots, n$, and desire to obtain $u_h(t_{n+1})$ of the IVP (7), hence we obtain the discretized formula for a corrector as follows.

$$u_h(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, u_h^P(t_{n+1})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, u_h(t_j)), \quad (9)$$

where

$$a_{j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^\alpha, & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n+1. \end{cases} \quad (10)$$

The approximation $u_h^P(t_{n+1})$ in Eq. (9) is called a predictor and is given by

$$u_h^P(t_{n+1}) = \sum_{k=0}^{m-1} \frac{t_{n+1}^k}{k!} u_0^{(k)} + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, u_h(t_j)), \quad (11)$$

in which

$$b_{j,n+1} = \frac{h^\alpha}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha). \quad (12)$$

B. The predictor-corrector scheme for delay fractional differential equation

To solve delay fractional differential equation, the Adams-Bashforth-Moulton type predictor-corrector scheme, described in the previous section, is modified as follows [12], [14], [15]. Consider the following fractional-order delay differential equation

$${}_C D_0^\alpha u(t) = f(t, u(t), u(t-\tau)), t \in [0, T], 0 < \alpha \leq 1 \quad (13)$$

$$u(t) = g(t), t \in [-\tau, 0]. \quad (14)$$

Discretizing a uniform grid

$$\{t_n = nh : n = -k, -k+1, \dots, -1, 0, 1, \dots, N\}, \quad (15)$$

where k and N are integers such that the step size h satisfies the equations $h = T/N$ and $h = \tau/k$. We denote the approximation of $u(t_j)$ by $u_h(t_j)$ for $j = -k, -k+1, \dots, -1, 0, 1, \dots, N$ and let

$$u_h(t_j) = g(t_j), j = -k, -k+1, \dots, -1, 0. \quad (16)$$

In addition, the delayed approximation $u_h(t_j - \tau)$ can be written as

$$u_h(t_j - \tau) = u_h(jh - kh) = u_h(t_{j-k}), j = 0, 1, \dots, N. \quad (17)$$

Taking ${}_{RL} J_0^\alpha$ on both sides of Eq. (13) and using Eq. (14), we obtain

$$u(t) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} f(\xi, u(\xi), u(\xi-\tau)) d\xi. \quad (18)$$

Evaluating Eq. (18) at $t = t_{n+1}$, we have

$$u(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha)} \int_0^{t_{n+1}} (t_{n+1} - \xi)^{\alpha-1} f(\xi, u(\xi), u(\xi - \tau)) d\xi. \quad (19)$$

Suppose we have already calculated the approximations $u_h(t_j)$ for $(j = -k, -k+1, \dots, -1, 0, 1, \dots, n)$, and we want to approximate $u(t_{n+1})$ in Eq. (19). This can be done by approximating the integral in Eq. (19) using the product trapezoidal quadrature formula. The resulting approximated solution $u_h(t_{n+1})$ of $u(t_{n+1})$ is called a corrector and can be expressed as

$$u_h(t_{n+1}) = g(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, u_h^P(t_{n+1}), u_h(t_{n+1} - \tau)) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, u_h(t_j), u_h(t_j - \tau)) = g(0) + \frac{h^\alpha}{\Gamma(\alpha+2)} f(t_{n+1}, u_h^P(t_{n+1}), u_h(t_{n+1-k})) + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{j,n+1} f(t_j, u_h(t_j), u_h(t_{j-k})), \quad (20)$$

where $a_{j,n+1}$ is given by the formula (10). The predictor term $u_h^P(t_{n+1})$ in Eq. (20) is evaluated by the product rectangle rule and can be written as

$$u_h^P(t_{n+1}) = g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, u_h(t_j), u_h(t_j - \tau)), = g(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{j,n+1} f(t_j, u_h(t_j), u_h(t_{j-k})), \quad (21)$$

where $b_{j,n+1}$ can be computed using the formula (12).

IV. NUMERICAL EXPERIMENT

This section will compare the computational results of the problem (1)–(2) with and without the time delay τ for $\alpha = 1, 0.9$. Firstly, the comparison is demonstrated for the first-order case using the simulation results obtained from the MATLAB commands, i.e., ODE45, DDE23 and the PECE method in Eqs. (9)–(12) and Eqs. (20)–(21). The objective of this comparison is to validate the CSOH model and to ensure the capability of PECE method for applying to the fractional delayed model (1)–(2). Next, the numerical results of the fractional delayed system (1)–(2), which are obtained using the PECE method in Eqs. (20)–(21), are graphically and numerically shown for $\alpha = 0.9$. The numerical experiments are as follows.

A. Simulation results of the problem (1)-(2) for $\alpha = 1$

We begin applying the Adams-Bashforth-Moulton type predictor-corrector scheme without a delay term as shown in Eqs. (9)–(12) to the problem (1)–(2) for $\alpha = 1, \tau = 0$. It can be noticed that the original problem is reduced to the initial value problem with the initial conditions $[C(0), S(0), O(0), H(0)]^T = [C_0, D_0, O_0, H_0]^T$. Then the discretized formulas for such a problem are written as

$$C_h^{n+1} = C_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\beta C_h^{P,n+1} \left(1 - \frac{C_h^{P,n+1}}{K} \right) - \gamma C_h^{P,n+1} S_h^{P,n+1} - \mu C_h^{P,n+1} \right] + \frac{h^\alpha}{\Gamma(\alpha+2)} \times \sum_{j=0}^n a_{1,j,n+1} \left[\beta C_h^j \left(1 - \frac{C_h^j}{K} \right) - \gamma C_h^j S_h^j - \mu C_h^j \right], \quad (22)$$

$$S_h^{n+1} = S_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\eta S_h^{P,n+1} \left(1 - \frac{S_h^{P,n+1}}{\omega C_h^{P,n+1} + Q} \right) + \lambda C_h^{P,n+1} S_h^{P,n+1} - \delta S_h^{P,n+1} O_h^{P,n+1} - \sigma S_h^{P,n+1} H_h^{P,n+1} - \rho S_h^{P,n+1} \right] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{2,j,n+1} \left[\eta S_h^j \times \left(1 - \frac{S_h^j}{\omega C_h^j + Q} \right) + \lambda C_h^j S_h^j - \delta S_h^j O_h^j - \sigma S_h^j H_h^j - \rho S_h^j \right], \quad (23)$$

$$O_h^{n+1} = O_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[\xi O_h^{P,n+1} \left(1 - \frac{O_h^{P,n+1}}{\kappa S_h^{P,n+1} + B} \right) + \zeta S_h^{P,n+1} O_h^{P,n+1} \right] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{3,j,n+1} \left[\xi O_h^j \left(1 - \frac{O_h^j}{\kappa S_h^j + B} \right) + \zeta S_h^j O_h^j \right], \quad (24)$$

$$H_h^{n+1} = H_0 + \frac{h^\alpha}{\Gamma(\alpha+2)} \left[c S_h^{P,n+1} - d H_h^{P,n+1} \right] + \frac{h^\alpha}{\Gamma(\alpha+2)} \sum_{j=0}^n a_{4,j,n+1} \left[c S_h^j - d H_h^j \right], \quad (25)$$

where

$$C_h^{P,n+1} = C_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{1,j,n+1} \left[\beta C_h^j \left(1 - \frac{C_h^j}{K} \right) - \gamma C_h^j S_h^j - \mu C_h^j \right], \quad (26)$$

$$S_h^{P,n+1} = S_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{2,j,n+1} \left[\eta S_h^j \left(1 - \frac{S_h^j}{\omega C_h^j + Q} \right) + \lambda C_h^j S_h^j - \delta S_h^j O_h^j - \sigma S_h^j H_h^j - \rho S_h^j \right], \quad (27)$$

$$O_h^{P,n+1} = O_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{3,j,n+1} \left[\xi O_h^j \left(1 - \frac{O_h^j}{\kappa S_h^j + B} \right) + \zeta S_h^j O_h^j \right], \quad (28)$$

$$H_h^{P,n+1} = H_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{4,j,n+1} \left[c S_h^j - d H_h^j \right], \quad (29)$$

where, for $l = 1, 2, 3, 4$,

$$a_{l,j,n+1} = \begin{cases} n^{\alpha+1} - (n-\alpha)(n+1)^\alpha, & \text{if } j = 0, \\ (n-j+2)^{\alpha+1} + (n-j)^{\alpha+1} \\ - 2(n-j+1)^{\alpha+1}, & \text{if } 1 \leq j \leq n, \\ 1, & \text{if } j = n+1, \end{cases} \quad (30)$$

and

$$b_{l,j,n+1} = \frac{h^\alpha}{\alpha} ((n+1-j)^\alpha - (n-j)^\alpha), \quad 0 \leq j \leq n. \quad (31)$$

To obtain the numerical solutions, we insert the following values

$$C_0 = 500, S_0 = 20, O_0 = 2, H_0 = 2, \beta = 3, \\ K = 3, 500, \gamma = 0.1, \mu = 0.3, \eta = 1, \omega = 0.2, Q = 100, \\ \lambda = 0.005, \delta = 0.2, \sigma = 0.1, \rho = 0.02, \xi = 1, \kappa = 0.05, \\ B = 10, \zeta = 0.002, c = 0.01, d = 0.06, \quad (32)$$

into the formulas (22)–(31) with the step size $h = 0.01$. The simulation results obtained using the above PECE method are plotted in Fig1. Numerical comparison between the results obtained by the PECE and the command ODE45 are shown in Table I.

Secondly, we impose the time delay $\tau = 0.1$ into the CSOH model (1)–(2) for $\alpha = 1$. To compare the numerical results obtained using the PECE method with the time delay to the command DDE23, only the the corrector and predictor of $S(t)$ in formulas (23), (27) must be modified using the discretized formulas (20), (21) as follows.

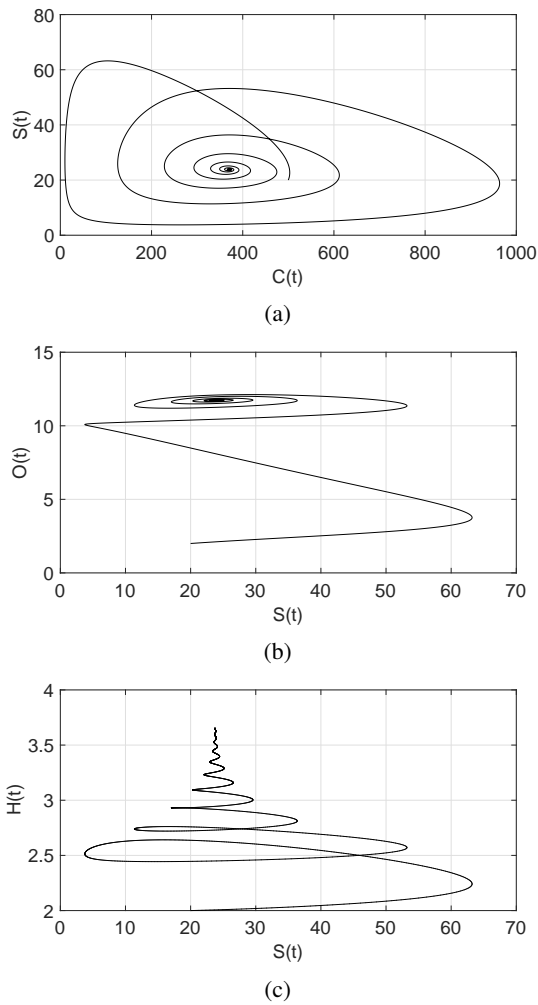


Fig. 1: Numerical results obtained using the PECE method with $\alpha = 1, \tau = 0, h = 0.01$, and the simulation time $T = 30$, (a) dynamics between $C(t)$ and $S(t)$, (b) dynamics between $S(t)$ and $O(t)$, (c) dynamics between $S(t)$ and $H(t)$.

$$\begin{aligned}
 S_h^{n+1} = & S_0 + \frac{h^\alpha}{\Gamma(\alpha + 2)} \left[\eta S_h^{P,n+1} \left(1 - \frac{S_h^{P,n+1}}{\omega C_h^{P,n+1} + Q} \right) \right. \\
 & + \lambda C_h^{P,n+1-k} S_h^{P,n+1} - \delta S_h^{P,n+1} O_h^{P,n+1} \\
 & \left. - \sigma S_h^{P,n+1} H_h^{P,n+1} - \rho S_h^{P,n+1} \right] \\
 & + \frac{h^\alpha}{\Gamma(\alpha + 2)} \sum_{j=0}^n a_{2,j,n+1} \\
 & \times \left[\eta S_h^j \left(1 - \frac{S_h^j}{\omega C_h^j + Q} \right) + \lambda C_h^{j-k} S_h^j \right. \\
 & \left. - \delta S_h^j O_h^j - \sigma S_h^j H_h^j - \rho S_h^j \right], \quad (33)
 \end{aligned}$$

$$\begin{aligned}
 S_h^{P,n+1} = & S_0 + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^n b_{2,j,n+1} \left[\eta S_h^j \left(1 - \frac{S_h^j}{\omega C_h^j + Q} \right) \right. \\
 & \left. + \lambda C_h^{j-k} S_h^j - \delta S_h^j O_h^j - \sigma S_h^j H_h^j - \rho S_h^j \right]. \quad (34)
 \end{aligned}$$

TABLE I: Discrepancy of the numerical results of the problem (1)–(2) for $\alpha = 1, \tau = 0$ obtained using the ODE45 solver and the PECE method ($h = 0.01$).

t	ODE45-PECE			
	$ \Delta C $	$ \Delta S $	$ \Delta O $	$ \Delta H $
0	0	0	0	0
3	0.53997	0.01918	0.00057	0.00022
6	1.24462	0.20278	0.00324	0.00111
9	1.07454	0.19353	0.00313	0.00138
12	0.64546	0.05613	0.00275	0.00119
15	0.46518	0.01336	0.00104	0.00100
18	0.16231	0.01158	0.00018	0.00088
21	0.07052	0.02192	0.00016	0.00075
24	0.28465	0.02560	0.00077	0.00070
27	0.30293	0.02458	0.00083	0.00060
30	0.47970	0.01272	0.00102	0.00056

TABLE II: Discrepancy of the numerical results of the problem (1)–(2) for $\alpha = 1, \tau = 0.1$ obtained using the DDE23 solver and the PECE method ($h = 0.01$).

t	DDE23-PECE			
	$ \Delta C $	$ \Delta S $	$ \Delta O $	$ \Delta H $
0	0	0	0	0
3	0.12882	0.01145	0.00067	0.00011
6	1.02404	0.33619	0.00428	0.00027
9	22.79801	1.19936	0.02216	0.00350
12	5.78926	0.62653	0.00887	0.00037
15	16.55247	0.40536	0.00074	0.00133
18	19.78738	0.33068	0.00646	0.00250
21	20.94212	0.30122	0.01251	0.00329
24	21.64938	0.30588	0.01686	0.00382
27	22.07041	0.34118	0.01869	0.00411
30	21.89598	0.41364	0.01799	0.00414

Using the same values of the parameters described in Eq. (32), the modified PECE method for the delayed first-order problem gives the graphical results in Fig2. The differences between the results obtained using the PECE method and the DDE23 solver are numerically shown in Table II. The impact of the time delay to the system is that it postpones the time of reaching the equilibrium points. Fig 3 shows that the numerical solutions are converging to their equilibrium points when $t \approx 140$. These convergences cannot be observed in Fig 2.

Referring to Table (II), it can be noticed that the differences of the numerical results obtained utilizing the two methods are quite large on the coconut yield $C(t)$. This can be overcome by reducing the step size h of the PECE method from $h = 0.01$ to $h = 0.001$. The numerical simulation consequently results in reducing the size of discrepancy as shown in Table III. Hence, reducing the step size h of the PECE method has a great impact on the accuracy of the numerical results.

B. Simulation results of the problem (1)-(2) for $\alpha = 0.9$

According to the above comparisons, we have the conformity among the used methods, i.e., the ODE45 solver, DDE23 solver, and PECE method on the problem (1)-(2) for $\alpha = 1$. In this section, we hence obtain numerical results of the fractional-order delayed CSOH model (1)-(2) for $\alpha = 0.9$. Applying the PECE method, which is described in Eqs. (22), (24), (25), (26), (28), (29), and the modified formulas (33), (34) with the step size $h = 0.001$ along with the history and parameter values described in Eq. (32), to

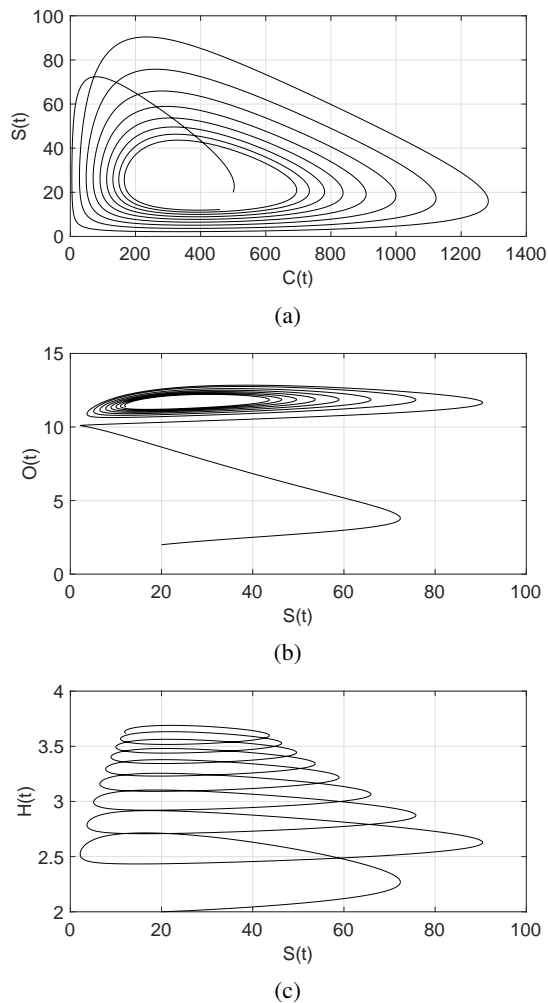


Fig. 2: Numerical results obtained using the PECE method with $\alpha = 1, \tau = 0.1, h = 0.01$, and the simulation time $T = 30$, (a) dynamics between $C(t)$ and $S(t)$, (b) dynamics between $S(t)$ and $O(t)$, (c) dynamics between $S(t)$ and $H(t)$.

TABLE III: Discrepancy of the numerical results of the problem (1)–(2) for $\alpha = 1, \tau = 0.1$ obtained using the DDE23 solver and the PECE method ($h = 0.001$).

t	DDE23-PECE			
	$ \Delta C $	$ \Delta S $	$ \Delta O $	$ \Delta H $
0	0	0	0	0
3	0.06723	0.00467	0.00002	0.00005
6	0.07294	0.33595	0.00283	0.00101
9	11.50632	0.73602	0.01110	0.00180
12	4.66667	0.29782	0.00290	0.00005
15	9.70057	0.18178	0.00156	0.00084
18	11.17735	0.14209	0.00554	0.00148
21	11.79295	0.12547	0.00909	0.00198
24	12.39969	0.12866	0.01198	0.00237
27	13.01634	0.14810	0.01349	0.00266
30	13.40185	0.19023	0.01348	0.00281

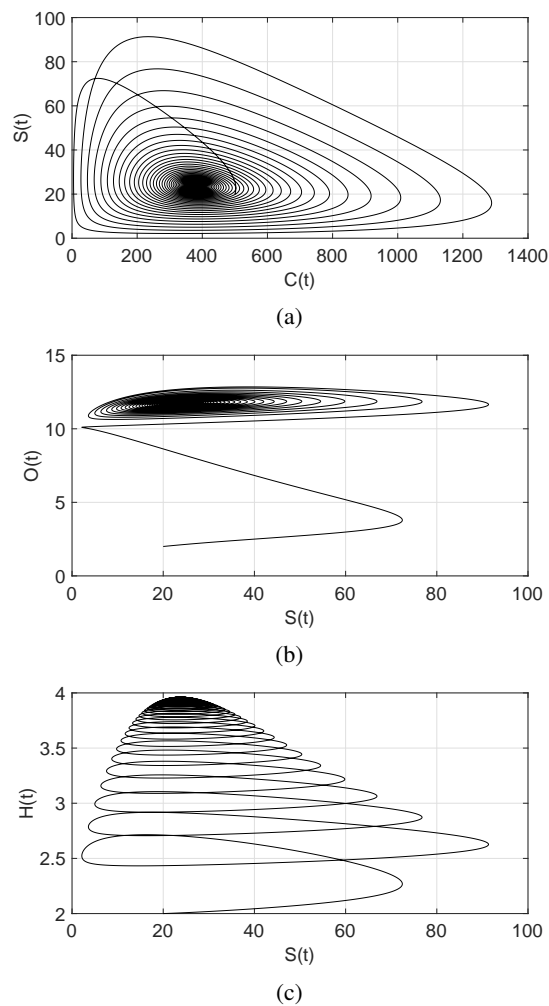


Fig. 3: Numerical results obtained using the PECE method with $\alpha = 1, \tau = 0.1, h = 0.01$, and the simulation time $T = 200$, (a) dynamics between $C(t)$ and $S(t)$, (b) dynamics between $S(t)$ and $O(t)$, (c) dynamics between $S(t)$ and $H(t)$.

TABLE IV: Numerical results of the problem (1)–(2) for $\alpha = 0.9, \tau = 0.1$ using the PECE method with $h = 0.001$.

t	PECE			
	$ \Delta C $	$ \Delta S $	$ \Delta O $	$ \Delta H $
0	500	20	2	2
3	173.23355	12.97739	9.41423	2.58536
6	311.95077	19.82819	11.22206	2.72312
9	357.43848	22.61053	11.48103	2.86142
12	361.10604	23.71782	11.57423	2.98823
15	360.45823	23.93775	11.61876	3.09766
18	361.54209	23.92124	11.64126	3.19126
21	363.34415	23.88780	11.65437	3.27187
24	365.05575	23.86833	11.66327	3.34179
27	366.50416	23.85648	11.66990	3.40275
30	367.72692	23.84725	11.67505	3.45609

the problem (1)–(2) for $\alpha = 0.9$, we obtain the approximate solutions as portrayed in Fig4 and the numerical results at some specified times t are shown in Table (IV).

V. CONCLUSIONS

To construct the mathematical model that represents the squirrel situation in coconut farm and to approximate for solution, we have used ODE 45 and DDE 23 as the baseline

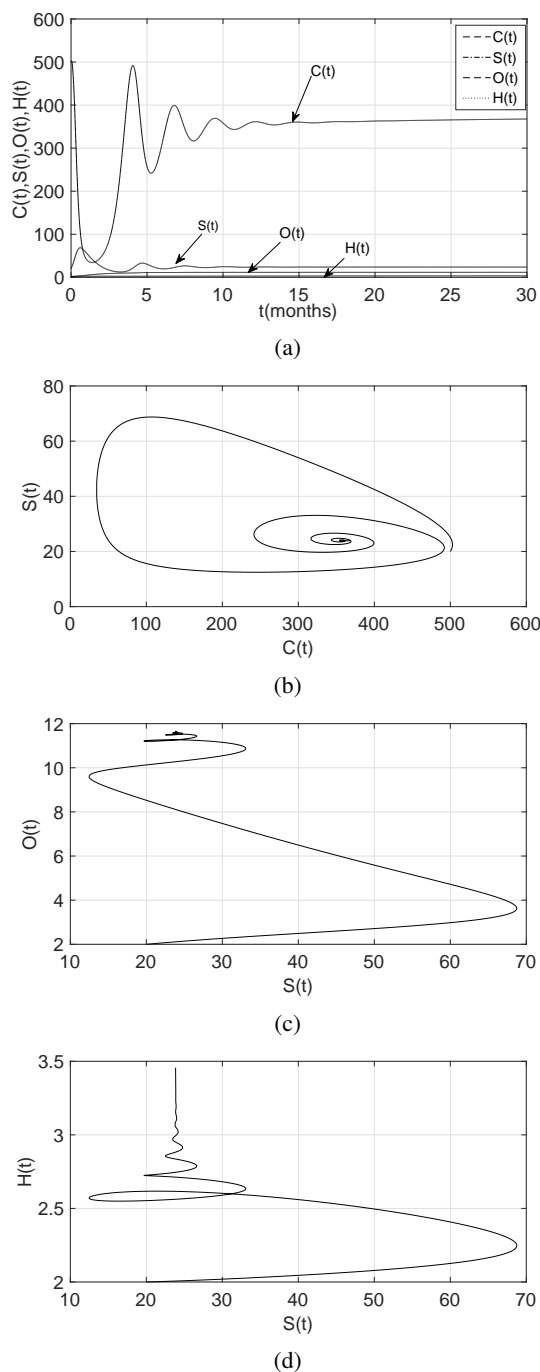


Fig. 4: Numerical results obtained using the PECE method with $\alpha = 0.9$, $\tau = 0.1$, $h = 0.001$, and the simulation time $T = 30$, (a) numerical solutions $C(t)$, $S(t)$, $O(t)$, $H(t)$, (b) dynamics between $C(t)$ and $S(t)$, (c) dynamics between $S(t)$ and $O(t)$, (d) dynamics between $S(t)$ and $H(t)$.

solution for benefit of developing the computation technique based on PECE method to approximate solution for fractional order. The comparison show that PECE method can approximate solution close to what approximated by ODE 45 and DDE 23, especially when step size is small, in the integer order case both for the model that impose and don't impose the time delay. Refer to the conformity of approximation from ODE 45, DDE 23, and PECE method, we thus adjust the computation detail of PECE method for approximating the solution of fractional order.

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