# Mixed H-infinity and Passivity Analysis for Neural Networks with Mixed Time-Varying Delays via Feedback Control

Sunisa Luemsai, Thongchai Botmart

***Abstract*—The problem of mixed $H_\infty$ and passivity analysis for neural networks with mixed time-varying delays which consist of discrete and distributed time-varying delays via feedback control is investigated. Based on designing feedback controller and constructing a Lyapunov-Krasovskii functional (LKF) comprising novel integral terms, the neural networks system is exponentially stable satisfying mixed $H_\infty$ and passivity performance. Moreover, improved Jensen inequalities and convex combination idea are utilized to derive sufficient conditions in terms of linear matrix inequalities (LMIs). A numerical example is employed to demonstrate the effectiveness of the proposed method.**

***Index Terms*—neural networks, mixed $H_\infty$ and passivity analysis, exponential stability, Lyapunov-Krasovskii functional, discrete and distributed time-varying delays**

## I. Introduction

In the past decades, neural networks have been studied by abundant researchers due to their fruitful applications in many areas such as pattern recognition, signal processing, optimization problems, automatic control engineering, parallel computation and so on [1], [2], [8], [20], [27]. Meanwhile, it is well known that time delay is a usual phenomenon that occurs in neural networks and existence of time delay which usually causes a source of poor performance, oscillation, divergence and even instability of the system [22]. Moreover, time delay is often encountered in a neural networks model due to communication time and the finite switching speed of the amplifiers in hardware implementation [28]. Thus stability analysis for neural networks with constant, discrete time-varying and distributed time-varying delays have received a great deal of attention [4], [7], [11], [12], [25].

On the other hand, the passivity play important roles of stability of neural networks with time delay and is more attractive to attention. The main key of passivity theory is that the passive properties can keep the system internally stable and represent the property of energy consumption. Particularly, it is effective tool related to the circuit analysis and has received much attention from the control areas. In addition, it has also been extensively applied in many physical systems such as fuzzy control, signal processing, network control [6] and sliding mode control [23]. Hence, the passivity of neural networks with time delay has studied in [10], [19], [24], [26]. Also, since $H_\infty$ control design expresses the control problem as a mathematical optimization problem to finding the controller solution, much attention of the $H_\infty$ approach becomes theoretical and practical importance [3], [18]. Furthermore, the $H_\infty$ approach has the merit over classical control techniques in that they are readily applicable to problems involving multivariate systems with cross-coupling between channels [16]. However, most recently, combined $H_\infty$ and passivity are more interested attention in the study and this problem with various systems becomes increasing interest among the researchers and it was first presented in [13], [14]. Particularly, the problems of mixed $H_\infty$ and passive synchronization for complex dynamical networks with sampled-data control have been studied in [17], [21]. Moreover, the problem of memristive neural networks analysis with mixed $H_\infty$ and passivity state estimation was investigated in [15].

Motivated by above discussions in this paper, we studied the problem of mixed $H_\infty$ and passivity analysis for neural networks with discrete and distributed time-varying delays via feedback control which researchers have not studied yet. The purpose is to focus on the exponential stability satisfying mixed $H_\infty$ and passivity performance for the neural networks such that a Lyapunov-Krasovskii functional comprising double, triple and quadruple integral terms and designing feedback controller are utilized. Moreover, improved Jensen inequalities and convex combination idea are employed to derive sufficient conditions in terms of linear matrix inequalities (LMIs). Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

Notation:
The following notations will be used in this paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the set of real numbers and the $n$-dimentionals real spaces, respectively. $P > 0$ or $P < 0$ denotes that the matrix $P$ is symmetric and positive definite or negative definite matrix. The notation $P^T$ and $P^{-1}$ denote the transpose and the inverse of $P$, respectively. $I$ denotes the identity matrix with appropriate dimensions. The symbol $*$ denotes the symmetric block in a symmetric matrix. diag$\{\ldots\}$ exhibits block diagonal matrix composed of elements in the bracket. $e_i$ denotes the unit column vector having one element on its $ith$ row and zeros elsewhere. For $x \in \mathbb{R}^n$, the norm of

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$x$, denoted by $\|x\|$, is defined by $\|x\|= \left(\sum_{i=1}^{n}|x_i|^2\right)^{1/2}$; $\|x(t+\epsilon)\|_{cl} = \max\{\sup_{-\max\{h_2,k_2\}\leq\epsilon\leq 0}\|x(t+\epsilon)\|^2, \sup_{-\max\{h_2,k_2\}\leq\epsilon\leq 0}\|\dot{x}(t+\epsilon)\|^2\}$.

## II. Preliminaries

Consider the following neural network model with both discrete time-varying delay and distributed time-varying delay:

$$\begin{aligned}
\dot{x}(t) &= -Ax(t) + Bf(x(t)) + Cg(x(t-h(t))) \\
&\quad + D\int_{t-k_2(t)}^{t-k_1(t)} j(x(s))\,ds + E\omega(t) + u(t), \\
z(t) &= C_1x(t) + C_2x(t-h(t)) \\
&\quad + C_3\int_{t-k_2(t)}^{t-k_1(t)} j(x(s))\,ds + C_4\omega(t), \\
x(t) &= \phi(t), t\in[-\varrho,0],
\end{aligned} \tag{1}$$

where $n$ denotes the number of neurons in the network, $x(t)=[x_1(t),x_2(t),\ldots,x_n(t)]^T\in\mathbb{R}^n$ is the neuron state vector, $z(t)\in\mathbb{R}^n$ is the output vector, $\omega(t)\in\mathbb{R}^n$ is the deterministic disturbance input which belongs to $\mathcal{L}_2[0,\infty)$, $u(t)\in\mathbb{R}^n$ is the control input, $f(x(t)),g(x(t)),j(x(t))\in\mathbb{R}^n$ are the neuron activation functions, $A=\mathrm{diag}\{a_1,a_2,\ldots,a_n\}$ is a positive diagonal matrix, $B,C,D$ are interconnection weight matrices, $E,C_1,C_2,C_3,C_4$ are given real matrices, $\phi(t)$ is the initial function. The variable $h(t)$ and $k_i(t)$ $(i=1,2)$ represent the discrete and distributed time-varying delays that satisfy $0\leq h_1\leq h(t)\leq h_2$, $0\leq k_1\leq k_1(t)\leq k_2(t)\leq k_2$ where $h_1,h_2,k_1,k_2,\varrho=\max\{h_2,k_2\}$ are known real constant scalars.

The following assumptions are made for later use.

**(H1)** The activation function $f$ is continuous and there exist constants $F_i^-$ and $F_i^+$ such that
$$F_i^-\leq\frac{f_i(\alpha_1)-f_i(\alpha_2)}{\alpha_1-\alpha_2}\leq F_i^+$$
for all $\alpha_1\neq\alpha_2$, and let $\tilde{F}_i=\max\{|F_i^-|,|F_i^+|\}$, where $f=[f_1,f_2,\ldots,f_n]^T$ and for any $i\in\{1,2,\ldots,n\}$, $f_i(0)=0$.

**(H2)** The activation function $g$ is continuous and there exist constants $G_i^-$ and $G_i^+$ such that
$$G_i^-\leq\frac{g_i(\alpha_1)-g_i(\alpha_2)}{\alpha_1-\alpha_2}\leq G_i^+$$
for all $\alpha_1\neq\alpha_2$, and let $\tilde{G}_i=\max\{|G_i^-|,|G_i^+|\}$, where $g=[g_1,g_2,\ldots,g_n]^T$ and for any $i\in\{1,2,\ldots,n\}$, $g_i(0)=0$.

**(H3)** The activation function $j$ is continuous and there exist constants $J_i^-$ and $J_i^+$ such that
$$J_i^-\leq\frac{j_i(\alpha_1)-j_i(\alpha_2)}{\alpha_1-\alpha_2}\leq J_i^+$$
for all $\alpha_1\neq\alpha_2$, and let $\tilde{J}_i=\max\{|J_i^-|,|J_i^+|\}$, where $j=[j_1,j_2,\ldots,j_n]^T$ and for any $i\in\{1,2,\ldots,n\}$, $j_i(0)=0$.

The state feedback controller takes the following form:
$$u(t)=Kx(t). \tag{2}$$

By substituting equation (2) into equation (1), we get

$$\begin{aligned}
\dot{x}(t) &= (K-A)x(t) + Bf(x(t)) + Cg(x(t-h(t))) \\
&\quad + D\int_{t-k_2(t)}^{t-k_1(t)} j(x(s))\,ds + E\omega(t), \\
z(t) &= C_1x(t) + C_2x(t-h(t)) \\
&\quad + C_3\int_{t-k_2(t)}^{t-k_1(t)} j(x(s))\,ds + C_4\omega(t), \\
x(t) &= \phi(t), t\in[-\varrho,0].
\end{aligned} \tag{3}$$

To derive the main results, we will introduce the following definitions and lemmas.

**Definition 1.** [21] The system (3) with $\omega(t)=0$ is exponentially stable, if there exist two constants $\eta>0$ and $\varpi>0$ such that
$$\|x(t)\|^2\leq\eta e^{-\varpi t}\|x(\epsilon)\|_{cl}. \tag{4}$$

**Definition 2.** [21] For given scalar $\sigma\in[0,1]$, the system (3) is exponentially stable and satisfies a mixed $H_\infty$/passivity performance index $\delta$, if the following two conditions can be guaranteed simultaneously:

(1) the system (3) is exponentially stable in view of Definition 1.
(2) under zero original condition, there exists a scalar $\delta>0$ such that the following inequality is satisfied:
$$\begin{aligned}
&\int_0^{\mathcal{T}_p}\left[-\sigma z^T(t)z(t)+2(1-\sigma)\delta z^T(t)\omega(t)\right]dt \\
&\quad\geq -\delta^2\int_0^{\mathcal{T}_p}\left[\omega^T(t)\omega(t)\right]dt,
\end{aligned} \tag{5}$$
for any $\mathcal{T}_p\geq 0$ and any non-zero $\omega(t)\in\mathcal{L}_2[0,\infty)$.

**Lemma 3.** [5], [9] Suppose $0\leq\eta_1<\eta_2$ and $x(t)\in\mathbb{R}^n$, for any positive definite matrix $P$ the following inequalities holds:

$$\begin{aligned}
&-(\eta_2-\eta_1)\int_{t-\eta_2}^{t-\eta_1}x^T(s)Px(s)\,ds \\
&\quad\leq -\int_{t-\eta_2}^{t-\eta_1}x^T(s)\,dsP\int_{t-\eta_2}^{t-\eta_1}x(s)\,ds \\
&-\frac{(\eta_2^2-\eta_1^2)}{2}\int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t}x^T(s)Px(s)\,ds\,d\beta \\
&\quad\leq -\int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t}x^T(s)\,ds\,d\beta P\int_{-\eta_2}^{-\eta_1}\int_{t+\beta}^{t}x(s)\,ds\,d\beta \\
&-\frac{\eta_2^3}{6}\int_{-\eta_2}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t}x^T(s)Px(s)\,ds\,d\lambda\,d\beta \\
&\quad\leq -\int_{-\eta_2}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t}x^T(s)\,ds\,d\lambda\,d\beta \\
&\qquad\times P\int_{-\eta_2}^{0}\int_{\beta}^{0}\int_{t+\lambda}^{t}x(s)\,ds\,d\lambda\,d\beta.
\end{aligned}$$

**Lemma 4.** [17] For a positive definite matrix $R>0$, any a differentiable function $\{x(u)|u\in[a,b]\}$, the following

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inequality hold:

$$\int_a^b \dot{x}^T(\alpha)R\dot{x}(\alpha) \geq \frac{1}{b-a}\left[x(b)-x(a)\right]^T R\left[x(b)-x(a)\right]$$
$$+\frac{3}{b-a}\left[x(b)+x(a)-\frac{2}{b-a}\int_a^b x(\alpha)\,d\alpha\right]^T$$
$$\times R\left[x(b)+x(a)-\frac{2}{b-a}\int_a^b x(\alpha)\,d\alpha\right].$$

## III. Main results

In this section, the sufficient conditions which ensure the neural networks system (3) to be exponentially stable satisfying mixed $H_\infty$ and passivity performance index $\delta$ are given. For the convenience of presentation, we denote

$$\begin{aligned}
F_1 =&\mathrm{diag}\{F_1^-F_1^+, F_2^-F_2^+, \ldots, F_n^-F_n^+\},\\
F_2 =&\mathrm{diag}\left\{\frac{F_1^-+F_1^+}{2}, \frac{F_2^-+F_2^+}{2}, \ldots, \frac{F_n^-+F_n^+}{2}\right\},\\
G_1 =&\mathrm{diag}\{G_1^-G_1^+, G_2^-G_2^+, \ldots, G_n^-G_n^+\},\\
G_2 =&\mathrm{diag}\left\{\frac{G_1^-+G_1^+}{2}, \frac{G_2^-+G_2^+}{2}, \ldots, \frac{G_n^-+G_n^+}{2}\right\},\\
J_1 =&\mathrm{diag}\{J_1^-J_1^+, J_2^-J_2^+, \ldots, J_n^-J_n^+\},\\
J_2 =&\mathrm{diag}\left\{\frac{J_1^-+J_1^+}{2}, \frac{J_2^-+J_2^+}{2}, \ldots, \frac{J_n^-+J_n^+}{2}\right\},
\end{aligned}$$

$$\begin{aligned}
\xi^T(t) = \Big[& x^T(t), \dot{x}^T(t), x^T(t-h_1), x^T(t-h_2), x^T(t-h(t)),\\
& f^T(x(t)), g^T(x(t-h(t))), j^T(x(t)), \tfrac{1}{h_1}\textstyle\int_{t-h_1}^t x^T(s)\,ds,\\
& \tfrac{1}{h_2}\textstyle\int_{t-h_2}^t x^T(s)\,ds, \tfrac{1}{h(t)-h_1}\int_{t-h(t)}^{t-h_1} x^T(s)\,ds,\\
& \tfrac{1}{h_2-h(t)}\textstyle\int_{t-h_2}^{t-h(t)} x^T(s)\,ds, \int_{t-k_2(t)}^{t-k_1(t)} j^T(x(s))\,ds,\\
& \textstyle\int_{-h(t)}^{-h_1}\int_{t+\beta}^t x^T(s)\,ds\,d\beta, \int_{-h_2}^{-h(t)}\int_{t+\beta}^t x^T(s)\,ds\,d\beta, \omega^T(t)\Big].
\end{aligned}$$

**Theorem 5.** For given scalars $h_1, h_2, k_1, k_2, \beta_1, \beta_2, \delta$ and $\sigma \in [0,1]$, if there exist eleven $n\times n$ matrices $P>0, Q_1>0, Q_2>0, R_1>0, R_2>0, U>0, L>0, X_1>0, X_2>0, N>0, Z$ and three $n\times n$ positive diagonal matrices $Y_1>0, Y_2>0, Y_3>0$ such that the following LMI holds:

$$\Xi+\Xi_1<0, \tag{6}$$
$$\Xi+\Xi_2<0, \tag{7}$$

wherein,

$$\begin{aligned}
\Xi_1 =& -e_{15}X_1e_{15}^T,\\
\Xi_2 =& -e_{14}X_1e_{14}^T,\\
\Xi =& \left[\pi_{(i,j)}\right]_{16\times 16},
\end{aligned}$$

with: $\pi_{(i,j)} = \pi_{(j,i)}$,

$$\begin{aligned}
\pi_{(1,1)} &= Q_1+Q_2-4R_1-4R_2+\sigma C_1^TC_1-F_1Y_1\\
&\quad -J_1Y_3+2\beta_1Z-2\beta_1N^TA+\tfrac{(h_2^2-h_1^2)^2}{4}X_1\\
&\quad -\tfrac{(h_2^2-h_1^2)^2}{4}X_2,\\
\pi_{(1,2)} &= P-\beta_1N^T+\beta_2Z^T-\beta_2N^TA,\\
\pi_{(1,3)} &= -2R_1,\\
\pi_{(1,4)} &= -2R_2,\\
\pi_{(1,5)} &= \sigma C_1^TC_2,\\
\pi_{(1,6)} &= F_2Y_1+\beta_1N^TB,\\
\pi_{(1,7)} &= \beta_1N^TC,\\
\pi_{(1,8)} &= J_2Y_3,\\
\pi_{(1,9)} &= 6R_1,\\
\pi_{(1,10)} &= 6R_2,\\
\pi_{(1,13)} &= \sigma C_1^TC_3+\beta_1N^TD,\\
\pi_{(1,14)} &= \tfrac{h_2^2-h_1^2}{2}X_2,\\
\pi_{(1,15)} &= \tfrac{h_2^2-h_1^2}{2}X_2,\\
\pi_{(1,16)} &= \sigma C_1^TC_4-(1-\sigma)\delta C_1^T+\beta_1N^TE,\\
\pi_{(2,2)} &= h_1^2R_1+h_2^2R_2+(h_2-h_1)^2U-2\beta_2N^T\\
&\quad +\tfrac{(h_2^3-h_1^3)^2}{36}X_2,\\
\pi_{(2,6)} &= \beta_2N^TB,\\
\pi_{(2,7)} &= \beta_2N^TC,\\
\pi_{(2,13)} &= \beta_2N^TD,\\
\pi_{(2,16)} &= \beta_2N^TE,\\
\pi_{(3,3)} &= -Q_1-4R_1-4U,\\
\pi_{(3,5)} &= -2U,\\
\pi_{(3,9)} &= 6R_1,\\
\pi_{(3,11)} &= 6U,\\
\pi_{(4,4)} &= -Q_2-4R_2-4U,\\
\pi_{(4,5)} &= -2U,\\
\pi_{(4,10)} &= 6R_2,\\
\pi_{(4,12)} &= 6U,\\
\pi_{(5,5)} &= -8U-G_1Y_2+\sigma C_2^TC_2,\\
\pi_{(5,7)} &= G_2Y_2,\\
\pi_{(5,11)} &= 6U,\\
\pi_{(5,12)} &= 6U,\\
\pi_{(5,13)} &= \sigma C_2^TC_3,\\
\pi_{(5,16)} &= \sigma C_2^TC_4-(1-\sigma)\delta C_2^T,\\
\pi_{(6,6)} &= -Y_1,\\
\pi_{(7,7)} &= -Y_2,\\
\pi_{(8,8)} &= (k_2-k_1)^2L-Y_3,\\
\pi_{(9,9)} &= -12R_1,\\
\pi_{(10,10)} &= -12R_2,\\
\pi_{(11,11)} &= -12U,\\
\pi_{(12,12)} &= -12U,\\
\pi_{(13,13)} &= -L+\sigma C_3^TC_3,\\
\pi_{(13,16)} &= \sigma C_3^TC_4-(1-\sigma)\delta C_3^T,\\
\pi_{(14,14)} &= -X_1-X_2,\\
\pi_{(14,15)} &= -X_2,\\
\pi_{(15,15)} &= -X_1-X_2,\\
\pi_{(16,16)} &= \sigma C_4^TC_4-2(1-\sigma)\delta C_4^T-\delta^2I,
\end{aligned}$$

another terms are 0,
then, the system (3) is exponentially stable and satisfies a mixed $H_\infty$/passivity performance index $\delta$. Furthermore, the desired controller gains can be given as:

$$K = N^{-1}Z. \tag{8}$$

**Proof.** We consider the following Lyapunov-Krasovskii functional candidate for the system (3) as

$$V(t) = \sum_{i=1}^{9} V_i(t), \tag{9}$$

where

$$\begin{aligned}
V_1(t) &= x^T(t)Px(t),\\
V_2(t) &= \int_{t-h_1}^{t} x^T(s)Q_1x(s)\,ds,\\
V_3(t) &= \int_{t-h_2}^{t} x^T(s)Q_2x(s)\,ds,\\
V_4(t) &= h_1\int_{-h_1}^{0}\int_{t+s}^{t} \dot{x}^T(\tau)R_1\dot{x}(\tau)\,d\tau\,ds,\\
V_5(t) &= h_2\int_{-h_2}^{0}\int_{t+s}^{t} \dot{x}^T(\tau)R_2\dot{x}(\tau)\,d\tau\,ds, \qquad (10)\\
V_6(t) &= (h_2-h_1)\int_{-h_2}^{-h_1}\int_{t+s}^{t} \dot{x}^T(\tau)U\dot{x}(\tau)\,d\tau\,ds,\\
V_7(t) &= (k_2-k_1)\int_{-k_2}^{-k_1}\int_{t+s}^{t} j^T(x(\tau))Lj(x(\tau))\,d\tau\,ds,\\
V_8(t) &= \frac{(h_2^2-h_1^2)}{2}\int_{-h_2}^{-h_1}\int_{\beta}^{0}\int_{t+\lambda}^{t} x^T(s)X_1x(s)\,ds\,d\lambda\,d\beta,\\
V_9(t) &= \frac{(h_2^3-h_1^3)}{6}\int_{-h_2}^{-h_1}\int_{\beta}^{0}\int_{\lambda}^{0}\int_{t+\varphi}^{t} \dot{x}^T(s)\\
&\quad \times X_2\dot{x}(s)\,ds\,d\varphi\,d\lambda\,d\beta.
\end{aligned}$$

Time derivatives of $V_i(t), i=1,2,\ldots,9$, along the trajectories of (3) are as follow:

$$\begin{aligned}
\dot{V}_1(t) &= x^T(t)P\dot{x}(t)+\dot{x}^T(t)Px(t), && (11)\\
\dot{V}_2(t) &= x^T(t)Q_1x(t)-x^T(t-h_1)Q_1x(t-h_1), && (12)\\
\dot{V}_3(t) &= x^T(t)Q_2x(t)-x^T(t-h_2)Q_2x(t-h_2), && (13)\\
\dot{V}_4(t) &= h_1^2\dot{x}^T(t)R_1\dot{x}(t)\\
&\quad -h_1\int_{t-h_1}^{t}\dot{x}^T(\alpha)R_1\dot{x}(\alpha)\,d\alpha, && (14)\\
\dot{V}_5(t) &= h_2^2\dot{x}^T(t)R_2\dot{x}(t)\\
&\quad -h_2\int_{t-h_2}^{t}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)\,d\alpha, && (15)\\
\dot{V}_6(t) &= (h_2-h_1)^2\dot{x}^T(t)U\dot{x}(t)\\
&\quad -(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(\alpha)U\dot{x}(\alpha)\,d\alpha, && (16)\\
\dot{V}_7(t) &= (k_2-k_1)^2j^T(x(t))Lj(x(t))\\
&\quad -(k_2-k_1)\int_{t-k_2}^{t-k_1} j^T(x(\alpha))Lj(x(\alpha))\,d\alpha\\
&\le (k_2-k_1)^2j^T(x(t))Lj(x(t))-(k_2(t)-k_1(t))\\
&\quad \times\int_{t-k_2(t)}^{t-k_1(t)} j^T(x(\alpha))Lj(x(\alpha))\,d\alpha, && (17)\\
\dot{V}_8(t) &= \frac{(h_2^2-h_1^2)^2}{4}x^T(t)X_1x(t)-\frac{(h_2^2-h_1^2)}{2}\\
&\quad \times\int_{-h_2}^{-h_1}\int_{t+\beta}^{t} x^T(s)X_1x(s)\,ds\,d\beta, && (18)\\
\dot{V}_9(t) &= \frac{(h_2^3-h_1^3)^2}{36}\dot{x}^T(t)X_2\dot{x}(t)-\frac{(h_2^3-h_1^3)}{6}\\
&\quad \times\int_{-h_2}^{-h_1}\int_{\beta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)X_2\dot{x}(s)\,ds\,d\lambda\,d\beta. && (19)
\end{aligned}$$

Utilizing Lemma 4., the following relations are easily obtained:

$$\begin{aligned}
&-h_1\int_{t-h_1}^{t}\dot{x}^T(\alpha)R_1\dot{x}(\alpha)\,d\alpha\\
&\quad \le -\xi^T(t)\left[e_1-e_3\right]R_1\left[e_1-e_3\right]^T\xi(t)\\
&\quad -3\xi^T(t)\left[e_1+e_3-2e_9\right]R_1\left[e_1+e_3-2e_9\right]^T\xi(t), \qquad (20)
\end{aligned}$$

$$\begin{aligned}
&-h_2\int_{t-h_2}^{t}\dot{x}^T(\alpha)R_2\dot{x}(\alpha)\,d\alpha\\
&\quad \le -\xi^T(t)\left[e_1-e_4\right]R_2\left[e_1-e_4\right]^T\xi(t)\\
&\quad -3\xi^T(t)\left[e_1+e_4-2e_{10}\right]R_2\left[e_1+e_4-2e_{10}\right]^T\xi(t), \qquad (21)
\end{aligned}$$

$$\begin{aligned}
&-(h_2-h_1)\int_{t-h_2}^{t-h_1}\dot{x}^T(\alpha)U\dot{x}(\alpha)\,d\alpha\\
&\le -\xi^T(t)\left[e_5-e_4\right]U\left[e_5-e_4\right]^T\xi(t)\\
&\quad -3\xi^T(t)\left[e_5+e_4-2e_{12}\right]U\left[e_5+e_4-2e_{12}\right]^T\xi(t)\\
&\quad -\xi^T(t)\left[e_3-e_5\right]U\left[e_3-e_5\right]^T\xi(t)\\
&\quad -3\xi^T(t)\left[e_3+e_5-2e_{11}\right]U\left[e_3+e_5-2e_{11}\right]^T\xi(t). \qquad (22)
\end{aligned}$$

On the other hand, we have the following relation from Lemma 3.

$$\begin{aligned}
&-(k_2(t)-k_1(t))\int_{t-k_2(t)}^{t-k_1(t)} j^T(x(\alpha))Lj(x(\alpha))\,d\alpha\\
&\quad \le -\xi^T(t)e_{13}Le_{13}^T\xi(t), \qquad (23)
\end{aligned}$$

$$\begin{aligned}
&-\frac{(h_2^2-h_1^2)}{2}\int_{-h_2}^{-h_1}\int_{t+\beta}^{t} x^T(s)X_1x(s)\,ds\,d\beta\\
&\quad \le -\xi^T(t)e_{15}X_1e_{15}^T\xi(t)-\varepsilon\xi^T(t)e_{15}X_1e_{15}^T\xi(t)\\
&\quad -(1-\varepsilon)\xi^T(t)e_{14}X_1e_{14}^T\xi(t)-\xi^T(t)e_{14}X_1e_{14}^T\xi(t), \qquad (24)
\end{aligned}$$

where $\varepsilon=\dfrac{h^2(t)-h_1^2}{h_2^2-h_1^2}$,

$$\begin{aligned}
&-\frac{(h_2^3-h_1^3)}{6}\int_{-h_2}^{-h_1}\int_{\beta}^{0}\int_{t+\lambda}^{t}\dot{x}^T(s)X_2\dot{x}(s)\,ds\,d\lambda\,d\beta\\
&\quad \le -\xi^T(t)\left[\frac{h_2^2-h_1^2}{2}e_1-e_{15}-e_{14}\right]X_2\\
&\quad \times\left[\frac{h_2^2-h_1^2}{2}e_1-e_{15}-e_{14}\right]^T\xi(t). \qquad (25)
\end{aligned}$$

Define $Y_1=\mathrm{diag}\{y_1,y_2,\ldots,y_n\}>0$, $Y_2=\mathrm{diag}\{\tilde{y}_1,\tilde{y}_2,\ldots,\tilde{y}_n\}>0$ and $Y_3=\mathrm{diag}\{\hat{y}_1,\hat{y}_2,\ldots,\hat{y}_n\}>0$, we get from assumptions **(H1)**, **(H2)** and **(H3)**, respectively, that

$$\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}^T\begin{bmatrix} -F_1Y_1 & F_2Y_1\\ F_2Y_1 & -Y_1\end{bmatrix}\begin{bmatrix} x(t)\\ f(x(t))\end{bmatrix}\ge 0, \qquad (26)$$

$$\begin{bmatrix} x(t-h(t))\\ g(x(t-h(t)))\end{bmatrix}^T\begin{bmatrix} -G_1Y_2 & G_2Y_2\\ G_2Y_2 & -Y_2\end{bmatrix}\begin{bmatrix} x(t-h(t))\\ g(x(t-h(t)))\end{bmatrix}\ge 0, \qquad (27)$$

$$\begin{bmatrix} x(t)\\ j(x(t))\end{bmatrix}^T\begin{bmatrix} -J_1Y_3 & J_2Y_3\\ J_2Y_3 & -Y_3\end{bmatrix}\begin{bmatrix} x(t)\\ j(x(t))\end{bmatrix}\ge 0. \qquad (28)$$

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We introduce some auxiliary equality as follows

$$\begin{aligned}0 &= 2\left[x^T(t)\beta_1 N^T + \dot{x}^T(t)\beta_2 N^T\right] \\ &\quad \times \Big[ -\dot{x}(t) + (N^{-1}Z - A)x(t) + Bf(x(t)) \\ &\quad + Cg(x(t-h(t))) + D\int_{t-k_2(t)}^{t-k_1(t)} j(x(s))\,ds + E\omega(t)\Big]. \end{aligned} \tag{29}$$

Adding the right-hand sides of (29) to $\dot{V}(t)$, we can get from Eq. (11)-(28) that

$$\begin{aligned}\dot{V}(t) &+ \sigma z^T(t)z(t) - 2(1-\sigma)\delta z^T(t)\omega(t) - \delta^2\omega^T(t)\omega(t) \\ &\le \xi^T(t)\left(\varepsilon\Xi^{(1)} + (1-\varepsilon)\Xi^{(2)}\right)\xi(t), \end{aligned} \tag{30}$$

where, $\Xi^{(i)} = \Xi + \Xi_i$ $(i = 1, 2)$ with $\Xi$ and $\Xi_i$ are defined in (6), (7). Since $0 \le \varepsilon \le 1$, the term $\varepsilon\Xi^{(1)} + (1-\varepsilon)\Xi^{(2)}$ is a convex combination of $\Xi^{(1)}$ and $\Xi^{(2)}$. These combinations are negative definite only if the following conditions hold simultaneously:

$$\Xi^{(1)} < 0, \tag{31}$$

$$\Xi^{(2)} < 0, \tag{32}$$

therefore, (31),(32) is equivalent to (6) and (7).
Thus, according to Eq. (6), (7) we have

$$\begin{aligned}\dot{V}(t) + \sigma z^T(t)z(t) - 2(1-\sigma)\delta z^T(t)\omega(t) \\ - \delta^2\omega^T(t)\omega(t) < 0. \end{aligned} \tag{33}$$

Then, under the zero original condition, it can be inferred that for any $\mathcal{T}_p$

$$\begin{aligned}&\int_0^{\mathcal{T}_p} \sigma z^T(t)z(t) - 2(1-\sigma)\delta z^T(t)\omega(t) - \delta^2\omega^T(t)\omega(t)\,dt \\ &\le \int_0^{\mathcal{T}_p} \dot{V}(t) + \sigma z^T(t)z(t) - 2(1-\sigma)\delta z^T(t)\omega(t) \\ &\quad - \delta^2\omega^T(t)\omega(t)\,dt < 0, \end{aligned}$$

which indicates that

$$\begin{aligned}&\int_0^{\mathcal{T}_p} \sigma z^T(t)z(t) - 2(1-\sigma)\delta z^T(t)\omega(t)\,dt \\ &\le \delta^2 \int_0^{\mathcal{T}_p} \omega^T(t)\omega(t)\,dt. \end{aligned}$$

In this case, the condition (5) is assured for any non-zero $\omega(t) \in \mathcal{L}_2[0,\infty)$. If $\omega(t) = 0$, in view of Eq. (33), we have

$$\dot{V}(t) < -\sigma z^T(t)z(t) \tag{34}$$

Applying the same method as in [11], we can find that the system (3) is exponentially stable. Therefore, according to Definition 2., the system (3) is exponentially stable and satisfies a mixed $H_\infty$/passivity performance index $\delta$. This completes the proof.

## IV. Numerical Example

In this section, a numerical example is presented to illustrate the effectiveness of our results through the maximum allowable delay bound, which is defined as the maximum delay value that retains the stability of the system.

**Example 6.** Consider the system (3) with the following parameters :

$k_1 = 0, k_2 = 2.0, \sigma = 0.1, \delta = 1, \beta_1 = 0.9, \beta_2 = 0.2,$

$$A = \begin{bmatrix}2.5 & 0\\ 0 & 2\end{bmatrix}, B = \begin{bmatrix}1.3 & 1\\ -0.5 & 0.5\end{bmatrix}, C = \begin{bmatrix}0.9 & 0.5\\ -0.3 & 0.4\end{bmatrix},$$

$$D = \begin{bmatrix}0.15 & 0.1\\ 0 & -0.3\end{bmatrix}, E = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, C_1 = \begin{bmatrix}-0.2 & 0\\ 0 & -0.4\end{bmatrix},$$

$$C_2 = \begin{bmatrix}0.1 & 0\\ 0 & 0.1\end{bmatrix}, C_3 = \begin{bmatrix}0.2 & 0.1\\ 0.1 & -0.4\end{bmatrix}, C_4 = \begin{bmatrix}0.1 & 0\\ 0 & 0.3\end{bmatrix},$$

$$F_1 = \begin{bmatrix}-0.2 & 0\\ 0 & -0.2\end{bmatrix}, F_2 = \begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix},$$

$$G_1 = \begin{bmatrix}-0.2 & 0\\ 0 & -0.1\end{bmatrix}, G_2 = \begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix},$$

$$J_1 = \begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}, J_2 = \begin{bmatrix}0.3 & 0\\ 0 & 0.1\end{bmatrix} \text{ and } I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}.$$

LMIs of (6), (7) in Theorem 5. are solved and the corresponding maximum allowable values of $h_2$ for different values of $h_1$. The maximum allowable values of $h_2$ are shown in TABLE I.

TABLE I
THE MAXIMUM ALLOWABLE VALUES OF $h_2$ FOR DIFFERENT VALUES OF $h_1$

| Method | $h_1 = 0$ | $h_1 = 0.5$ | $h_1 = 1$ | $h_1 = 2$ | $h_1 = 3$ |
|---|---|---|---|---|---|
| Theorem 5. | 1.7133 | 2.2133 | 2.7133 | 3.7133 | 4.7133 |

## V. Conclusion

In this paper, a new criterion has been derived for the mixed $H_\infty$ and passivity analysis of neural networks with discrete and distributed time-varying delays via feedback control. By designing feedback controller and a Lyapunov-Krasovskii functional comprising double, triple and quadruple integral terms have been utilized, the sufficient conditions guaranteed exponential stability satisfying mixed $H_\infty$ and passivity performance for the neural networks have been obtained. Eventually, a numerical example has been presented to demonstrate the effectiveness of the proposed method.

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