Improved Exponential Stability Analysis of Certain Nonlinear Neutral Differential Equation with Time-Varying Delays

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Abstract—**In this paper we study the problem of delay dependent exponential stability criteria for certain nonlinear neutral differential equation with discrete and distributed time-varying delays. By using Lyapunov-krasovskii functional (LKF), model transformation, Leiniz-Newton formula and utilization of zero equation with guarantee exponentially stable of the neutral equation, the exponential stability criterion is in the form of linear matrix inequality (LMI). The numerical examples are given to illustrate the present result.**

Index Terms—**exponential stability, neutral equation, linear matrix inequality, discrete time-varying delays**

I. Introduction

THE neutral differential equation is in the form

$$\frac{d}{dt}[x(t)+px(t-\tau(t))] = -ax(t) + b\tanh x(t-\sigma(t)), \quad t \geq 0, \quad (1)$$

where a, b are positive real constants and $|p| < 1$. $\tau(t)$ and $\sigma(t)$ are neutral and discrete time-varying delays respectively,

$$0 \leq \tau(t) \leq \tau_2, \quad \dot{\tau}(t) < \tau_d,$$
$$0 \leq \sigma(t) \leq \sigma_2, \quad \dot{\sigma}(t) < \sigma_d,$$

where τ, σ, τ_d and σ_d are given positive real constants. For each solution $x(t)$ of (1), we assume the initial condition

$$x_0(t) = \phi(t), \quad t \in [-r, 0],$$

where $\phi \in C([-r, 0]; R)$.

Time-delay was discovered in many real-world engineering systems either in the state, the control input, or the measurements. Delays are more concerned with challenging areas of communication and information technologies: in the stabilization of networked controlled systems and in the high-speed communication network. Time-delay is an origin of instability. However, the attendance of delay can have a stabilizing effect on some systems. The stability analysis of time-delay systems (TDSs) is the importance of theoretical and practical.

Neutral differential equation was discovered in scientific and engineering field such as aircraft, chemical and process control system, biological system [6],[13],[16]. Therefore, many researchers have focused on the stability analysis of neutral differential equation. The asymptotic stability of the neutral differential equation with constant delay have been discussed in [1],[3]-[5],[9],[10],[17]-[21] and the exponential stability of neutral differential equation with time-varying delay have been discussed in [2],[3],[14],[15]. It is well know on the problem of stability for delay system that can be divided into two categories, namely delay-independent stability criteria and delay-dependent stability criteria. Since, the former are more conservative than the letter when the delay is small. Thus, much attention has been paid to delay-dependent stability criteria. Lyapunov stability method, the Lyapunov function is powerful tool for stability analysis of time delay systems. Delay dependent stability criteria of these system are established in term of linear matrix inequalities (LMIs).

In this research, we study the exponential stability of the neutral differential equation with discrete time-varying delays. By using Lyapunov-Krasovskii functional, model transformation, Leibniz-Newton formula and utilization of zero equation with guarantee exponential stable of the neutral equation, the stability criterion is in the form of LMI. Finally, numerical examples are presented to show the effectiveness of the proposed criterion by comparing the upper bounds of the delay $\sigma(t)$ with other existing works.

Notation:
The following notations will be accounted in this paper: R^+ denotes the set of all non-negative real numbers, R^n denotes the n-dimensional Euclidean space, $R^{n\times r}$ denotes the set of $(n \times r)$ real matrices, $C([-r, 0]; \mathbb{R})$ denotes the space of all continuous vector functions mapping $[-r, 0]$ into $\mathbb{R}$, $\|x\|$ denotes the Euclidean vector norm of $x \in R^n$, I denotes the identity matrix.

II. Preliminaries

Definition 1. [14] The equilibrium point $x = 0$ of the equation (1) is *exponentially stable* if there exist positive real constants K, λ such that

$$\|x(t)\| \leq Ke^{-\lambda t} \sup_{-r\leq s\leq 0} \|x(s)\| = Ke^{-\lambda t}\|x_0\|,$$

where $\|x_t\|_s = \sup_{-r\leq s\leq 0} \|x(t + s)\|$.

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Lemma 2. [8]**(Jensen's Inequality)** For any symmetric positive definite matrix Q, positive real number h, and vector function $\dot{x}(t) : [-h, 0] \to R^n$ such that the following integral is well defined, then

$$-h \int_{-h}^{0} \dot{x}^T(s+t) Q \dot{x}(s+t) ds \tag{2}$$
$$\leq -\Big(\int_{-h}^{0} \dot{x}(s+t) ds \Big)^T Q \Big(\int_{-h}^{0} \dot{x}(s+t) ds \Big).$$

Lemma 3. (Cauchy's inequality) For any constant symmetric positive definite matrix $P \in R^{n\times n}$ and $a, b \in R^n$,

$$\pm 2a^T b \leq a^T P a + b^T P^{-1} b.$$

Lemma 4. [11],[12] **(Peng - Park's integral inequality)** For any matrix $\begin{bmatrix} Z & S \\ * & Z \end{bmatrix} \geq 0$, positive scalars τ and $\tau(t)$ satisfying $0 < \tau(t) < \tau$, vector function $\dot{x}[-\tau, 0] \to R^n$ such that the concerned integrations are well defined, then

$$-\tau \int_{t-\tau}^{t} \dot{x}^T(s) Z \dot{x}(s) ds \leq \omega^T(t) \ominus \omega(t)$$

where $\omega = [x^T(t), x^T(t-\tau(t)), x^T(t-\tau)]$ and $\ominus = \begin{bmatrix} -Z & Z-S & S \\ * & 2Z+S+S^T & Z-S \\ * & * & -Z \end{bmatrix}$.

III. Main results

In this section, we analyze the exponential stability problem for the neutral differential equation (1) with time-varying delays. From model transformation method, we have the Leibniz-Newton formula in the form

$$0 = x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t} \dot{x}(s) ds. \tag{3}$$

By utilizing the zero equation, we obtain

$$0 = r_1 x(t) - r_1 x(t-\tau(t)) - r_1 \int_{t-\tau(t)}^{t} \dot{x}(s) ds, \tag{4}$$
$$0 = (1-r_2)x(t) - (1-r_2)x(t-\tau(t)) - (1-r_2) \int_{t-\tau(t)}^{t} \dot{x}(s) ds, \tag{5}$$

where $r_1, r_2 \in R$ will be chosen to guarantee the exponential stability of the equation (1). By (3)-(5) the equation (1) can be formulated in the form

$$\dot{x}(t) = (r_1 - a)x(t) + b \tanh x(t-\sigma(t)) - (p + \dot{\tau}(t) - r_2\dot{\tau}(t))\dot{x}(t-\tau(t)) - r_1 x(t-\tau(t)) - r_1 \int_{t-\tau(t)}^{t} \dot{x}(s) ds. \tag{6}$$

The exponential stability for the neutral differential equation with time-varying delays in equation (1) will be presented as follows.

Theorem 5. For given positive real constants τ_2, τ_d, σ_2 and σ_d, equation (1) is exponential stable with a delay rate $\alpha > 0$ if there exist positive real constants k_i where $i = 1, 2, \ldots, 10$ and real constants w_k, m_k where $k = 1, 2, \ldots, 6$ such that

$$\Xi < 0, \tag{7}$$

where $\Xi = [\Omega_{(i,j)}]$,

$$\Omega_{(1,1)} = -2w_2 + k_4 + k_5 + k_8\tau_2\tau_2 - 2w_6,$$
$$\Omega_{(1,2)} = -w_1 + w_2 r_1 - w_2 a + m_1 - w_6 a,$$
$$\Omega_{(1,3)} = -w_2 r_1 - m_1,$$
$$\Omega_{(1,4)} = 0,$$
$$\Omega_{(1,5)} = -w_2 r_1 - w_5 - m_1,$$
$$\Omega_{(1,6)} = w_2 b - w_3 + w_6 b,$$
$$\Omega_{(1,7)} = -w_2 p - w_4 - w_6 p,$$
$$\Omega_{(1,8)} = \Omega_{(1,9)} = \Omega_{(1,10)}\Omega_{(1,11)} = \Omega_{(1,12)} = 0,$$
$$\Omega_{(2,2)} = 2\alpha k_1 + 2k_1 r_1 - 2k_1 a + 2w_1 r_1 - 2w_1 a + k_2 + k_3 + k_6\tau_2\tau_2 + k_7\sigma_2\sigma_2 - k_8 e^{-2\alpha\tau_2} + k_9 + k_{10}\sigma_2\sigma_2 + 2m_2,$$
$$\Omega_{(2,3)} = -k_1 r_1 - w_1 r_1 + k_8 e^{-2\alpha\tau_2} - s1 - m_2 + m_3,$$
$$\Omega_{(2,4)} = s_1,$$
$$\Omega_{(2,5)} = -k_1 r_1 - w_1 r_1 + w_5 r_1 - a w_5 - m_2 + m_4,$$
$$\Omega_{(2,6)} = k_1 b + w_1 b + w_3 r_1 - w_3 a + m_5,$$
$$\Omega_{(2,7)} = -k_1 p - w_1 p + w_4 r_1 - w_4 a,$$
$$\Omega_{(2,8)} = \Omega_{(2,9)} = \Omega_{(2,10)} = \Omega_{(2,11)} = 0,$$
$$\Omega_{(2,12)} = m_6,$$
$$\Omega_{(3,3)} = -2k_8 e^{-2\alpha\tau_2} + 2s_1 - 2m_3,$$
$$\Omega_{(3,4)} = k_8 e^{-2\alpha\tau_2} - s_1,$$
$$\Omega_{(3,5)} = -w_5 r_1 - m_3 - m_4,$$
$$\Omega_{(3,6)} = -w_3 r_1 - m_5,$$
$$\Omega_{(3,7)} = -w_4 r_1,$$
$$\Omega_{(3,8)} = \Omega_{(3,9)} = \Omega_{(3,10)} = \Omega_{(3,11)} = 0,$$
$$\Omega_{(3,12)} = -m_6,$$
$$\Omega_{(4,4)} = -k_2 e^{-2\alpha\tau_2} - k_8 e^{-2\alpha\tau_2},$$
$$\Omega_{(4,5)} = \Omega_{(4,6)} = \Omega_{(4,7)} = \Omega_{(4,8)} = \Omega_{(4,9)} = \Omega_{(4,10)} = \Omega_{(4,11)} = \Omega_{(4,12)} = 0,$$
$$\Omega_{(5,5)} = -2w_5 r_1 - 2m_4,$$
$$\Omega_{(5,6)} = -w_3 r_1 + w_5 b - m_5,$$
$$\Omega_{(5,7)} = -w_4 r_1 - w_5 p,$$
$$\Omega_{(5,8)} = \Omega_{(5,9)} = \Omega_{(5,10)} = \Omega_{(5,11)} = 0,$$
$$\Omega_{(5,12)} = -m_6,$$
$$\Omega_{(6,6)} = 2w_3 b - k_9 e^{-2\alpha\sigma_2} + k_9\sigma_d,$$
$$\Omega_{(6,7)} = -w_3 p + w_4 b,$$
$$\Omega_{(6,8)} = \Omega_{(6,9)} = \Omega_{(6,10)} = \Omega_{(6,11)} = \Omega_{(6,12)} = 0,$$
$$\Omega_{(7,7)} = -2w_4 p + \tau_d k_4 - k_4 e^{-2\alpha\tau_2},$$
$$\Omega_{(7,8)} = \Omega_{(7,9)} = \Omega_{(7,10)} = \Omega_{(7,11)} = \Omega_{(7,12)} = 0,$$
$$\Omega_{(8,8)} = -k_3 e^{-2\alpha\sigma_2}$$
$$\Omega_{(8,9)} = \Omega_{(8,10)} = \Omega_{(8,11)} = \Omega_{(8,12)} = 0,$$
$$\Omega_{(9,9)} = -k_5 e^{-2\alpha\sigma_2} + \sigma_d k_5,$$
$$\Omega_{(9,10)} = \Omega_{(9,11)} = \Omega_{(9,12)} = 0,$$
$$\Omega_{(10,10)} = -k_6 e^{-2\alpha\tau_2},$$
$$\Omega_{(10,11)} = \Omega_{(10,12)} = 0,$$
$$\Omega_{(11,11)} = -k_7 e^{-2\alpha\sigma_2},$$
$$\Omega_{(11,12)} = 0,$$
$$\Omega_{(12,12)} = -k_{10} e^{-2\alpha\sigma_2}$$

Proof: Consider the Lyapunov-Krasovskii functional candidates:

$$V(t,x_t)=\sum_{i=1}^{6}V_i(t,x_t),$$

where

$$\begin{aligned}
V_1(t,x_t) &= k_1e^{2\alpha t}x^2(t),\\
V_2(t,x_t) &= k_2\int_{t-\tau_2}^{t}e^{2\alpha s}x^2(s)ds+k_3\int_{t-\sigma_2}^{t}e^{2\alpha s}x^2(s)ds\\
&\quad +k_4\int_{t-\tau(t)}^{t}e^{2\alpha s}\dot{x}^2(s)ds\\
&\quad +k_5\int_{t-\sigma(t)}^{t}e^{2\alpha s}\dot{x}^2(s)ds,\\
V_3(t,x_t) &= k_6\tau_2\int_{-\tau_2}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-\theta)}x^2(s)dsd\theta\\
&\quad +k_7\sigma_2\int_{-\sigma_2}^{o}\int_{t+\theta}^{t}e^{2\alpha(s-\theta)}x^2(s)dsd\theta,\\
V_4(t,x_t) &= k_8\tau_2\int_{-\tau_2}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-\theta)}\dot{x}^2(s)dsd\theta,\\
V_5(t,x_t) &= k_9\int_{t-\sigma t}^{t}e^{2\alpha s}\tanh^2 x(s)ds\\
&\quad +k_{10}\sigma_2\int_{-\sigma_2}^{0}\int_{t+\theta}^{t}e^{2\alpha(s-\theta)}\tanh^2 x(s)dsd\theta,\\
V_6(t,x_t) &= \gamma e^{2\alpha t}x^2(t),
\end{aligned}$$

where γ is a positive number that will be be determined later. Calculating the time derivatives of $V(t,x_t)$ along the solution of (6), we get

$$\dot{V}(t,x_t) = \sum_{i=1}^{6}\dot{V}_i(t,x_t), \tag{8}$$

where

$$\begin{aligned}
\dot{V}_1 &= 2\alpha k_1e^{2\alpha t}x^2(t)+2k_1e^{2\alpha t}x(t)\dot{x}(t)\\
&= 2\alpha k_1e^{2\alpha t}x^2(t)+2k_1e^{2\alpha t}x(t)\left[r_1x(t)-ax(t)\right.\\
&\quad -r_1x(t-\tau(t))-r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]\\
&\quad +2w_1e^{2\alpha t}x(t)\left[r_1x(t)-ax(t)-\dot{x}(t)-r_1x(t-\tau(t))\right.\\
&\quad -r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]\\
&\quad +2w_2e^{2\alpha t}\dot{x}(t)\left[r_1x(t)-ax(t)-\dot{x}(t)-r_1x(t-\tau(t))\right.\\
&\quad -r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]\\
&\quad +2w_3e^{2\alpha t}\tanh x(t-\sigma(t))\left[r_1x(t)-ax(t)-\dot{x}(t)\right.\\
&\quad -r_1x(t-\tau(t))-r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]\\
&\quad +2w_4e^{2\alpha t}\dot{x}(t-\tau(t))\left[r_1x(t)-ax(t)-\dot{x}(t)\right.\\
&\quad -r_1x(t-\tau(t))-r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]\\
&\quad +2w_5e^{2\alpha t}\int_{t-\tau(t)}^{t}\dot{x}sds\left[r_1x(t)-ax(t)-\dot{x}(t)\right.\\
&\quad -r_1x(t-\tau(t))-r_1\int_{t-\tau(t)}^{t}\dot{x}(s)ds+b\tanh x(t-\sigma(t))\\
&\quad \left.-p\dot{x}(t-\tau(t))+r_2\dot{\tau}(t)\dot{x}(t-\tau(t))\right]
\end{aligned}\tag{9}$$

$$\begin{aligned}
\dot{V}_2 &= k_2\left[e^{2\alpha t}x^2(t)-e^{2\alpha(t-\tau_2)}x^2(t-\tau_2)\right]\\
&\quad +k_3\left[e^{2\alpha t}x^2(t)-e^{2\alpha(t-\sigma_2)}x^2(t-\sigma_2)\right]\\
&\quad +k_4\left[e^{2\alpha t}\dot{x}^2(t)-\left(e^{2\alpha(t-\tau(t))}\dot{x}^2(t-\tau(t))\right)(1-\dot{\tau}(t))\right]\\
&\quad +k_5\left[e^{2\alpha t}\dot{x}^2(t)-\left(e^{2\alpha(t-\sigma(t))}\dot{x}^2(t-\sigma(t))\right)(1-\dot{\sigma}(t))\right].
\end{aligned}$$

Since $\tau(t)\le\tau_2,\sigma(t)\le\sigma_2,\dot{\tau}(t)<\tau_d$ and $\dot{\sigma}(t)<\sigma_d$, we have

$$\begin{aligned}
\dot{V}_2 &\le e^{2\alpha t}\left[k_2x^2(t)-k_2e^{-2\alpha\tau_2}x^2(t-\tau_2)+k_3x^2(t)\right.\\
&\quad -k_3e^{-2\alpha\sigma_2}x^2(t-\sigma_2)+k_4\dot{x}^2(t)-k_4e^{-2\alpha\tau_2}x^2(t-\tau(t))\\
&\quad +\tau_dk_4\dot{x}^2(t-\tau(t))+k_5\dot{x}^2(t)-k_5e^{-2\alpha\sigma_2}x^2(t-\sigma(t))\\
&\quad \left.+\sigma_dk_5\dot{x}^2(t-\sigma(t))\right]
\end{aligned}\tag{10}$$

$$\begin{aligned}
\dot{V}_3 &= k_6\tau_2\left[\int_{-\tau_2}^{0}e^{2\alpha(t-\theta)}x^2(t)d\theta-\int_{-\tau_2}^{0}e^{2\alpha t}x^2(t+\theta)d\theta\right]\\
&\quad +k_7\sigma_2\left[\int_{-\sigma_2}^{0}e^{2\alpha(t-\theta)}x^2(t)d\theta-\int_{-\sigma_2}^{0}e^{2\alpha t}x^2(t+\theta)d\theta\right].
\end{aligned}$$

By lemma 2, we get

$$\begin{aligned}
\dot{V}_3 &\le e^{2\alpha t}\left[k_6\tau_2^2e^{2\alpha\tau_2}x^2(t)-k_6\left(\int_{t-\tau_2}^{t}x^2(s)ds\right)^2\right.\\
&\quad \left.+k_7\sigma_2^2e^{2\alpha\sigma_2}x^2(t)-k_7\left(\int_{t-\sigma_2}^{t}x^2(s)ds\right)^2\right]
\end{aligned}\tag{11}$$

$$\dot{V}_4 = k_8\tau_2\left[\int_{-\tau_2}^{0}e^{2\alpha(t-\theta)}\dot{x}^2(t)d\theta-\int_{-\tau_2}^{0}e^{2\alpha t}\dot{x}^2(t+\theta)d\theta\right].$$

By lemma (4), we get

$$\begin{aligned}
\dot{V}_4 &\le e^{2\alpha t}k_8\tau_2^2e^{2\alpha\tau_2}\dot{x}^2(t)\\
&\quad +\begin{bmatrix}x(t) & x(t-\tau(t)) & x(t-\tau_2)\end{bmatrix}\\
&\quad \begin{bmatrix}-k_8 & k_8-s & s\\ * & -2k_8+2s & k_8-s\\ * & * & -k_8\end{bmatrix}\begin{bmatrix}x(t)\\ x(t-\tau(t))\\ x(t-\tau_2)\end{bmatrix}
\end{aligned}\tag{12}$$

$$\begin{aligned}
\dot{V}_5 &= k_9e^{2\alpha t}\tanh^2 x(t)-k_9e^{2\alpha(t-\sigma(t)}\tanh^2 x(t-\sigma(t))\\
&\quad +k_9\dot{\sigma}(t)e^{2\alpha t}\tanh^2 x(t-\sigma(t))\\
&\quad +k_{10}\sigma_2\int_{-\sigma_2}^{0}e^{2\alpha(t-\theta)}\tanh^2 x(t)d\theta\\
&\quad -k_{10}\sigma_2\int_{-\sigma_2}^{0}e^{2\alpha t}\tanh^2 x(t+\theta)d\theta.
\end{aligned}$$

Since $\tanh^2 x(t) \leq x^2(t)$ and using lemma (2), we get

$$\begin{aligned}\dot{V}_5 \leq e^{2\alpha t}\Big[&k_9 x^2(t) - k_9 - k_9 e^{-2\alpha\sigma(t)}\tanh^2 x(t-\sigma(t)) \\ &+k_9\sigma_d \tanh^2 x(t-\sigma(t)) + k_{10}\sigma_2^2 e^{2\alpha\sigma_2)} x^2(t) \\ &-k_{10}\left(\int_{t-\sigma_2}^{t} \tanh^2 x(s)ds\right)^2\Big]. \end{aligned} \tag{13}$$

From the Leibniz-Newton formula, we have

$$\begin{aligned}&2\Big[m_1\dot{x}(t) + m_2 x(t) + m_3 x(t-\tau(t)) + m_4\int_{t-\tau(t)}^{t}\dot{x}(s)ds \\ &+m_5\tanh x(t-\sigma(t)) + m_6\int_{t-\sigma(t)}^{t}\tanh x(s)ds\Big] \\ &\Big[x(t) - x(t-\tau(t)) - \int_{t-\tau(t)}^{t}\dot{x}(s)ds\Big]. \end{aligned} \tag{14}$$

Combining equation (9)-(14), we have

$$\dot{V}^* = \sum_{i=1}^{5} V_i(t, x_t) \leq e^{2\alpha t}\omega^T(t)\Xi\omega(t),$$

where $\omega(t) = [\dot{x}(t), x(t), x(t-\tau(t)), x(t-\tau_2),$ $\int_{t-\tau(t)}^{t}\dot{x}(s)ds, \tanh x(t-\sigma(t)), \dot{x}(t-\tau(t)), x(t-\sigma_2),$ $\dot{x}(t-\sigma_2), \int_{t-\tau_2}^{t} x(s)ds, \int_{t-\sigma_2}^{t} x(s)ds, \int_{t-\sigma_2}^{t}\tanh x(s)ds]^T$ and Ξ is define in equation (7). Since $\Xi < 0$, we have $\dot{V}^* \leq e^{2\alpha t}\omega^T(t)\Xi\omega(t) < 0$. Therefore, there is a positive number λ such that

$$\begin{aligned}\dot{V}^* \leq& -\lambda e^{2\alpha t}\big(||\dot{x}(t)||^2 + ||x(t)||^2 \\ &+||x(t-\tau(t))||^2 + ||x(t-\tau_2)||^2 \\ &+||\int_{t-\tau(t)}^{t}\dot{x}(s)ds||^2 + ||\tanh x(t-\sigma(t))||^2 \\ &+||\dot{x}(t-\tau(t))||^2 + ||x(t-\sigma_2)||^2 \\ &+||\dot{x}(t-\sigma_2)||^2 + ||\int_{t-\tau_2}^{t} x(s)ds||^2 \\ &+||\int_{t-\sigma_2}^{t} x(s)ds||^2 + ||\int_{t-\sigma_2}^{t}\tanh x(s)ds||^2\big) \\ \leq& -\lambda e^{2\alpha t}||x(t)||^2. \end{aligned}$$

Taking the derivative of V_6 along trajectory of equation (1) and utilizing the Cauchy inequality (lemma 3), we have

$$\begin{aligned}\dot{V}_6 =& 2\gamma e^{2\alpha t}[x(t)\dot{x}(t) + \alpha x^2(t)] \\ =& 2\gamma e^{2\alpha t}\big[x(t)(-ax(t) + b\tanh x(t-\sigma(t)) \\ &-p\dot{x}(t-\tau(t))) + \alpha x^2(t)\big] \\ =& 2\gamma e^{2\alpha t}\big[-ax^2(t) + bx(t)\tanh x(t-\sigma(t)) \\ &-px(t)\dot{x}(t-\tau(t)) + \alpha x^2(t)\big] \\ \leq& \gamma e^{2\alpha t}\big[(-a + 2\alpha + 2)x^2(t) + b^2\tanh^2 x(t-\sigma(t)) \\ &+p^2\dot{x}^2(t-\tau(t))\big] \end{aligned}$$

We chose

$$\gamma = \begin{cases}\frac{\lambda}{2}\min\{\frac{1}{b^2}, \frac{1}{p^2}\}, & \text{if } -2a + 2\alpha + 2 \leq 0; \\ \frac{\lambda}{2}\min\{\frac{1}{-2a+2\alpha+2}, \frac{1}{b^2}, \frac{1}{p^2}\}, & \text{if otherwise.}\end{cases}$$

We obtain $\dot{V}(t) = \sum_{i=1}^{6} V_i(t, x_t) \leq -\frac{\lambda}{2}e^{2\alpha t}||x(t)||^2 < 0$. From the condition that $\dot{V}(t)$ is negative definite and $0 \leq \tau(t) \leq \tau_2, 0 \leq \sigma(t) \leq \sigma_2$, we have $V(x(t)) \leq V(x(0))$, for all $t \geq 0$, with

$$\begin{aligned}V(x(0)) =& \sum_{i=1}^{6} V_i(x(0)) \\ =& k_1 x^2(0) + k_2\int_{-\tau_2}^{0} e^{2\alpha s}x^2(s)ds \\ &+ k_3\int_{-\sigma_2}^{0} e^{2\alpha s}x^2(s)ds + k_4\int_{-\tau(0)}^{0} e^{2\alpha s}\dot{x}^2(s)ds \\ &+ k_5\int_{-\sigma(0)}^{0} e^{2\alpha s}\dot{x}^2(s)ds \\ &+ k_6\tau_2\int_{-\tau_2}^{0}\int_{\theta}^{0} e^{2\alpha(s-\theta)}x^2(s)dsd\theta \\ &+ k_7\sigma_2\int_{-\sigma_2}^{0}\int_{\theta}^{0} e^{2\alpha(s-\theta)}x^2(s)dsd\theta \\ &+ k_8\tau_2\int_{-\tau_2}^{0}\int_{\theta}^{0} e^{2\alpha(s-\theta)}\dot{x}^2(s)dsd\theta \\ &+ k_9\int_{-\sigma 0}^{0} e^{2\alpha s}\tanh^2 x(s)ds \\ &+ k_{10}\sigma_2\int_{-\sigma_2}^{0}\int_{\theta}^{t} e^{2\alpha(s-\theta)}\tanh^2 x(s)dsd\theta \\ &+ \gamma x^2(0) \\ \leq& [k_1 + k_2\tau_2 + k_3\sigma_2 + k_4\tau_2 + k_5\sigma_2 + k_6\frac{\tau_2^3}{2} \\ &+ k_7\frac{\sigma_2^3}{2} + k_8\frac{\tau_2^3}{2} + k_9\sigma_2 + k_{10}\frac{\sigma_2^3}{2} \\ &+ \gamma]\max\{||x_0||_s^2, ||\dot{x}_0||_s^2\} \\ =& \Delta\max\{||x_0||_s^2, ||\dot{x}_0||_s^2\}, \end{aligned}$$

where $\Delta = k_1 + k_2\tau_2 + k_3\sigma_2 + k_4\tau_2 + k_5\sigma_2 + k_6\frac{\tau_2^3}{2} + k_7\frac{\sigma_2^3}{2} + k_8\frac{\tau_2^3}{2} + k_9\sigma_2 + k_{10}\frac{\sigma_2^3}{2} + \gamma$.
From $\gamma e^{2\alpha t}x(t) \leq V(x(t)) \leq \Delta||x_0||_s^2$, we obtain

$$||x(t)|| \leq Me^{-\alpha t}; M = \sqrt{\frac{\Delta}{\gamma}}||x_0||_s.$$

This implies that the zero solution of equation (1) is exponentially stable.

■

IV. Numerical example

Example 6. Consider the following equation studied in [3], [14] :

$$\begin{aligned}\frac{d}{dt}[x(t) + 0.2x(t-\tau(t))] =& -0.6x(t) \\ &+ 0.5\tanh x(t-\sigma(t)), \\ &t \geq 0,\end{aligned}$$

when $\tau(t) = \frac{\sin^2(t)}{10}$ and $\sigma_2 = 0.2$.
Solving our criterion (7), guaranteeing uniformly exponential stability, when $\alpha = 0.0038$ is given, allows the upper bound of the delay $\sigma(t) = 9.00$. For exponential stability of this

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TABLE I
THE UPPER BOUND OF TIME DELAY $\sigma(t)$ FOR EXAMPLE 6 .

Methods	$\alpha = 0.0038$	$\alpha = 0.028$
[3] (2011)	infeasible	infeasible
[14] (2014)	7.5231	0.0321
Our result (7)	9.00	0.1850

TABLE II
THE UPPER BOUND OF TIME DELAY σ FOR EXAMPLE 7 .

Methods	$\alpha = 0.0038$
[2] (2012)	175.2890
[3] (2011)	10^{21} (No α)
[20] (2010)	1.9470
[14] (2014)	175.3540
Our result (7)	175.3543

example is listed in the comparison in table 1. We can see that our results are much lass than conservative than [3], [14].

Example 7. Consider the following equation studied in [2],[3],[20],[14] :

$$\frac{d}{dt}[x(t) + 0.2x(t - 0.1] = -0.6x(t) + 0.3 \tanh x(t - \sigma), \quad t \geq 0,$$

Solving our criterion (7), guaranteeing uniformly exponential stability, when $\alpha = 0.0038$ is given, allows the upper bound of the delay $\sigma(t) = 175.3543$. For exponential stability of this example is listed in the comparison in table 2. We can see that our results are much lass than conservative than other existing work.

V. CONCLUSION

In this paper has improved delay-dependent exponential stability for nonlinear neutral differential equations with discrete time-varying delays. From information given above, it seems that our delay-dependent sufficient conditions obtained are much less conservative than some existing results. Two numerical examples are given to demonstrate the power of our result.

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REFERENCES

[1] R.P. Agarwal and S.R. Grace, "Asymptotic Stability of Certain Neutral Differential Equations," *Mathematical and Computer Modelling,* Vol. 31, pp. 9-15, 2000.
[2] H. Chen, "Some Improved Criteria on Exponential Stability of Neutral Differential Equation," *Advances in Difference Equations,* Vol. 170, 9 pages, 2012.
[3] H. Chen and X. Meng, "An Improved Exponential Stability Criterion for a class of neutral delayed differential equations," *Applied Mathematics Letters,* Vol. 24, pp. 1763-1767, 2011.
[4] S. Deng, X. Liao and S. Guo, "Asymptotic Stability Analysis of Certain Neutral Differential Equations: a Descriptor System Approach," *Mathematics and Computers in Simulation,* Vol. 71, pp. 4297-4308, 2009.
[5] H. A. El-Morshedy and K. Gopalsamy, "Nonoscillation, Oscillation and Convergence of a Class of Neutral Equations," *Nonlinear Analysis,* Vol.40, pp.173-183, 2000.
[6] E. Fridman, "Stability of Linear Descriptor Systems with Delays: a Lyapunov-Based Approach," *Mathematical Analysis and Applications,* Vol.273 pp.24-44, 2002.
[7] E. Fridman, "Introduction to time- Delay Systems Analysis and Control," Birkhäuser, Israel, 2010.
[8] K. Gu, V.L. Kharitonov and J. Chen, "Stability of Time-Delay Systems," Birkhäuser, Berlin, 2003.
[9] J.K. Hale and S.M. Verduyn Lunel, "Introduction to Functional Differential Equations," Springer, New York, 1993.
[10] O. M. Kwon and J. H. Park, "On Improved Delay-Dependent Stability Criterion of Certain Neutral Differential Equations," *Applied Mathematics and Computation,* Vol.199, pp.385-391, 2008.
[11] P.G. Park, J.W. Ko, C.K. Jeong, "Reciprocally Convex Approach to Stability of Systems with Time-Varying Delays," *Automatica,* Vol. 47, pp. 235-238, 2011.
[12] C. Peng, M.R. Fei, "An Improved Result on the Stability of Uncertain T-S Fuzzy Systems with Interval Time-Varying Delay," *Fuzzy Sets and Systems,* Vol. 212, pp. 97-109, 2013.
[13] O.M. Kwon, J.H. Park and S.M. Lee, "Augmented Lyapunov Functional Approach to Stability of Uncertain Neutral Systems with Time-Varying Delays," *Applied Mathematics and Computation,* Vol. 207, pp. 202-212, 2009.
[14] P. Keadnarmol and T. Rojsiraphisal, "Globally Exponential Stability of a Certain Neutral Differential Equation with Time-Varying Delays," *Advances in Difference Equations,* Vol. 32, pp. 10 pages, 2014.
[15] X. Li, "Global Exponential Stability for a Class of Neural Networks," *Applied Mathematics Letters,* Vol. 22, pp. 1235-1239, 2009.
[16] X. Liao, Y. Liu, S. Guo and H. Mai, "Asymptotic Stability of Delayed Neural Networks: a Descriptor System Approach," *Communications in Nonlinear Science and Numerical Simulation* Vol. 14, pp. 3120-3133, 2009.
[17] P.T. Nam and V.N. Phat, "An Improved Stability Criterion for a Class of Neutral Differential Equations," *Applied Mathematics Letters,* Vol. 22, pp. 31-35, 2009.
[18] J.H. Park, "Delay-Dependent Criterion for Asymptotic Stability of a class of Neutral Equations," *Applied Mathematics Letters,* Vol. 17, pp. 1203-1206, 2004.
[19] J.H. Park and O.M. Kwon, "Stability Analysis of Certain Nonlinear Differential Equation," *Chaos Solitons Fractals,* Vol. 27, pp. 450-453, 2008.
[20] T. Rojsiraphisal and P. Niamsup, "Exponential Stability of Certain Neutral Differential Equations," *Applied Mathematics Letters,* Vol. 17, pp. 3875-3880, 2010.
[21] Y. G. Sun and L. Wang, "Note on Asymptotic Stability of a Class of Neutral Differential Equations," *Applied Mathematics Letters,* Vol. 19, pp. 949-953, 2006.