Numerical Integral Equation Methods of Average Run Length on Modified EWMA Control Chart for Exponential AR(1) Process

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Abstract— **The purpose of this research is to develop the numerical integral equation (NIE) methods for evaluating the average run length (ARL) on modified exponentially weighted moving average (modified EWMA) control chart for the first-order autoregressive process in the case of exponential white noise. The performance comparison of explicit formulas and the numerical integral techniques is measured with the average run length. The results showed that the performance of all methods are hardly different. However, the NIE methods are easier to calculate the ARL.**

Index Terms—**Modified EWMA control chart, AR(1), average run length, numerical integral equation**

I. Introduction

THE Statistical Process Control (SPC) are widely used in the manufacture industry for monitoring, controlling and improvement processes. A control chart is one of the tools for SPC. It can be applied to other fields such as finance, economics, industry, health and medicine (see [1]-[4]).

The common control charts are often used for detecting shifts such as the Shewhart chart, the Exponentially Weighted Moving Average (EWMA) chart and the Cumulative Sum (CUSUM) chart. The Shewhart control chart is useful for detecting large changes in process mean. In case of small shifts, the suitable control charts are the CUSUM control chart and the EWMA control chart, as in [5].

Alpaben and Jyoti [6] developed a modified EWMA chart, which combined the features of the Shewhart chart and EWMA chart. It can detect a small shift and is effective for auto-correlated data. This chart used past observations and additionally considered the latest past changes in the process.

The popular measure of a control chart's performance is the expected value of the run-length distribution, called the average run length (ARL) which is used to detect a change in the process (see [7]). The evaluation methods of ARL have been described in previous literature such as in the Monte Carlo simulation (MC), Markov Chain approach (MCA), martingale approach, explicit formulas and the numerical integral equation (NIE) method. Chou, Chan and Liu [8] simulated ARL for evaluating the control chart performance and studied the independent normal data, non-normal data, and auto-correlated data. Chananet, Areepong, Sukparungsee [9] proposed the Markov Chain approach to evaluate ARL of EWMA control chart. Sukparungsee and Novikov [10] used Martingale approach for analytic approximation of ARL on EWMA control chart. Areepong [11] presented explicit formulas of ARL on moving average control chart. Peerajit [12] studied the numerical integral equation method of ARL on CUSUM chart.

Frequently, an autoregressive model is used on control charts. This process is a difference equation determined by random variables on time series analysis. The order of an autoregressive model is the number of immediately previous values that are used to predict the present value. The first order autoregressive model is considered in this research, written as AR(1) which can be applied with real data, such as environmental, Economic, and Industrial data (see [13]-[15]).

This research presents the numerical integral equation (NIE) methods to evaluate the average run length (ARL) for the modified EWMA control chart for the first-order autoregressive process in the case of exponential white noise.

II. Modified EWMA Control Chart with Exponential AR(1) Process

The modified EWMA statistic Z_t (see [6]) based on AR(1) process with exponential white noise is expressed by the recursion:

$$Z_t = (1-\lambda) Z_{t-1} + \lambda X_t + (X_t - X_{t-1}),\ t = 1,2,3,... \qquad (1)$$

where λ is an exponential smoothing parameter $(0<\lambda<1)$, X_t is a general form of the AR(1) process (see [16]) and $Z_0 = u$, $X_0 = v$ is an initial value. The AR(1) process is assumed to be as follow

$$X_t = \eta + \phi X_{t-1} + \varepsilon_t,\ \varepsilon_t \sim Exp(\beta) \qquad (2)$$

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where η is a constant, and ϕ is an autoregressive coefficient $(|\phi|<1)$. The corresponding stopping time τ_b for showing out-of-control can be written as:

$$\tau_b = \inf\{t>0;\ Z_t > b\} \quad (3)$$

where b is the upper control limit. The ARL of the upper-sided modified EWMA control chart for AR(1) process is given by

$$ARL = E_\theta(\tau_b) \quad (4)$$

where θ is the change-point time, $E_\theta(.)$ is the expectation under the assumption that the change-point occurs at time θ, and $ARL_0 = E_\infty(\tau_b) = L(u)$.

III. Numerical Integral Equation (NIE) Methods of ARL on Modified EWMA Control Chart

The ARL can be derived a Fredholm integral equation of the second kind (see [17]). In this research, the numerical integral equation (NIE) method is used to solve ARL values (4). If ε_1 is in-control limit, then

$$0 \le (1-\lambda)u + (\lambda\phi+\phi-1)v + (1+\lambda)\varepsilon_1 + (1+\lambda)\eta \le b.$$

The solution of integral equation can be written in the form

$$L(u) = 1 + \int_{\frac{-(1-\lambda)u-(\lambda\phi+\phi-1)v}{(1+\lambda)}-\eta}^{\frac{b-(1-\lambda)u-(\lambda\phi+\phi-1)v}{(1+\lambda)}-\eta} L\begin{bmatrix}(1-\lambda)u+(\lambda\phi+\phi-1)v\\ +(1+\lambda)y+(1+\lambda)\eta\end{bmatrix} f(y)dy. \quad (5)$$

Let $k = (1-\lambda)u + (\lambda\phi+\phi-1)v + (1+\lambda)y + (1+\lambda)\eta,$ it is obtained that

$$L(u) = 1 + \frac{1}{1+\lambda}\int_0^b L(k)\, f\left(\frac{k-(1-\lambda)u-(\lambda\phi+\phi-1)v}{(1+\lambda)} - \eta\right)dk. \quad (6)$$

Equation (6) can be approximated by the use of numerical quadrature rules (see [18]) which can be calculated using many methods. This study selects the composite midpoint rule, the composite trapezoidal rule and the composite Simpson's rule.

A. Midpoint Rule

Given $f(A_j) = f\left(\frac{a_j-(1-\lambda)u-(\lambda\phi+\phi-1)v}{(1+\lambda)} - \eta\right).$ The Integral Equation (6) can be approximated by

$$\tilde{L}_M(u) \approx 1 + \frac{1}{1+\lambda}\sum_{j=1}^{m} w_j L(a_j)\, f(A_j) \quad (7)$$

where $w_j = \frac{b}{m}$ and $a_j = \left(j-\frac{1}{2}\right)w_j$; $j = 1,2,...,m.$

B. Trapezoidal Rule

Similarly, it can be written as follow

$$\tilde{L}_T(u) \approx 1 + \frac{1}{1+\lambda}\sum_{j=1}^{m+1} w_j L(a_j)\, f(A_j) \quad (8)$$

where $a_j = jw_j$ and $w_j = \frac{b}{m}$; $j = 1,2,...,m-1,$

in other cases, $w_j = \frac{b}{2m}.$

C. Simpson's Rule

By using the Simpson's Rule, ARL can be solved as follow

$$\tilde{L}_S(u) \approx 1 + \frac{1}{1+\lambda}\sum_{j=1}^{2m+1} w_j L(a_j)\, f(A_j) \quad (9)$$

where $a_j = jw_j$ and $w_j = \frac{4}{3}\left(\frac{b}{2m}\right)$; $j = 1,3,...,2m-1,$

$w_j = \frac{2}{3}\left(\frac{b}{2m}\right)$; $j = 2,4,...,2m-2,$

in other cases, $w_j = \frac{1}{3}\left(\frac{b}{2m}\right).$

IV. Comparison Results

This section compares the performance between the numerical integral equation (NIE) methods and explicit formulas for ARL of AR(1) processes on the modified EWMA control chart. ARL results obtained from the NIE method and used the division point m = 500 nodes.

TABLE I
Comparison of ARL values when given $u=v=1,\ \eta=2$ for ARL=370.

$\lambda=0.05$		Explicit Formula	$\tilde{L}_M(u)$	$\tilde{L}_T(u)$	$\tilde{L}_S(u)$
ϕ	b				
0.1	0.33399	370.57033	370.57033	370.57035	370.57033
0.2	0.30195	369.97819	369.97818	369.97820	369.97819
0.3	0.27301	370.45192	370.45191	370.45193	370.45192
0.5	0.22323	370.80353	370.80353	370.80354	370.80353
0.8	0.16512	371.17985	371.17985	371.17986	371.17985

TABLE II
Comparison of ARL values when given $u=v=1,\ \eta=2$ for ARL=500.

$\lambda=0.20$		Explicit Formula	$\tilde{L}_M(u)$	$\tilde{L}_T(u)$	$\tilde{L}_S(u)$
ϕ	b				
0.1	0.34678	500.57335	500.57326	500.57352	500.57335
0.2	0.31289	500.10193	500.10186	500.10207	500.10193
0.3	0.28239	499.00233	499.00227	499.00244	499.00233
0.5	0.23019	498.27362	498.27358	498.27369	498.27362
0.8	0.16967	503.29883	503.29881	503.29887	503.29883

In Table I and II, the performance comparison of explicit formulas and the NIE methods that used the composite midpoint rule (7), the composite trapezoidal rule (8) and the composite Simpson's rule (9) is ranged levels of autocorrelation, i.e. $\phi = 0.1, 0.2, 0.3, 0.5$ and 0.8. The results shows that ARL of the four methods are very closed and the NIE method with the composite Simpson's rule is as effective as the explicit formulas.

TABLE III
COMPARISON OF ARL VALUES WHEN GIVEN $u = v = 1,\ \eta = 2$ FOR ARL=370.

$\lambda = 0.05$		Explicit Formula	$\tilde{L}_M(u)$	$\tilde{L}_T(u)$	$\tilde{L}_S(u)$
ϕ	δ				
0.1	0.00	370.57033	370.57033	370.57035	370.57033
	0.01	78.404286	78.404284	78.404288	78.404286
	0.05	18.899453	18.899453	18.899454	18.899453
	0.08	12.085533	12.085533	12.085534	12.085533
0.3	0.00	370.45192	370.45191	370.45193	370.45192
	0.01	74.501939	74.501938	74.501941	74.501939
	0.05	17.760579	17.760579	17.760579	17.760579
	0.08	11.338447	11.338446	11.338447	11.338447

TABLE IV
COMPARISON OF ARL VALUES WHEN GIVEN $u = v = 1,\ \eta = 2$ FOR ARL=500.

$\lambda = 0.20$		Explicit Formula	$\tilde{L}_M(u)$	$\tilde{L}_T(u)$	$\tilde{L}_S(u)$
ϕ	δ				
0.2	0.00	500.10193	500.10186	500.10207	500.10193
	0.01	65.551813	65.551811	65.551817	65.551813
	0.03	24.139495	24.139495	24.139496	24.139495
	0.09	8.5667006	8.5667005	8.5667007	8.5667006
0.5	0.00	498.27362	498.27358	498.27369	498.27362
	0.01	60.192951	60.192950	60.192953	60.192951
	0.03	21.998375	21.998375	21.998376	21.998375
	0.09	7.7891573	7.7891572	7.7891573	7.7891573

For Table III and IV, the procedures are explained with variant shifting in the mean (δ). Likewise, ARL values of the NIE methods are similarly to that of explicit formulas.

V. CONCLUSION

The numerical integral equation (NIE) methods was able to analyze average run length (ARL) values on modified EWMA control chart for exponential AR(1) Process. This research's results can be applied to other fields such as economics, industry, and environment.

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