New Results on Passivity Criteria for a Class of Neural Networks with Interval and Distributed Time-Varying Delays

Sorphorn Noun, Thongchai Botmart

***Abstract*—New results on passivity criteria for a class of neural networks with interval and distributed time-varying delays, as well as generalized activation function, is investigated. Based on refined Jensen's inequalities, improved Lyapunov-Krasovskii functional (LKF) with double integrals, triple integrals, and quadruple integrals, all of them are picked up in terms of linear matrix inequalities (LMIs) which can be checked numerically using the effective LMIs toolbox in MATLAB. Additionally, the effectiveness of approach considered in this paper is illustrated by numerical examples which is less conservative than the previous results.**

***Index Terms*—passivity analysis, neural networks, linear matrix inequality, discrete and distributed time-varying delays**

I. Introduction

In the recent past, neural networks have attracted considerable attention in various fields of science and engineering applications such as signal processing, pattern recognition, static image processing, associative memory, and combinatorial optimization [7], [27]. Anyway, time-delays are common in many physical and biological phenomena, as is demonstrated by using of mathematical modeling incorporating time-delay in a wide range of applications such as mechanical transmission, fluid transmission, metallurgical processes and the networked control systems which are often a source of instability, periodic oscillatory, chaos and poor control performance. These applications are largely dependent upon the stability of the equilibrium of neural networks, ie. Stability is much importance in dynamical properties about neural networks when neural networks are designed. As research results, the stability problem and the performance of the neural networks with time-delay have been improved [4], [16], [18], [25]. However, the most results were discussed the only discrete delay in the neural networks. In contrast, the distributed delay should be associated into model of a system that there exist a distribution of propagation delays over a period of time in some cases as discussed in [11], [22]. When modeling a realistic neural network, both of discrete and distributed delays should be included. Moreover, they recently add more leakage delay with those of delays to analyze on neural networks as in [2], [12], [13], [15], [17], [19].

On the other hand, the passivity is more interested attention and is closely related to the circuit analysis method. The main scope of passivity theory is that the passive properties of the system can keep the system internally stable [10]. Specifically, the passive system utilizes the product of input and output as the energy provision and embodies the energy attenuation character. A passive system only burns energy without energy production, and thus passivity represents the property of energy consumption [23]. The problem of passivity performance analysis has been extensively applied in many areas such as signal processing, fuzzy control, sliding mode control [29] and networked control. Because of these features, the passivity problems have been an active area of research in the past decades with neural networks.

The most authors also studied about the passivity analysis of neural networks with time-varying delay, see in [21], [24], [30]. In the same way, in [1], [20], [28] also studied the passivity analysis of neural networks with discrete and distributed delays. However, the delay neural networks could be classified into two categories: delay-independent and delay-dependent. Also, the delay-dependent passivity of neural networks with various time-delays has also focused in [3], [5], [14], [26]. According to these literatures, we can conclude that the most of those studies with the passivity of neural networks with discrete and distributed delays were focused on differentiable delay ($\dot{\tau}(t) \leq \mu$). To the best our knowledge, in this paper concern with the new results on passivity criteria for a class of neural networks with interval and distributed time-varying delays as non-differentiable delay. By conducting suitable Lyapunov-Krasovskii functional with double integrals, triple integrals, quadruple integrals and using refined Jensen's inequality, and linear matrix framework, which guarantees stability for the passivity of addressed neural networks. Additionally, the effectiveness of approach proposed in this paper is illustrated by numerical examples which is less conservative than the previous results in the literature.

Notation:
The following notations will be accounted in this paper: let $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denotes n-dimensional Euclidean space with vector norm $||.||$ and set of $n\times m$ matrices, respectively. Matrices $A, B \in \mathbb{R}^{n\times m}$, $col\{A, B\}$ and diag$\{A, B\}$ denote

Manuscript received December 7, 2017; revised January 29, 2018. The first author was supported by royal scholarship under her royal highness Princess Mahachakri Sirindhorn education project, Thailand to Cambodia education and Research Publication Scholarship Fiscal Year of 2018, Graduate School of Khon Kaen University and Department of Mathematics, Faculty of Science, Khon Kaen University. The second author was supported by the National Research Council of Thailand and Faculty of Science, Khon Kaen University 2018 and the Thailand Research Fund (TRF), the Office of the Higher Education Commission (OHEC) (grant number : MRG6080239).

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the block matrices $\begin{bmatrix} A \\ B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, $\mathrm{Sym}(A)$ stand for $A+A^T$. A matrix P is symmetric positive definite, write $P>0$, if $P^T=P$ and $x^TPx>0$ for all $x\in\mathbb{R}^n$,$x\neq 0$. Let $\mathbb{S}_n^+$ denote the set of symmetric positive definite matrices in $\mathbb{R}^{n\times n}$. We also denoted by $\mathbb{D}_n^+$ the set of positive diagonal matrices. A matrix $D=\mathrm{diag}\{d_1,d_2,....,d_n\}\in\mathbb{D}_n^+$ if $d_i>0(i=1,2,...,n)$. I represent the identity matrix.

II. Preliminaries

Consider the following the class of neural networks with interval and distributed time-varying delays in the form:

$$\begin{aligned}\dot{x}(t) &= -Ax(t)+Wg(x(t))+W_1g(x(t-\tau(t)))\\ &\quad +W_2\int_{t-k(t)}^{t} g(x(s))\,ds+u(t), t\geq 0,\\ y(t) &= C_1g(x(t))+C_2g(x(t-\tau(t)))\\ &\quad +C_3\int_{t-k(t)}^{t} g(x(s))\,ds+C_4u(t), t\geq 0,\\ x(t) &= \phi(t), t\in[-d,0],\quad d=\max\{\tau_2,k\},\end{aligned} \tag{1}$$

where n denotes the number of neurons in the network, $x(t)=[x_1(t),\ x_2(t),...,\ x_n(t)]^T\in\mathbb{R}^n$ is the neurons state vector, $y(t)\in\mathbb{R}^n$ is the output vector and $u(t)$ is the external input of the network, $g(x(t))=[g_1(x_1(t)),\ g_2(x_2(t)),....,\ g_n(x_n(t))]^T\in\mathbb{R}^n$ denote the activation function, $g(x(t-\tau(t)))=[g_1(x_1(t-\tau(t))),\ g_2(x_2(t-\tau(t))),....,\ g_n(x_n(t-\tau(t)))]^T\in\mathbb{R}^n$. $A=\mathrm{diag}\{a_1,a_2,...,a_n\}$ is a positive diagonal matirx and W,W_1,W_2 are interconnection weight matrices. C_1,C_2,C_3,C_4 are given real matrices, and $\phi(t)\in\mathbb{R}^n$ is initial condition.

In this paper, we assume the delay $\tau(t)$ and $k(t)$ represent unknown discrete interval and distributed delay of the systems with

$$0\leq\tau_1\leq\tau(t)\leq\tau_2 \quad\text{and}\quad 0\leq k(t)\leq k,\ \ \forall t\geq 0, \tag{2}$$

where $\tau_1,\ \tau_2,$ and k are constants.

The neural activation functions $g_i(x_i(t))$ are continuous $g_i(0)=0$ and there exist constants $l_i^-,l_i^+\quad(i=1,2,...,n)$ such that

$$l_i^-\leq\frac{g_i(x)-g_i(y)}{x-y}\leq l_i^+,\quad \forall x,y\in\mathbb{R},x\neq y. \tag{3}$$

Definition 1. [6] The network (1) is said to be passive if there exists a scalar $\gamma>0$ such that, under zero initial condition, the following inequality holds for all $t_f\geq 0$

$$2\int_0^{t_f} y^T(s)u(s)\,ds\geq-\gamma\int_0^{t_f} u^T(s)u(s)\,ds. \tag{4}$$

Lemma 2. [9] (refined Jensen-based inequalities, Hien and Trinh) For a given matrix $R\in\mathbb{S}_n^+$ and a function $\varphi:[a,b]\to\mathbb{R}^n$ whose derivative $\dot\varphi\in PC([a,b],\mathbb{R}^n)$, the following inequalities hold

$$\int_a^b\dot\varphi^T(s)R\dot\varphi(s)\,ds\geq\frac{1}{b-a}\hat\chi^T\bar R\hat\chi, \tag{5}$$

$$\int_a^b\int_s^b\dot\varphi^T(u)R\dot\varphi(u)\,du\,ds\geq 2\hat\Omega^T\hat R\hat\Omega, \tag{6}$$

where $\bar R=\mathrm{diag}\{R,3R,5R\}$, $\hat R=\mathrm{diag}\{R,2R\}$, $\hat\chi=[\chi_1^T,\chi_2^T,\chi_3^T]^T$, $\hat\Omega=[\Omega_1^T,\Omega_2^T]^T$ and

$$\begin{aligned}\chi_1 &= \varphi(b)-\varphi(a),\\ \chi_2 &= \varphi(b)+\varphi(a)-\frac{2}{b-a}\int_a^b\varphi(s)\,ds,\\ \chi_3 &= \varphi(b)-\varphi(a)+\frac{6}{b-a}\int_a^b\varphi(s)\,ds\\ &\quad -\frac{12}{(b-a)^2}\int_a^b\int_s^b\varphi(u)\,du\,ds,\\ \Omega_1 &= \varphi(b)-\frac{1}{b-a}\int_a^b\varphi(s)\,ds,\\ \Omega_2 &= \varphi(b)+\frac{2}{b-a}\int_a^b\varphi(s)\,ds\\ &\quad -\frac{6}{(b-a)^2}\int_a^b\int_s^b\varphi(u)\,du\,ds.\end{aligned}$$

Lemma 3. [8] (Jensen inequality) For a positive definite matrix $R>0$, and an integral function $\{w(u)|u\in[a,b]\}$, the following inequalities hold:

$$\begin{aligned}&\int_a^b w^T(\alpha)Rw(\alpha)\,d\alpha\\ &\geq\frac{1}{b-a}\left(\int_a^b w(\alpha)\,d\alpha\right)^T R\left(\int_a^b w(\alpha)\,d\alpha\right),\\ &\int_a^b\int_\beta^b\int_s^b w^T(\alpha)Rw(\alpha)\,d\alpha\,d\beta\,ds\\ &\geq\frac{6}{(b-a)^3}\left(\int_a^b\int_\beta^b\int_s^b w(\alpha)\,d\alpha\,d\beta\,ds\right)^T R\\ &\quad\left(\int_a^b\int_\beta^b\int_s^b w(\alpha)\,d\alpha\,d\beta\,ds\right).\end{aligned} \tag{7}$$

III. Main results

In this section, new results on passivity criteria for a class of neural networks with interval and distributed time-varying delays will be established. Now, we have the following theorem of passivity of system (1).

theorem 1. The delay neural networks in system (1) is passive in the sense of defintion (1) for any delays $\tau(t)$ and $k(t)$ satisfying $0\leq\tau_1\leq\tau(t)\leq\tau_2$ and $0\leq k(t)\leq k$ if there exists matrices $P\in\mathbb{S}_{4n}^+;Q_1,S_3\in S_{2n}^+;Q_2,Q_3,S_1,S_2,S_4,R_1,R_2,U_1,U_2\in\mathbb{S}_n^+;\Lambda_k,D_j\in\mathbb{D}_n^+$, $(k=1,2,......,8;j=1,2)$, and a scalar $\gamma>0$ such that the following LMIs holds

$$\psi(\tau)=\mathrm{Sym}(\Lambda_0)-\Xi_1(\tau)-\sum_{n=2}^{6}\Xi_n-\Gamma(\tau)<0. \tag{8}$$

with

$\Sigma_1=\mathrm{diag}\{l_1^-,l_2^-,...,l_n^-\}$, $\Sigma_2=\mathrm{diag}\{l_1^+,l_2^+,...,l_n^+\}$,
$\bar S_i=\mathrm{diag}\{S_i,3S_i,5S_i\}$ $(i=1,2)$,
$G_1(\tau)=[e_1^T\ \ \tau_1e_9^T\ \ (\tau-\tau_1)e_{10}^T+(\tau_2-\tau)e_{11}^T\ \ \frac{\tau_1^2}{2}e_{12}^T]^T$,
$G_2=[\mathcal{A}^T\ \ (e_1-e_2)^T\ \ (e_2-e_4)^T\ \ \tau_1(e_1-e_9)^T]^T$,
$G_3=[e_2^T\ \ e_8^T]^T$,
$G_4=[e_4^T\ \ e_7^T]^T$,

ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$G_5 = [e_1^T \ \ e_5^T]^T$,
$G_6 = [e_1^T \ \ e_2^T \ \ e_9^T \ \ e_{12}^T]^T$,
$G_7 = [e_3^T \ \ e_4^T \ \ e_{11}^T \ \ e_{14}^T]^T$,
$G_8 = [e_2^T \ \ e_3^T \ \ e_{10}^T \ \ e_{13}^T]^T$,
$G_9(\tau) = [((\tau-\tau_1)e_{10} + (\tau_2-\tau)e_{11})^T \ \ (e_{16}+e_{17})^T]^T$,
$G_{10} = \frac{\tau_1^2}{2}e_1 - \frac{\tau_2^2}{2}e_{12}$,
$G_{11}(\tau) = \frac{\tau_{12}^2}{2}e_2 - \frac{(\tau-\tau_1)^2}{2}e_{13} - \frac{(\tau_2-\tau)^2}{2}e_{14}$,
$G_{12} = e_5 - \Sigma_1 e_1, \quad G_{13} = \Sigma_2 e_1 - e_5$,
$G_{14} = e_6 - \Sigma_1 e_3, \quad G_{15} = \Sigma_2 e_3 - e_6$,
$G_{16} = e_5 - e_6 - \Sigma_1(e_1-e_3), \quad G_{17} = \Sigma_2(e_1-e_3) - e_5 + e_6$,
$G_{18} = e_5 - e_8 - \Sigma_1(e_1-e_2), \quad G_{19} = \Sigma_2(e_1-e_2) - e_5 + e_8$,
$G_{20} = e_5 - e_7 - \Sigma_1(e_1-e_4), \quad G_{21} = \Sigma_2(e_1-e_4) - e_5 + e_7$,
$G_{22} = e_7 - e_8 - \Sigma_1(e_4-e_2), \quad G_{23} = \Sigma_2(e_4-e_2) - e_7 + e_8$,
$G_{24} = e_6 - e_8 - \Sigma_1(e_3-e_2), \quad G_{25} = \Sigma_2(e_3-e_2) - e_6 + e_8$,
$G_{26} = e_7 - e_6 - \Sigma_1(e_4-e_3), \quad G_{27} = \Sigma_2(e_4-e_3) - e_7 + e_6$,
$\Pi_0(\tau) = G_1^T(\tau)PG_2 + e_1^T(\Sigma_2 D_2 - \Sigma_1 D_1)\mathcal{A} - e_5^T(D_2-D_1)\mathcal{A}$,
$\Pi_1 = e_1^T(Q_2+Q_3)e_1 - e_2^T Q_2 e_2 - e_4^T Q_3 e_4 + G_3^T Q_1 G_3 - G_4^T Q_1 G_4 + \tau_{12}^2 G_5^T S_3 G_5 + K^2 e_5^T S_4 e_5$,
$\Pi_2 = \mathcal{A}^T[\tau_1^2 S_1 + \tau_{12}^2 S_2 + \frac{\tau_1^2}{2}R_1 + \frac{\tau_{12}^2}{2}R_2 + \frac{\tau_1^6}{36}U_1 + \frac{(\tau_2^3-\tau_1^3)(\tau_2-\tau_1)^3}{36}U_2]\mathcal{A}$,
$\Pi_3 = e_{15}^T(\gamma I_n + 2C_4)e_{15} + \mathrm{Sym}(e_{15}^T(C_1 e_5 + C_2 e_6 + C_3 e_{18}))$,
$\Xi_1(\tau) = \mathrm{Sym}(\Pi_0(\tau) + \Pi_1 + \Pi_2 - \Pi_3$,
$\Xi_2 = G_6^T F^T \bar{S}_1 F G_6$,
$\Xi_3 = G_7^T F^T \bar{S}_2 F G_7$,
$\Xi_4 = G_8^T F^T \bar{S}_2 F G_8$,
$\Xi_5 = 2(e_1-e_9)^T R_1(e_1-e_9) + 4(e_1+2e_9-3e_{12})^T R_1(e_1+2e_9-3e_{12})$,
$\Xi_6 = 2(e_2-e_{10})^T R_2(e_2-e_{10}) + 4(e_2+2e_{10}-3e_{13})^T R_2(e_2+2e_{10}-3e_{13}) + 2(e_3-e_{11})^T R_2(e_3-e_{11}) + 4(e_3+2e_{11}-3e_{14})^T R_2(e_3+2e_{11}-3e_{14})$,
$\Gamma(\tau) = G_9^T(\tau)S_3 G_9(\tau) + G_{10}^T U_1 G_{10} + G_{11}^T(\tau)U_2 G_{11}(\tau)$,
$\Lambda_0 = G_{12}^T\Lambda_1 G_{13} + G_{14}^T\Lambda_2 G_{15} + G_{16}^T\Lambda_3 G_{17} + G_{18}^T\Lambda_4 G_{19} + G_{20}^T\Lambda_5 G_{21} + G_{22}^T\Lambda_6 G_{23} + G_{24}^T\Lambda_7 G_{25} + G_{26}^T\Lambda_8 G_{27}$.

$\tau_{12} = \tau_2 - \tau_1$ and $F = \begin{bmatrix} I_n & -I_n & 0 & 0 \\ I_n & I_n & -2I_n & 0 \\ I_n & -I_n & 6I_n & -6I_n \end{bmatrix}$.

Proof: Let $\Lambda_k = \mathrm{diag}\{\lambda_{k1}, \lambda_{k2}, ..., \lambda_{kn}\}$ $(k = 1, 2, ..., 8)$, $D_j = \mathrm{diag}\{d_{j1}, d_{j2}, ..., d_{jn}\}$ $(j = 1, 2)$ and denote $e_i = [0_{n\times(i-1)n} \ \ I_n \ \ 0_{n\times(18-i)n}]$ $(i = 1, 2, ..., 18)$, $\mathcal{A} = Ae_1 + We_5 + W_1 e_6 + W_2 e_{18} + e_{15}$.

Consider the Lyapunov-Krasovskii functional candidates:

$$V(t, x_t) = \sum_{i=1}^{5} V_i(t, x_t),$$

where

$$\begin{aligned} V_1(t,x_t) &= \eta_1^T(t)P\eta_1(t) \\ &\quad +2\sum_{i=1}^{n} d_{1i}\int_0^{x_i(t)} (g_i(s) - l_i^- s)\,ds \\ &\quad +2\sum_{i=1}^{n} d_{2i}\int_0^{x_i(t)} (l_i^+ s - g_i(s))\,ds, \\ V_2(t,x_t) &= \int_{t-\tau_2}^{t-\tau_1} \eta_2^T(s)Q_1\eta_2(s)\,ds \\ &\quad +\int_{t-\tau_1}^{t} x^T(s)Q_2 x(s)\,ds \\ &\quad +\int_{t-\tau_2}^{t} x^T(s)Q_3 x(s)\,ds, \end{aligned}$$

$$\begin{aligned} V_3(t,x_t) &= \tau_1\int_{-\tau_1}^{0}\int_{t+s}^{t} \dot{x}^T(u)S_1\dot{x}(u)\,du\,ds \\ &\quad +\tau_{12}\int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t} \dot{x}^T(u)S_2\dot{x}(u)\,du\,ds \\ &\quad +\tau_{12}\int_{-\tau_2}^{-\tau_1}\int_{t+s}^{t} \eta_2^T(u)S_3\eta_2(u)\,du\,ds \\ &\quad +k\int_{-k}^{0}\int_{t+s}^{t} g^T(x(u))S_4 g(x(u))\,du\,ds, \\ V_4(t,x_t) &= \int_{t-\tau_1}^{t}\int_{s}^{t}\int_{u}^{t} \dot{x}^T(\theta)R_1\dot{x}(\theta)\,d\theta\,du\,ds \\ &\quad +\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{t+u}^{t} \dot{x}^T(\theta)R_2\dot{x}(\theta)\,d\theta\,du\,ds, \\ V_5(t,x_t) &= \frac{\tau_1^3}{6}\int_{t-\tau_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\int_{\varphi}^{t} \dot{x}^T(\theta)U_1\dot{x}(\theta)\,d\theta\,d\varphi\,d\lambda\,ds \end{aligned}$$

$$+\frac{(\tau_2^3-\tau_1^3)}{6}\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{\lambda}^{-\tau_1}\int_{t+\varphi}^{t} \dot{x}^T(\theta)U_2\dot{x}(\theta)\,d\theta\,d\varphi\,d\lambda\,ds.$$

Let $\eta_1(t) = [x^T(t) \ \ \int_{t-\tau_1}^{t} x^T(s)\,ds \ \ \int_{t-\tau_2}^{t-\tau_1} x^T(s)\,ds \ \ \int_{t-\tau_1}^{t}\int_s^t x^T(u)\,du\,ds]^T$ and $\eta_2(t) = [x^T(t) \ \ g^T(x(t))]^T$.
$\xi(t) = [x^T(t),\ x^T(t-\tau_1),\ x^T(t-\tau(t)),\ x^T(t-\tau_2),\ g^T(x(t)),\ g^T(x(t-\tau(t))),\ g^T(x(t-\tau_2)),\ g^T(x(t-\tau_1)),\ \frac{1}{\tau_1}\int_{t-\tau_1}^{t} x^T(s)\,ds,\ \frac{1}{\tau(t)-\tau_1}\int_{t-\tau(t)}^{t-\tau_1} x^T(s)\,ds,\ \frac{1}{\tau_2-\tau(t)}\int_{t-\tau_2}^{t-\tau(t)} x^T(s)\,ds,\ \frac{2}{\tau_1^2}\int_{t-\tau_1}^{t}\int_s^t x^T(u)\,du\,ds,\ \frac{2}{(\tau(t)-\tau_1)^2}\int_{t-\tau(t)}^{t-\tau_1}\int_s^{t-\tau_1} x^T(u)\,du\,ds,\ \frac{2}{(\tau_2-\tau(t))^2}\int_{t-\tau_2}^{t-\tau(t)}\int_s^{t-\tau(t)} x^T(u)\,du\,ds,\ u^T(t),\ \int_{t-\tau(t)}^{t-\tau_1} g^T(x(s))\,ds, \int_{t-\tau_2}^{t-\tau(t)} g^T(x(s))\,ds,\ \int_{t-k(t)}^{t} g^T(x(s))\,ds]^T$.

Calculating the time derivatives of $V(t, x_t)$ along the solution of (1), we get

$$\begin{aligned} \dot{V}(t,x_t) &= \xi^T(t)\Big(\mathrm{Sym}(\Pi_0(\tau) + \Pi_1 + \Pi_2\Big)\xi(t) \\ &\quad -\tau_1\int_{t-\tau_1}^{t} \dot{x}^T(s)S_1\dot{x}(s)\,ds \\ &\quad -\tau_{12}\int_{t-\tau_2}^{t-\tau_1} \dot{x}^T(s)S_2\dot{x}(s)\,ds \\ &\quad -\tau_{12}\int_{t-\tau_2}^{t-\tau_1} \eta_2^T(s)S_3\eta_2(s)\,ds \\ &\quad -k(t)\int_{t-k(t)}^{t} g^T(x(s))S_4 g(x(s))\,ds \\ &\quad -\int_{t-\tau_1}^{t}\int_{s}^{t} \dot{x}^T(u)R_1\dot{x}(u)\,du\,ds \\ &\quad -\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1} \dot{x}^T(u)R_2\dot{x}(u)\,du\,ds \\ &\quad -\frac{\tau_1^3}{6}\int_{t-\tau_1}^{t}\int_{s}^{t}\int_{\lambda}^{t} \dot{x}^T(\varphi)U_1\dot{x}(\varphi)\,d\varphi\,d\lambda\,ds \end{aligned}$$

$$-\frac{(\tau_2^3-\tau_1^3)}{6}\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{t+\lambda}^{t-\tau_1} \dot{x}^T(\varphi)U_2\dot{x}(\varphi)\,d\varphi\,d\lambda\,ds. \tag{9}$$

Using inequality (5) in lemma (2), we have

$$\begin{aligned} -\tau_1\int_{t-\tau_1}^{t} \dot{x}^T(s)S_1\dot{x}(s)\,ds &\le -\xi^T(t)G_6^T F^T\bar{S}_1 F G_6\xi(t) \\ &= -\xi^T(t)\Xi_2\xi(t). \end{aligned} \tag{10}$$

By splitting

$$-\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\dot{x}^T(s)S_2\dot{x}(s)\,ds = -\tau_{12}\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)S_2\dot{x}(s)\,ds - \tau_{12}\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)S_2\dot{x}(s)\,ds. \quad (11)$$

Applying inequality (5) in Lemma (2) yields

$$-\tau_{12}\int_{t-\tau_2}^{t-\tau(t)}\dot{x}^T(s)S_2\dot{x}(s)\,ds \le -\xi^T(t)G_7^TF^T\bar{S}_2FG_7\xi(t) = -\xi^T(t)\Xi_3\xi(t). \quad (12)$$

and

$$-\tau_{12}\int_{t-\tau(t)}^{t-\tau_1}\dot{x}^T(s)S_2\dot{x}(s)\,ds \le -\xi^T(t)G_8^TF^T\bar{S}_2FG_8\xi(t) = -\xi^T(t)\Xi_4\xi(t). \quad (13)$$

Now, by employing inequality (6) to estimate the second integral terms in (9), we hold

$$-\int_{t-\tau_1}^{t}\int_{s}^{t}\dot{x}^T(u)R_1\dot{x}(u)\,du\,ds \le -\xi^T(t)\Xi_5\xi(t). \quad (14)$$

and

$$-\int_{t-\tau_2}^{t-\tau_1}\int_{s}^{t-\tau_1}\dot{x}^T(u)R_2\dot{x}(u)\,du\,ds \le -\xi^T(t)\Xi_6\xi(t). \quad (15)$$

In the same way, the following inequality in lemma (3) then,

$$\begin{aligned}&-\tau_{12}\int_{t-\tau_2}^{t-\tau_1}\eta_2^T(s)S_3\eta_2(s)\,ds\\ &\le -\Big(\int_{t-\tau_2}^{t-\tau_1}\eta_2(s)\,ds\Big)^TS_3\Big(\int_{t-\tau_2}^{t-\tau_1}\eta_2(s)\,ds\Big)\\ &\le -\xi^T(t)G_9^T(\tau)S_3G_9(\tau)\xi(t). \end{aligned}\quad (16)$$

$$\begin{aligned}&-k(t)\int_{t-k(t)}^{t}g(x(s))^T(s)S_4g(x(s))\,ds\\ &\le -\Big(\int_{t-k(t)}^{t}g(x(s))\,ds\Big)^TS_4\Big(\int_{t-k(t)}^{t}g(x(s))\,ds\Big)\\ &\le -\xi^T(t)e_{18}^TS_4e_{18}\xi(t). \end{aligned}\quad (17)$$

$$\begin{aligned}&-\frac{\tau_1^3}{6}\int_{t-\tau_1}^{t}\int_{s}^{t}\int_{\lambda}^{t}\dot{x}^T(\varphi)U_1\dot{x}(\varphi)\,d\varphi\,d\lambda\,ds\\ &\le -\xi^T(t)\Big(\frac{\tau_1^2}{2}e_1-\frac{\tau_1^2}{2}e_{12}\Big)^TU_1\Big(\frac{\tau_1^2}{2}e_1-\frac{\tau_1^2}{2}e_{12}\Big)\xi(t)\\ &\le -\xi^T(t)G_{10}^TU_1G_{10}\xi(t). \end{aligned}\quad (18)$$

$$\begin{aligned}&-\frac{(\tau_2^3-\tau_1^3)}{6}\int_{-\tau_2}^{-\tau_1}\int_{s}^{-\tau_1}\int_{t+\lambda}^{t-\tau_1}\dot{x}^T(\varphi)U_2\dot{x}(\varphi)\,d\varphi\,d\lambda\,ds\\ &\le -\xi^T(t)G_{11}^T(\tau)U_2G_{11}(\tau)\xi(t). \end{aligned}\quad (19)$$

Since $\lambda_{1i} > 0$ $(i = 1, 2, .., n)$, it follows from condition (3) that

$$2(g_i(x_i(t)) - l_i^- x_i(t))\lambda_{1i}(l_i^+ x_i(t) - g_i(x_i(t))) \ge 0$$

and thus

$$\xi^T(t)\mathrm{Sym}(G_{12}^T\Lambda_1G_{13})\xi(t) \ge 0. \quad (20)$$

By the same arguments used in deriving (20) we obtain

$$\xi^T(t)\mathrm{Sym}(\Lambda_0)\xi(t) \ge 0. \quad (21)$$

Then, to show the passivity of system (1), we set $J(t_f) = \int_0^{t_f}[-\gamma u^T(t)u(t) - 2y^T(t)u(t)]\,dt$ for $t_f \ge 0$. under the zero initial condition, we can deduce that

$$\begin{aligned}J(t_f) &= \int_0^{t_f}[-\gamma u^T(t)u(t) - 2y^T(t)u(t) + \dot{V}(x_t)]\,dt - V(x_{t_f})\\ &\le \int_0^{t_f}[\dot{V}(x_t) - \gamma u^T(t)u(t) - 2y^T(t)u(t)\,dt].\end{aligned}$$

Combining estimates from (9) to (21) we then obtain

$$\dot{V}(x_t) - \gamma u^T(t)u(t) - 2y^T(t)u(t) \le \xi^T(t)\psi(\tau)\xi(t).$$

where $\psi(\tau)$ defined in theorem (1) since $\psi(\tau)$ is an affine function in τ, $\psi(\tau) < 0$ for all $\tau \in [\tau_1, \tau_2]$ if and only if $\psi(\tau_1) < 0$ and $\psi(\tau_2) < 0$. if (8) holds for $\tau = \tau_1$ and $\tau = \tau_2$ we have $\psi(\tau) < 0$, then

$$\dot{V}(x_t) - \gamma u^T(t)u(t) - 2y^T(t)u(t) \le 0.$$

We have $J(t_f) < 0$ for any $t_f \ge 0$ when conditions (3) is satisfied.
Thus, neural network (1) is passive. the proof is completed. ■

IV. Numerical Example

In this section, the following numerical examples demonstrate the effectiveness and applicability of our results.
Example 1. Consider a neural network (1) with the following parameters:

$$A = \begin{bmatrix} 2.2 & 0 \\ 0 & 1.8 \end{bmatrix}, W_0 = \begin{bmatrix} 1.2 & 1 \\ -0.2 & 0.3 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 0.8 & 0.4 \\ -0.2 & 0.1 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$\Sigma_1 = \mathrm{diag}\{0, 0\}$, $\Sigma_2 = \mathrm{diag}\{1, 1\}$, and $C_1 = I$, $C_2 = C_3 = C_4 = 0$.
The activation functions are assumed to be
$g_i(x_i) = 0.5(|x_i + 1| - |x_i - 1|)$ $(i = 1, 2)$.

Using Theorem 1, we get the upper bound of τ_2 is shown in table 4.1 which give comparision those results obtained in existing approaches in the literature.

ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

TABLE I
UPPER BOUNDS OF τ_2 FOR EXAMPLE 1

	μ=0.5	unknown μ
[23]	0.5227	-
[20]	1.3752	-
[25]	1.8090	-
[27]	3.0430	-
[1]	3.6566	-
Theorem 1	-	4.1010

V. CONCLUSION

In this paper, we focused on new results on passivity criteria for a class of neural networks with interval and distributed time-varying delays. Based on refined Jensen's inequalities, and by constructing the Lyapunov-Krasovskii functional with double, triple, quadruple integrals, all of them were picked up in terms of linear matrix inequalities which can be checked using LMI toolbox in MATLAB. Furthermore, This result is less conservative than existing results in literature and can also be the effective proposed method.

ACKNOWLEDGMENT

The authors thank anonymous reviewers for their valuable comments and suggestions.

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