Optimal Guaranteed Cost Control for Exponential Stability of Nonlinear System with Mixed Time-Varying Delays via Feedback Control

Nuchira Khongja, Thongchai Botmart

Abstract—**The problem of optimal guaranteed cost control for exponential stability of nonlinear system with mixed time-varying delays via feedback control is considered. The mixed time-varying delays consisting of both discrete and distributed delays are considered without assuming the differentiability of the time-varying delays. Based on an improved Lyapunov-Krasovskii functional with triple integral terms and employing Newton-Leibniz formula, Jensen's inequality and reciprocal convex combination technique, new delay-dependent sufficient conditions for the existence of guaranteed cost feedback control for the system are given in terms of linear matrix inequalities (LMIs), which can be checked numerically by using the effective LMI toolbox in MATLAB. Finally, a numerical example is given to illustrate the effectiveness and improve over some existing results in the literature.**

Index Terms—**exponential stability analysis, nonlinear system, guaranteed cost control, feedback control, discrete and distributed time-varying delays**

I. Introduction

In the scope of functional differential equations, stability and control problem has been the subject of investigable research attention. In most control engineering practice, it is always desirable to design a control system which is not only stabilizable but also guarantees an adequate level of performance. The guaranteed cost control was first put forward by Chang and Peng [5] and introduced by a lot of authors, which different design approaches have been proposed for systems with delay [11-14]. In [15], author designed state feedback guaranteed cost control of nonlinear systems with time-varying delay. By applying Lyapunov-Krasovskii functional method and linear matrix inequality technique, new delay-dependent sufficient conditions for designing the state feedback guaranteed cost control are derived. Optimal cost controller for linear system with mixed time-varying delays state and control has been considered in [18]. By improving Lyapunov-Krasovskii functionals with Newton-Leibniz formula, the sufficient conditions for the existence of an optimal cost state feedback control for the system have been derived in term of LMIs.

Manuscript received December 7, 2017; revised January 29, 2018. The first author was supported by Research Publication Scholarship Fiscal Year of 2018, Graduate School of Khon Kaen University and Department of Mathematics, Faculty of Science, Khon Kaen University. The second author was supported by the National Research Council of Thailand and Faculty of Science, Khon Kaen University 2018 and the Thailand Research Fund (TRF), the Office of the Higher Education Commission (OHEC) (grant number : MRG6080239).

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Time-delay systems have actually been studied by several researchers. This may be the new emerging applications in engineering (such as network controlled systems) compounded with new theoretical results, that allowed one to solve some open problems (decoupling problems, stabilization, robustness, H_∞ control, etc.) and less conservative results. Absolutely, applications motivate the need of theory, which in return makes the control applications feasible.

Stability criteria for time-delay systems can be divided into two types: delay-dependent and delay-independent. Delay-dependent stability criteria are concerned with the size of the delay and usually provide a maximal delay size. On the other hand, delay-independent stability criteria tend to be more conservative, especially for small size delay, such criteria do not give any information on the size of the delay. There are many different methods given to deal with the stability problem. Among the well-known Lyapunov stability method, the Lyapunov functional is a powerful tool for stability analysis of time delay systems. Delay-dependent stability criteria for these systems are established in terms of linear matrix inequalities (LMIs).

In this research, we have considered the optimal guaranteed cost control problem for a class of nonlinear system with mixed time-varying delays. The mixed time-varying delays consisting of both discrete and distributed delays are considered without assuming the differentiability of the time-varying delays. Based on an improved Lyapunov-Krasovskii functional with triple integral terms and employing Jensen's inequality, Newton-Leibniz formula and reciprocal convex combination technique. A performance measure for the system is considered by a quadratic cost function. The feedback stabilizing controllers are designed to satisfy with exponential stability. We give sufficient conditions for existence of the feedback guaranteed cost control in terms of LMIs, which can be determined by utilizing MATLABs LMI control toolbox. A numerical example is presented to show the effectiveness of the proposed method.

II. Preliminaries

Notations

The following notation will be used in this paper : $\mathbb{R}^+$ denotes the set of all real nonnegative numbers; $\mathbb{R}^n$ denotes the $n-$dimensional space and $\|\cdot\|$ denotes the Euclidean vector norm; $A^{n\times m}$ denotes the space of all matrices of $(n\times m)$ -dimensions; A^T denotes the transpose of matrix A; A is symmetric if $A = A^T$; I denotes the identity matrix;

ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$\lambda(A)$ denotes the set of all eigenvalues of A; $\lambda_{\min}(A) = \min\{\text{Re}\lambda; \lambda \in \lambda(A)\}$; $\lambda_{\max}(A) = \max\{\text{Re}\lambda; \lambda \in \lambda(A)\}$; $x_t = \{x(t+s) : s \in [-h, 0]\}$; $||x_t|| = \sup_{s\in[-h,0]} ||x(t+s)||$; $C([0,t], \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$-valued continuous functions on $[0,t]$; $L_2([0,t], \mathbb{R}^m)$ denotes the set of all $\mathbb{R}^m$-valued sguare integrable functions on $[0,t]$; Matrix A is called positive definite $(A > 0)$ if $x^T Ax > 0$ for all $x \in \mathbb{R}^n, x \neq 0$; Matrix A is called semi-positive definite $(A \geq 0)$ if $x^T Ax \geq 0$ for all $x \in \mathbb{R}^n$; $A > B$ means $A - B > 0$. The symmetric term in a matrix is denoted by $*$.

Consider a nonlinear system with mixed time-varying delay of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bx(t-h_1(t)) + C\int_{t-d_1(t)}^{t} x(s)ds \\ &\quad + f(t, x(t), x(t-h_1(t)), \int_{t-d_1(t)}^{t} x(s)ds, u(t)) \\ &\quad + U(t), \qquad (1)\\ U(t) &= D_1u(t) + D_2u(t-h_2(t)) + D_3\int_{t-d_2(t)}^{t} u(s)ds, \\ x(t) &= \phi(t), \quad t \in [-d_3, 0], \quad d_3 = \max\{h_{1M}, h_2, d_1, d_2\},\end{aligned}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ are the state and control, respectively, the control $u(\cdot) \in L_2([0,t], \mathbb{R}^m)$, $u(t) = Kx(t), K$ is constant matrix gain, $\phi(t) \in C([-d_3, 0], \mathbb{R}^n)$ is the initial function with the norm

$$||\phi|| = sup_{t\in[-d_3,0]}\sqrt{||\phi(t)||^2 + ||\dot{\phi}(t)||^2},$$

A, B, C, D_1, D_2, D_3 are given constant matrices with appropriate dimensions, the delay functions $h_i(t), d_i(t), i = 1, 2$ satisfy the condition

$$\begin{gathered}0 \leq h_{1m} \leq h_1(t) \leq h_{1M}, \quad 0 \leq h_2(t) \leq h_2, \\ 0 \leq d_1(t) \leq d_1, \quad 0 \leq d_2(t) \leq d_2, \\ \eta = h_{1M} - h_{1m}\end{gathered}$$

and $f(\cdot) : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a given continuous function satisfying $f(t, 0, 0, 0, 0) = 0, \forall t \in \mathbb{R}^+$, and $f(t, x, y, z, u)$ satisfy Lipschitz condition with respect to (x, y, z, u), such that $\exists a, b, c, d > 0$:

$$||f(t, x, y, z, u)|| \leq a||x|| + b||y|| + c||z|| + d||u||. \qquad (2)$$

Define the following quadratic cost function of the associated system (1) as follows :

$$J = \int_0^\infty L(t, x(t), \quad x(t-h_1(t)), \int_{t-d_1(t)}^{t} x(s)ds, u(t))dt, \qquad (3)$$

where

$$\begin{aligned}L(\cdot) &\leq x^T(t)Z_1x(t) + x^T(t-h_1(t))Z_2x(t-h_1(t)) \\ &\quad + \Big(\int_{t-d_1(t)}^{t} x^T(s)ds\Big)Z_3\Big(\int_{t-d_1(t)}^{t} x(s)ds\Big) \\ &\quad + u^T(t)Y_1u(t),\end{aligned}$$

$Z_1, Z_2, Z_3 \in \mathbb{R}^{n\times n}$ and $Y_1 \in \mathbb{R}^{m\times m}$ are positive definite matrices.

The objective of this paper is to design a feedback controller $u(t) = Kx(t)$ and a finite number $J^* > 0$, such that the resulting closed-loop system

$$\begin{aligned}\dot{x}(t) &= (A + D_1K)x(t) + Bx(t-h_1(t)) \\ &\quad + C\int_{t-d_1(t)}^{t} x(s)ds + f(t, x(t), x(t-h_1(t)), \\ &\quad \int_{t-d_1(t)}^{t} x(s)ds, Kx(t)) + D_2u(t-h_2(t)) \\ &\quad + D_3\int_{t-d_2(t)}^{t} u(s)ds \qquad (4)\end{aligned}$$

is exponentially stable and the value $J(u) \leq J^*$.

Definition 1. Given $\alpha > 0$. The zero solution of closed-loob system (4) is α-exponentially stabilizable if there exists a positive number $N > 0$, such that every solution $x(t, \phi)$ satisfies the following condition

$$||x(t, \phi)|| \leq Ne^{-\alpha t}||\phi||, \quad \forall t \in \mathbb{R}^+.$$

Definition 2. Consider the control system (1). If there exist a continuous stabilizing state feedback control law $u^*(t) = Kx(t)$ and a positive number J^* such that the zero solution of the closed-loop system (4) is exponentially stable and the value (3) satisfies $J(u^*) \leq J^*$ then the cost value J^* is a guaranteed cost value, $u^*(t)$ is a guaranteed cost controller of the system.

Proposition 3. [9]. (Cauchy inequality) For any symmetric positive definite matrix $N \in M^{n\times n}$ and $x, y \in \mathbb{R}^n$ we have

$$\pm 2x^T y \leq x^T Nx + y^T N^{-1}y.$$

Proposition 4. [9]. (Schur complement lemma) Given constant symmetric matrices X, Y and Z with appropriate dimensions satisfying $X = X^T, Y = Y^T > 0$, then $X + Z^T Y^{-1}Z < 0$ if and only if

$$\begin{bmatrix} X & Z^T \\ * & -Y \end{bmatrix} < 0 \quad \text{or} \quad \begin{bmatrix} -Y & Z \\ * & X \end{bmatrix} < 0.$$

Proposition 5. [17]. For any constant matrix $Z = Z^T > 0$ and positive numbers $h, \bar{h}$ such that the following integrals are well defined, then

$$\begin{aligned}(i) &- \int_{t-h}^{t} x(s)^T Zx(s)ds \\ &\leq -\frac{1}{h}\left(\int_{t-h}^{t} x(s)ds\right)^T Z\left(\int_{t-h}^{t} x(s)ds\right). \\ (ii) &- \int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)^T Zx(\tau)d\tau ds \leq -\frac{2}{\bar{h}^2 - h^2} \times \\ &\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)d\tau ds\right)^T Z\left(\int_{-\bar{h}}^{-h}\int_{t+s}^{t} x(\tau)d\tau ds\right).\end{aligned}$$

Proposition 6. [17]. Let $f_1, f_2, ..., f_N : \mathbb{R}^m \to \mathbb{R}$ have positive values in an open subset D of $\mathbb{R}^m$. Then, the reciprocally convex combination of f_i over D satisfies

$$\begin{aligned}&min_{\{r_i|r_i>0,\sum_i r_i=1\}}\sum_i \frac{1}{r_i}f_i(t) \\ &\qquad = \sum_i f_i(t) + max_{g_{i,j}(t)}\sum_{i\neq j} g_{i,j}(t)\end{aligned}$$

subject to

$$g_{i,j}:\mathbb{R}^m\to\mathbb{R}, g_{j,i}(t)=g_{i,j}(t),\quad \begin{bmatrix} f_i(t) & g_{i,j}(t)\\ g_{i,j}(t) & f_j(t)\end{bmatrix}\geq 0\ .$$

III. Main results

The following theorem gives sufficient conditions for designing a guaranteed cost controller for system (1).

Theorem 7. Given $\alpha>0$. Consider the system (1) and the cost function (3). If there exist symmetric positive definite matrices P, Q_1, Q_2, R_1, R_2, R_3, R_4, S_1, T_1, T_2, W_1, W_2 and W_3 satisfying the following LMIs

$$\Phi=\begin{bmatrix}\Phi_{11} & \Phi_{12} & \Phi_{13}\\ * & \Phi_{22} & \Phi_{23}\\ * & * & \Phi_{33}\end{bmatrix}<0, \tag{5}$$

then

$$u(t)=-\frac{1}{2}D_1^TP^{-1}x(t),\qquad t\in\mathbb{R}^+ \tag{6}$$

is a guaranteed cost controller and the guaranteed cost value is

$$J^*=\lambda_2||\phi||^2.$$

Moreover, the solution $x(t,\phi)$ satisfies

$$||x(t,\phi)||\leq\sqrt{\frac{\lambda_2}{\lambda_1}}e^{-\alpha t}||\phi||,\qquad \forall t\in\mathbb{R}^+,$$

where

$$\Phi_{11}=\begin{bmatrix} M_{1,1} & M_{1,2} & M_{1,3} & M_{1,4} & M_{1,5}\\ * & M_{2,2} & 0 & 0 & 0\\ * & * & M_{3,3} & 0 & 0\\ * & * & * & M_{4,4} & 0\\ * & * & * & * & M_{5,5}\end{bmatrix},$$

$$\Phi_{12}=\begin{bmatrix} 0 & M_{1,7} & M_{1,8} & M_{1,9} & 0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ M_{5,6} & M_{5,7} & M_{5,8} & 0 & 0\end{bmatrix},$$

$$\Phi_{13}=\begin{bmatrix} M_{1,11} & M_{1,12} & M_{1,13} & M_{1,14} & 0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0 & 0 & 0 & M_{5,14} & 0\end{bmatrix},$$

$$\Phi_{22}=\begin{bmatrix} M_{6,6} & 0 & 0 & 0 & 0\\ * & M_{7,7} & M_{7,8} & 0 & 0\\ * & * & M_{8,8} & 0 & 0\\ * & * & * & M_{9,9} & M_{9,10}\\ * & * & * & * & M_{10,10}\end{bmatrix},$$

$$\Phi_{23}=\begin{bmatrix} 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&0&0\\ 0&0&0&M_{9,14}&0\\ 0&0&0&0&0\end{bmatrix},$$

$$\Phi_{33}=\begin{bmatrix} M_{11,11} & 0 & 0 & 0 & 0\\ * & M_{12,12} & 0 & 0 & 0\\ * & * & M_{13,13} & 0 & 0\\ * & * & * & M_{14,14} & M_{14,15}\\ * & * & * & * & M_{15,15}\end{bmatrix},$$

$$\begin{aligned}
M_{1,1}=&[A+\alpha I]P+P[A+\alpha I]^T+(a+b+c+0.5d)I\\
&-D_1D_1^T+dD_1D_1^T+0.25D_1Y_1D_1^T+Q_1+Q_2\\
&+d_1^2T_2+3e^{2\alpha h_2}D_2S_1D_2^T+2e^{2\alpha d_2}D_3T_1D_3^T\\
&-e^{-2\alpha h_{1\mathrm{m}}}R_1-e^{-2\alpha h_{1\mathrm{M}}}R_2-2e^{-4\alpha h_{1\mathrm{m}}}W_1\\
&-2e^{-4\alpha h_{1\mathrm{M}}}W_2-\frac{2e^{-4\alpha h_{1\mathrm{M}}}(h_{1\mathrm{M}}-h_{1\mathrm{m}})}{(h_{1\mathrm{M}}+h_{1\mathrm{m}})}W_3,\\
M_{1,2}=&P,\ M_{1,3}=D_1,\ M_{1,4}=d_2^2D_1,\ M_{1,5}=BP,\\
M_{1,7}=&e^{-2\alpha h_{1\mathrm{m}}}R_1,\ M_{1,8}=e^{-2\alpha h_{1\mathrm{M}}}R_2,\ M_{1,9}=CP,\\
M_{1,11}=&\frac{2e^{-4\alpha h_{1\mathrm{m}}}}{h_{1\mathrm{m}}}W_1,\ M_{1,12}=\frac{2e^{-4\alpha h_{1\mathrm{M}}}}{h_{1\mathrm{M}}}W_2,\\
M_{1,13}=&\frac{2e^{-4\alpha h_{1\mathrm{M}}}}{(h_{1\mathrm{M}}+h_{1\mathrm{m}})}W_3,\ M_{1,14}=PA^T-0.5D_1D_1^T,\\
M_{2,2}=&-(2aI+Z_1)^{-1},\ M_{3,3}=-2e^{2\alpha h_2}S_1,\\
M_{4,4}=&-4d_2^2T_1,\\
M_{5,5}=&-2e^{-2\alpha h_{1\mathrm{M}}}R_3+e^{-2\alpha h_{1\mathrm{M}}}(R_4+R_4^T),\\
M_{5,6}=&P,\ M_{5,7}=e^{-2\alpha h_{1\mathrm{M}}}R_3-e^{-2\alpha h_{1\mathrm{M}}}R_4,\\
M_{5,8}=&e^{-2\alpha h_{1\mathrm{M}}}R_3-e^{-2\alpha h_{1\mathrm{M}}}R_4^T,\ M_{5,14}=PB^T,\\
M_{6,6}=&-(2bI+Z_2)^{-1},\\
M_{7,7}=&-e^{-2\alpha h_{1\mathrm{m}}}Q_1-e^{-2\alpha h_{1\mathrm{m}}}R_1-e^{-2\alpha h_{1\mathrm{M}}}R_3,\\
M_{7,8}=&e^{-2\alpha h_{1\mathrm{M}}}R_4^T,\\
M_{8,8}=&-e^{-2\alpha h_{1\mathrm{M}}}Q_2-e^{-2\alpha h_{1\mathrm{M}}}R_2-e^{-2\alpha h_{1\mathrm{M}}}R_3,\\
M_{9,9}=&-e^{-2\alpha d_1}T_2,\quad M_{9,10}=P,\quad M_{9,14}=PC^T,\\
M_{10,10}=&-(2cI+Z_3)^{-1},\ M_{11,11}=\frac{-2e^{-4\alpha h_{1\mathrm{m}}}}{h_{1\mathrm{m}}^2}W_1,\\
M_{12,12}=&\frac{-2e^{-4\alpha h_{1\mathrm{M}}}}{h_{1\mathrm{M}}^2}W_2,\ M_{13,13}=\frac{-2e^{-4\alpha h_{1\mathrm{M}}}}{(h_{1\mathrm{M}}^2-h_{1\mathrm{m}}^2)}W_3,\\
M_{14,14}=&h_{1\mathrm{m}}^2R_1+h_{1\mathrm{M}}^2R_2+(h_{1\mathrm{M}}-h_{1\mathrm{m}})^2R_3+h_{1\mathrm{m}}^2W_1\\
&+h_{1\mathrm{M}}^2W_2+(h_{1\mathrm{M}}-h_{1\mathrm{m}})h_{1\mathrm{M}}W_3\\
&+3e^{2\alpha h_2}D_2S_1D_2^T+2e^{2\alpha d_2}D_3T_1D_3^T\\
&+(a+b+c+0.5d)I-2P,\\
M_{14,15}=&h_2^2D_1, M_{15,15}=-4h_2^2S_1,
\end{aligned}$$

$$\begin{aligned}
\lambda_1=&\lambda_{\text{min}}(P^{-1}),\\
\lambda_2=&\lambda_{\text{max}}(P^{-1})+h_{1\mathrm{m}}\lambda_{\text{max}}(P^{-1}Q_1P^{-1})\\
&+h_{1\mathrm{M}}\lambda_{\text{max}}(P^{-1}Q_2P^{-1})\\
&+h_{1\mathrm{m}}^3\lambda_{\text{max}}(P^{-1}R_1P^{-1})\\
&+h_{1\mathrm{M}}^3\lambda_{\text{max}}(P^{-1}R_2P^{-1})\\
&+(h_{1\mathrm{M}}-h_{1\mathrm{m}})^3\lambda_{\text{max}}(P^{-1}R_3P^{-1})\\
&+\frac{1}{4}h_2^3\lambda_{\text{max}}(P^{-1}D_1S_1^{-1}D_1^TP^{-1})\\
&+\frac{1}{4}d_2^3\lambda_{\text{max}}(P^{-1}D_1T_1^{-1}D_1^TP^{-1})\\
&+d_1^3\lambda_{\text{max}}(P^{-1}T_2P^{-1})\\
&+h_{1\mathrm{m}}^3\lambda_{\text{max}}(P^{-1}W_1P^{-1})\\
&+h_{1\mathrm{M}}^3\lambda_{\text{max}}(P^{-1}W_2P^{-1})\\
&+(h_{1\mathrm{M}}-h_{1\mathrm{m}})h_{1\mathrm{M}}^2\lambda_{\text{max}}(P^{-1}W_3P^{-1}).
\end{aligned}$$

Proof: Let $Y=P^{-1}$, $y(t)=Yx(t)$. Using the feedback control (3), we consider the LyapunovKrasovskii functional for the closed-loop system (4),

$$V(t,x_t)=\sum_{i=1}^{12}V_i(t,x_t),$$

where

$$\begin{aligned}
V_1(\cdot) &= x^T(t)Yx(t),\\
V_2(\cdot) &= \int_{t-h_{1\mathrm{m}}}^{t} e^{2\alpha(s-t)}x^T(s)YQ_1Yx(s)ds,\\
V_3(\cdot) &= \int_{t-h_{1\mathrm{M}}}^{t} e^{2\alpha(s-t)}x^T(s)YQ_2Yx(s)ds,\\
V_4(\cdot) &= h_{1\mathrm{m}}\int_{-h_{1\mathrm{m}}}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta)YR_1Y\dot{x}(\theta)d\theta ds,\\
V_5(\cdot) &= h_{1\mathrm{M}}\int_{-h_{1\mathrm{M}}}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta)YR_2Y\dot{x}(\theta)d\theta ds,\\
V_6(\cdot) &= \eta\int_{-h_{1\mathrm{M}}}^{-h_{1\mathrm{m}}}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{x}^T(\theta)YR_3Y\dot{x}(\theta)d\theta ds,\\
V_7(\cdot) &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}\dot{u}^T(\theta)S_1^{-1}\dot{u}(\theta)d\theta ds,\\
V_8(\cdot) &= d_2\int_{-d_2}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}u^T(\theta)T_1^{-1}u(\theta)d\theta ds,\\
V_9(\cdot) &= d_1\int_{-d_1}^{0}\int_{t+s}^{t} e^{2\alpha(\theta-t)}x^T(\theta)YT_2Yx(\theta)d\theta ds,\\
V_{10}(\cdot) &= \int_{-h_{1\mathrm{m}}}^{0}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta)YW_1Y\dot{x}(\theta)\times\\
&\quad d\theta ds d\tau,\\
V_{11}(\cdot) &= \int_{-h_{1\mathrm{M}}}^{0}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta)YW_2Y\dot{x}(\theta)\times\\
&\quad d\theta ds d\tau,\\
V_{12}(\cdot) &= \int_{-h_{1\mathrm{M}}}^{-h_{1\mathrm{m}}}\int_{\tau}^{0}\int_{t+s}^{t} e^{2\alpha(\theta+s-t)}\dot{x}^T(\theta)YW_3Y\dot{x}(\theta)\times\\
&\quad d\theta ds d\tau.
\end{aligned}$$

It is easy to check that

$$\lambda_1||x(t)||^2 \le V(t,x_t) \le \lambda_2||x_t||^2, \quad \forall t\ge 0. \tag{7}$$

Taking the derivative of $V_i(t,x_t)$ along the solution of the system by using Newton-Leibniz formula, condition (2), Proposition 3, Proposition 5, Proposition 6, we have

$$\dot{V}(t,x_t)+2\alpha V(t,x_t)\le \xi^T(t)\ \Pi\ \xi(t)-L(\cdot) \tag{8}$$

where

$$\begin{aligned}
\xi^T(t) = \Big[& y^T(t), y^T(t-h_1(t)), y^T(t-h_{1\mathrm{m}}), y^T(t-h_{1\mathrm{M}}),\\
&\int_{t-d_1(t)}^{t} y^T(s)ds, \int_{t-h_{1\mathrm{m}}}^{t} y^T(\tau)d\tau, \int_{t-h_{1\mathrm{M}}}^{t} y^T(\tau)d\tau,\\
&\int_{t-h_{1\mathrm{M}}}^{t-h_{1\mathrm{m}}} y^T(\tau)d\tau, \dot{y}^T(t)\Big],
\end{aligned}$$

$$\Pi=\begin{bmatrix}\Pi_{11} & \Pi_{12} & \Pi_{13}\\ * & \Pi_{22} & \Pi_{23}\\ * & * & \Pi_{33}\end{bmatrix}<0, \tag{9}$$

$$\Pi_{11}=\begin{bmatrix}N_{1,1} & N_{1,2} & N_{1,3}\\ * & N_{2,2} & N_{2,3}\\ * & * & N_{3,3}\end{bmatrix},$$

$$\Pi_{12}=\begin{bmatrix}N_{1,4} & N_{1,5} & N_{1,6}\\ N_{2,4} & 0 & 0\\ N_{3,4} & 0 & 0\end{bmatrix},$$

$$\Pi_{13}=\begin{bmatrix}N_{1,7} & N_{1,8} & N_{1,9}\\ 0 & 0 & N_{2,9}\\ 0 & 0 & 0\end{bmatrix},$$

$$\Pi_{22}=\begin{bmatrix}N_{4,4} & 0 & 0\\ * & N_{5,5} & 0\\ * & * & N_{6,6}\end{bmatrix},$$

$$\Pi_{23}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & N_{5,9}\\ 0 & 0 & 0\end{bmatrix},$$

$$\Pi_{33}=\begin{bmatrix}N_{7,7} & 0 & 0\\ * & N_{8,8} & 0\\ * & * & N_{9,9}\end{bmatrix},$$

$$\begin{aligned}
N_{1,1} =& [A+\alpha I]P+P[A+\alpha I]^T+(a+b+c+0.5d)I\\
&- D_1D_1^T + dD_1D_1^T + 0.25D_1Y_1D_1^T + Q_1 + Q_2\\
&+ d_1^2T_2 + 3e^{2\alpha h_2}D_2S_1D_2^T + 2e^{2\alpha d_2}D_3T_1D_3^T\\
&- e^{-2\alpha h_{1\mathrm{m}}}R_1 - e^{-2\alpha h_{1\mathrm{M}}}R_2 - 2e^{-4\alpha h_{1\mathrm{m}}}W_1\\
&- 2e^{-4\alpha h_{1\mathrm{M}}}W_2 - \frac{2e^{-4\alpha h_{1\mathrm{M}}}(h_{1\mathrm{M}}-h_{1\mathrm{m}})}{(h_{1\mathrm{M}}+h_{1\mathrm{m}})}W_3\\
&+ P(2aI+Z_1)P + 0.5e^{-2\alpha h_2}D_1S_1^{-1}D_1^T\\
&+ 0.25d_2^2D_1T_1^{-1}D_1^T,\\
N_{1,2} =& BP,\ N_{1,3} = e^{-2\alpha h_{1\mathrm{m}}}R_1,\ N_{1,4} = e^{-2\alpha h_{1\mathrm{M}}}R_2,\\
N_{1,5} =& CP,\ N_{1,6} = \frac{2e^{-4\alpha h_{1\mathrm{m}}}}{h_{1\mathrm{m}}}W_1,\\
N_{1,7} =& \frac{2e^{-4\alpha h_{1\mathrm{M}}}}{h_{1\mathrm{M}}}W_2,\quad N_{1,8} = \frac{2e^{-4\alpha h_{1\mathrm{M}}}}{(h_{1\mathrm{M}}+h_{1\mathrm{m}})}W_3,\\
N_{1,9} =& PA^T - 0.5D_1D_1^T,\\
N_{2,2} =& -2e^{-2\alpha h_{1\mathrm{M}}}R_3 + e^{-2\alpha h_{1\mathrm{M}}}(R_4+R_4^T)\\
&+ P(2bI+Z_2)P,\\
N_{2,3} =& e^{-2\alpha h_{1\mathrm{M}}}R_3 - e^{-2\alpha h_{1\mathrm{M}}}R_4,\\
N_{2,4} =& e^{-2\alpha h_{1\mathrm{M}}}R_3 - e^{-2\alpha h_{1\mathrm{M}}}R_4^T,\quad N_{2,9} = PB^T,\\
N_{3,3} =& -e^{-2\alpha h_{1\mathrm{m}}}Q_1 - e^{-2\alpha h_{1\mathrm{m}}}R_1 - e^{-2\alpha h_{1\mathrm{M}}}R_3,\\
N_{3,4} =& e^{-2\alpha h_{1\mathrm{M}}}R_4^T,\\
N_{4,4} =& -e^{-2\alpha h_{1\mathrm{M}}}Q_2 - e^{-2\alpha h_{1\mathrm{M}}}R_2 - e^{-2\alpha h_{1\mathrm{M}}}R_3,\\
N_{5,5} =& -e^{-2\alpha d_1}T_2 + P(2cI+Z_3)P,\quad N_{5,9} = PC^T,\\
N_{6,6} =& \frac{-2e^{-4\alpha h_{1\mathrm{m}}}}{h_{1\mathrm{m}}^2}W_1,\quad N_{7,7} = \frac{-2e^{-4\alpha h_{1\mathrm{M}}}}{h_{1\mathrm{M}}^2}W_2,\\
N_{8,8} =& \frac{-2e^{-4\alpha h_{1\mathrm{M}}}}{(h_{1\mathrm{M}}^2-h_{1\mathrm{m}}^2)}W_3,\\
N_{9,9} =& h_{1\mathrm{m}}^2R_1 + h_{1\mathrm{M}}^2R_2 + (h_{1\mathrm{M}}-h_{1\mathrm{m}})^2R_3 + h_{1\mathrm{m}}^2W_1\\
&+ h_{1\mathrm{M}}^2W_2 + (h_{1\mathrm{M}}-h_{1\mathrm{m}})h_{1\mathrm{M}}W_3\\
&+ 3e^{2\alpha h_2}D_2S_1D_2^T + 2e^{2\alpha d_2}D_3T_1D_3^T\\
&+ (a+b+c+0.5d)I - 2P + 0.25h_2^2D_1S_1^{-1}D_1^T,
\end{aligned}$$

Using Proposition 4 (Schur complement lemma), condition (5) is equivalent to the condition $\Pi < 0$. Thus, from $(5)-(9)$,we obtain

$$\dot{V}(t, x_t) + 2\alpha V(t, x_t) \leq -L(\cdot), \qquad \forall t \in \mathbb{R}^+. \qquad (10)$$

Since $L(\cdot) > 0$, we have

$$\dot{V}(t, x_t) \leq -2\alpha V(t, x_t), \qquad \forall t \in \mathbb{R}^+. \qquad (11)$$

Integrating both sides of (11) from 0 to t, we obtain

$$V(t, x_t) \leq V(0, x_0)e^{-2\alpha t}, \qquad \forall t \in \mathbb{R}^+.$$

Furthermore, taking condition (7) into account, we have

$$\lambda_1 ||x(t, \phi)||^2 \leq V(t, x_t) \leq V(0, x_0)e^{-2\alpha t} \leq \lambda_2 e^{-2\alpha t}||\phi||^2,$$

then

$$||x(t, \phi)|| \leq \sqrt{\frac{\lambda_2}{\lambda_1}} e^{-\alpha t}||\phi||, \qquad \forall t \geq 0,$$

which implies the exponential stability of the closed-loop system (4). To find the upper bound of the cost function (3), we consider the derived condition (10) and $V(t, x_t) > 0$,we have

$$\dot{V}(t, x_t) \leq -L(\cdot), \qquad \forall t \in \mathbb{R}^+. \qquad (12)$$

Integrating both sides of (12) from 0 to t, we obtain

$$\int_0^t L(\cdot)\ dt \leq V(0, x_0) - V(t, x_t) \leq V(0, x_0), \quad \forall t \in \mathbb{R}^+,$$

because of $V(t, x_t) > 0$. Hence, letting $t \to \infty$, we finally obtain that

$$J = \int_0^\infty L(\cdot)\ dt \leq V(0, x_0) \leq \lambda_2 ||\phi||^2 = J^*.$$

This completes the proof of the theorem.

■

IV. Numerical Examples

In this section, we now provide an example to show the effectiveness of the result in Theorem 7.

Example.1 Consider a nonlinear system and mixed time-varying delay using feedback control with the following :

$$\begin{aligned} \dot{x}(t) =&\ Ax(t) + Bx(t - h_1(t)) + C\int_{t-d_1(t)}^t x(s)ds \\ &+ f(t, x(t), x(t - h_1(t)), \int_{t-d_1(t)}^t x(s)ds, u(t)) \\ &+ U(t), \qquad (1) \end{aligned}$$

$$U(t) = D_1u(t) + D_2u(t - h_2(t)) + D_3\int_{t-d_2(t)}^t u(s)ds,$$

$$x(t) = \phi(t), \quad t \in [-d_3, 0], \quad d_3 = \max\{h_{1M}, h_2, d_1, d_2\},$$

where

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & -0.5 \\ 0 & -1 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.2 & 0 \\ 0 & -0.1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad D_3 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$Z_1 = Z_2 = Z_3 = Y_1 = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix},$$

$$a = b = c = d = 0.001, \quad \alpha = 0.01,$$
$$h_2 = 0.3, \quad h_{1m} = 0.1, \quad h_{1M} = 0.3,$$
$$d_1 = 0.03, \quad d_2 = 0.02.$$

By using the LMI Toolbox in MATLAB, we obtain

$$P = \begin{bmatrix} 4.5599 & -0.6159 \\ -0.6159 & 3.8500 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 3.8029 & 0.5375 \\ 0.5375 & 1.8969 \end{bmatrix},$$

$$Q_2 = \begin{bmatrix} 4.2538 & 0.4146 \\ 0.4146 & 1.0766 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 17.9365 & -15.9555 \\ -15.9555 & 48.7317 \end{bmatrix},$$

$$R_2 = \begin{bmatrix} 0.3552 & -0.7884 \\ -0.7884 & 2.3270 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 20.7980 & -3.6989 \\ -3.6989 & 18.9409 \end{bmatrix},$$

$$R_4 = \begin{bmatrix} 1.0501 & -2.6087 \\ -2.6087 & 7.6735 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 5.2109 & -1.8802 \\ -1.8802 & 12.7356 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 1.0972 & -2.2022 \\ -2.2022 & 6.1546 \end{bmatrix},$$

$$T_2 = \begin{bmatrix} 82.1632 & -80.0082 \\ -80.0082 & 205.1834 \end{bmatrix},$$

$$W_1 = \begin{bmatrix} 2.4264 & -4.0157 \\ -4.0157 & 12.5592 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 0.5094 & -1.1200 \\ -1.1200 & 3.3355 \end{bmatrix},$$

$$W_3 = \begin{bmatrix} 0.1800 & -0.2595 \\ -0.2595 & 0.7044 \end{bmatrix},$$

$$K = \begin{bmatrix} -0.2241 & -0.0359 \\ -0.0538 & -0.3982 \end{bmatrix},$$

and the feedback control is

$$u(t) = \begin{bmatrix} -0.2241 & -0.0359 \\ -0.0538 & -0.3982 \end{bmatrix} x(t),$$

$$\lambda_1 = 0.2034, \quad \lambda_2 = 0.4143.$$

We take the initial condition

$$\phi(t) = \begin{bmatrix} 0.5\sin\ t \\ 0.5\cos\ t \end{bmatrix},$$

$$||\phi|| = 1$$

and the guaranteed cost value is

$$J^* = 0.4143||\phi||^2 = 0.4143.$$

By Theorem 1, the system is exponentially stable and solution $x(t, \phi(t))$ satisfies

$$||x(t, \phi(t)|| \leq 1.4272e^{-0.01t}, \quad t \geq 0.$$

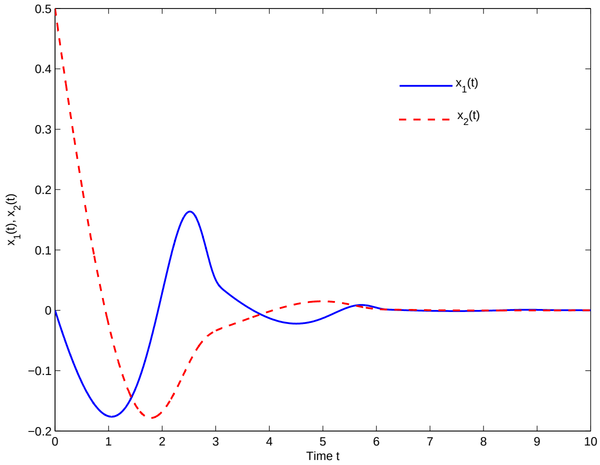

Fig. 1. The trajectories $x_1(t)$, and $x_2(t)$ of closed-loop system

V. Conclusion

In this paper, we have investigated the problem of optimal guaranteed cost control for exponential stability of nonlinear system with mixed time-varying delays via feedback control. The mixed time-varying delays consisting of both discrete and distributed delays are considered without assuming the differentiability of the time-varying delays. Based on an improved Lyapunov-Krasovskii functional with triple integral terms, new delay-dependent sufficient conditions for the existence of guaranteed cost feedback control for the system are given in terms of linear matrix inequalities (LMIs). A performance measure for the system is considered by a quadratic cost function. Finally, a numerical example is given to illustrate the effectiveness and improve over some existing results in the literature.

Acknowledgment

The authors thank anonymous reviewers for their valuable comments and suggestions.

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ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)