Proceedings of the International MultiConference of Engineers and Computer Scientists 2018 Vol II
IMECS 2018, March 14-16, 2018, Hong Kong

Generalizations of Non-strict Alpha-matrices with Nonzero Elements Chain

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***Abstract*—The main theorems of this paper, which extend the previous generalizations of α-matrices, provide some criteria for nonsingular H-matrices by the theories of non-strict α-matrices with nonzero elements chain. Meanwhile, the effectiveness of the result is illustrated by one numerical example.**

***Index Terms*—nonzero elements chain, H-matrices, α-matrices, non-strict α-matrices.**

I. Introduction

NONSINGULAR H-matrices arise in many practical applications, such as the Linear Complementarity Problem (LCP, see [1]), the numerical solution of Euler equations in fluid dynamics and many other problems. Thus, how to investigate criteria for H-matrices is of great significance. However, in most of the cases, it is not practical to answer this question by using the classical definitions. In this paper, we extend the previous generalizations of α-matrices and give some criteria for nonsingular H-matrices based on the theories of non-strictly α-matrices with nonzero elements chain, which are easier to check than the original definitions.

First, we will recall some notations and definitions. Consider the set of the first n positive integers denoted by $\mathbf{N}=\{1,2,\ldots,n\}$ and the pair set $\mathbf{M}=\{(i,j): i\neq j;\ i,j\in\mathbf{N}\}$. Let $\mathbf{C}^{m\times n}$ ($\mathbf{R}^{n\times n}$) denote the set of all $n\times n$, complex (real) matrices. For $A=[a_{ij}]\in\mathbf{C}^{n\times n}$ $(n\geq 2)$, we write $R_i=R_i(A)=\sum\limits_{j\neq i}^{n}\mid a_{ij}\mid$ $(i\in\mathbf{N})$ the ith deleted absolute row sums and $C_i=C_i(A)=\sum\limits_{j\neq i}^{n}\mid |a_{ji}\mid$ $(i\in\mathbf{N})$ the ith deleted absolute column sums of A, for the sake of simplicity. $A=[a_{ij}]\in\mathbf{C}^{n\times n}$ is a (row) diagonally dominant matrix (D) if $\mid a_{ii}\mid\geq R_i,\ \forall i\in\mathbf{N}$, and A is further said to be a strictly (row) diagonally dominant matrix (SD) if $\mid a_{ii}\mid> R_i,\ \forall i\in\mathbf{N}$.

The comparison matrix of a given matrix $A=[a_{ij}]\in\mathbf{C}^{n\times n}$, denoted by $\mu(A)=(\mu_{ij})$, is defined by

$$\mu_{ij}=\begin{cases}|a_{ij}|,\ i=j;\\-|a_{ij}|,\ i\neq j.\end{cases}$$

If $A=\mu(A)$, and can be written in the form $A=\beta I-P$ where P is a nonnegative matrix and $\beta>\rho(A)$, the spectral radius of A, we call A a (nonsingular) M-matrix. We say A is a (nonsingular) H-matrix if $\mu(A)$ is an M-matrix. Since well-known characterization of H-matrices is given by the fact that a matrix A is an H-matrix if and only if there exists a positive diagonal matrix $X=\operatorname{diag}(x_1,x_2,\ldots,x_n)$, such that $AX\in SD$ (i.e., $x_i\mid a_{ii}\mid> x_j\sum\limits_{j\neq i}^{n}|a_{ij}|,\ i\in N$). So, we assume that $a_{ii}\neq 0$ for all $i\in\mathbf{N}$ during this paper.

Next, we will present some already known subclasses of H-matrix.

The first result is the well-known Lévy-Desplanques Theorem (see[2], [3]).

Lemma 1.1. Let $A=[a_{ij}]\in\mathbf{C}^{n\times n}$, $n\geq 2$. If $A\in SD$, then A is nonsingular, more over A is an H-matrix.

Many generalizations of Lévy-Desplanques Theorem have occurred in the literature. Ostrowski (see [4]) extended the Lemma 1.1 by using generalized geometric means of row and column sums as are given below.

Lemma 1.2. Let $A=[a_{ij}]\in\mathbf{C}^{n\times n}$, $n\geq 2$, and let

$$\mid a_{ii}\mid>R_i^{\alpha}C_i^{1-\alpha},\ (i\in\mathbf{N}),$$

hold for some $\alpha\in[0,1]$, then A is nonsingular, more over A is an H-matrix.

From the generalized arithmetic-geometric mean inequality (see [5])

$$\alpha\tau+(1-\alpha)\sigma\geq\tau^{\alpha}\sigma^{1-\alpha},\tag{1}$$

where $\sigma,\ \tau\geq 0,\ \alpha\in[0,1]$, with equality holding for $\tau=\sigma$ or $\alpha=0$, else or $\alpha=1$, the following results were given in [5] .

Lemma 1.3. Let $A=[a_{ij}]\in\mathbf{C}^{n\times n}$, $n\geq 2$, and let

$$\mid a_{ii}\mid>\alpha R_i+(1-\alpha)C_i,\ (i\in\mathbf{N}),$$

hold for some $\alpha\in[0,1]$, then A is nonsingular, more over A is an H-matrix.

The matrices that fulfill conditions of the Lemma 1.2 are known as (strict) α_2-matrices , while (strict) α_1-matrices are the matrices that fulfill conditions of the Lemma 1.3, respectively.

Authors extended the Lemmas 1.1, 1.2 and 1.3 by letting all but at least one of the considered inequalities not to be strict with irreducible matrices (see [6], [7] and [8]).

In what follows, we are interested in the fact: the matrices remain to be H-matrices if we change the irreducibility with the existence of nonzero element chains. P. N. Shivakumar and K. H. Chew [9] introduced the most basic concept of diagonally dominant matrix associated with nonzero element chains as follows.

Lemma 1.4. $A=[a_{ij}]\in\mathbf{C}^{n\times n}$, $n\geq 2$, is called a diagonally dominant matrix with nonzero elements chain if $\mid a_{ii}\mid\geq R_i,\ \forall i\in\mathbf{N}$, at least one strict inequality holds, and for every vertex i with $\mid a_{ii}\mid= R_i$ there exists a nonzero elements chain $a_{ij_1},a_{j_1j_2},\ldots,a_{j_{k-1}j_k}$ such that $\mid a_{j_kk}\mid> R_{j_k}$.

Manuscript received January 9, 2018; revised February 3, 2018. This work was supported in part by the National Natural Science Foundation under Grant No. 81460656 of China, by Natural Science Foundation of Inner Mongolia Autonomous Region under Grant No. 2016MS0118 of China, and by the Science Research Funding of the Inner Mongolia University for the Nationalities under Grant No. NMDYB1778 of China.

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ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

It is well known that a diagonally dominant matrix with nonzero elements chain is an H-matrix ([1], [9]).

II. Non-strict α-matrices with nonzero elements chain

We call α_1-matrices and α_2-matrices as α(alpha)-matrices. It is well known that the class of α-matrices play a central role in identifying H-matrices, and characterizations of α-matrices were given in [5] and [10]. Non-strict α-matrices with nonzero elements chain are extension of α-matrices. So, we will start this section with its definitions.

Definition 2.1. A matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, $n \geq 2$, is said to be a non-strict α_2-matrix with nonzero elements chain if there exists $\alpha \in [0, 1]$, such that

$$| a_{ii} | \geq R_i^{\alpha} C_i^{1-\alpha}, \ (\forall\ i \in \mathbf{N}),$$

and for each vertex i of A with $| a_{ii} | = R_i^{\alpha} C_i^{1-\alpha}$, there exists a nonzero elements chain $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_g j}$ such that

$$j \in J = \{i \in \mathbf{N} | :| a_{ii} | > R_i^{\alpha} C_i^{1-\alpha}\} \neq \emptyset.$$

Definition 2.2. A matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, $n \geq 2$, is said to be a non-strict α_1-matrix with nonzero elements chain if there exists $\alpha \in [0, 1]$, such that

$$| a_{ii} | \geq \alpha R_i + (1-\alpha) C_i, \ (\forall\ i \in \mathbf{N}),$$

and for each vertex i of A with $| a_{ii} | = \alpha R_i + (1-\alpha) C_i$, there exists a nonzero elements chain $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_k j}$ such that

$$j \in J_0 = \{i \in \mathbf{N} | :| a_{ii} | > \alpha R_i + (1-\alpha) C_i\} \neq \emptyset.$$

As shown in [11], non-strict α_2-matrices with nonzero elements chain are nonsingular, moreover are subclass of H-matrices. And by the generalized arithmetic-geometric mean inequality (1), we easily get that non-strict α_1-matrices with nonzero elements chain are nonsingular, and are subclass of H-matrices, too.

The following notations are useful in the sequel.

$$\Re_0 = \{i | : R_i > C_i, \ i \in \mathbf{N}\};$$

$$\mathcal{C}_0 = \{i | : C_i > R_i, \ i \in \mathbf{N}\};$$

$$\varepsilon_0 = \{i | : C_i = R_i, i \in \mathbf{N}\}.$$

Then, the following theorems hold.

Theorem 2.1. A matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, $n \geq 2$, satisfies the following three conditions:

(i) $| a_{ii} | \geq \min\{R_i, \ C_i\}$ for all $i \in \mathbf{N}$;

(ii)
$$\log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} \geq \log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |}, \qquad (2)$$
for all $i \in \Re_0 \setminus \{l : C_l = 0\}$, and for all $j \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$;

(iii) for each vertex $s \in \mathbf{N}$ with $| a_{ss} | = R_s = C_s$, and each vertex $i \in \Re_0$, $j \in \mathcal{C}_0$ with
$$\log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} = \log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |}, \qquad (3)$$
if they exist, there are three nonzero elements chains $a_{ss_1}, a_{s_1 s_2}, \ldots, a_{s_k p}$; $a_{ii_1}, a_{i_1 i_2}, \ldots a_{i_l q}$ and $a_{jj_1}, a_{j_1 j_2}, \ \ldots, a_{j_m t}$ such that
$$p, q, t \in J' = \{i \in \Re_0 | : \ \log_{\frac{R_i}{C_i}} \frac{|a_{ii}|}{C_i} > \log_{\frac{C_j}{R_j}} \frac{C_j}{|a_{jj}|}, \ j \in \mathcal{C}_0\} \neq \emptyset. \qquad (4)$$

Then A is nonsingular, more over A is an H-matrix.

Proof: For each $i \in \varepsilon_0$, condition (i) directly implies the inequality $| a_{ii} | \geq R_i^{\alpha} C_i^{1-\alpha}$ for any $\alpha \in [0, 1]$, with equality holding only for $| a_{ii} |= R_i = C_i$. Recalling the assumption that $a_{ii} \neq 0$ for all $i \in \mathbf{N}$ during this paper, then for $i \in \Re_0$ such that $C_i = 0$, or $i \in \mathcal{C}_0$ such that $R_i = 0$, inequality $| a_{ii} | > R_i^{\alpha} C_i^{1-\alpha}$ for any $\alpha \in (0, 1)$, follows immediately. Thus, it remains to show that $| a_{ii} | \geq R_i^{\alpha} C_i^{1-\alpha}$ holds for vertexes from the set $\Re_0 \setminus \{l : C_l = 0\}$ and the set $\mathcal{C}_0 \setminus \{l : R_l = 0\}$.

Next, we consider the following two cases.

Case 1. If for all $i \in \Re_0 \setminus \{l : C_l = 0\}$, and all $j \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$,

$$\log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} > \log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |}. \qquad (5)$$

Noting that for each $i \in \Re_0 \setminus \{l : C_l = 0\}$, we have $R_i > C_i$, i.e., $\frac{R_i}{C_i} > 1$, and, thus by condition (i), leads to $| a_{ii} |\geq C_i$, i.e., $\frac{|a_{ii}|}{C_i} \geq 1$. Now, using the properties of the log function for the base greater than one, we obtain

$$\log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} \geq 0. \qquad (6)$$

With similar arguments, for each $j \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$, we deduce

$$\log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |} \leq 1. \qquad (7)$$

Together the inequalities (6), (6) and (7) imply that there is some $\alpha \in [0, 1]$, such that

$$\log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |} < \alpha < \log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} \qquad (8)$$

for each $i \in \Re_0 \setminus \{l : C_l = 0\}$, $j \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$.

From the left inequality and right inequality of inequality (8), we have, respectively, that $j \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$, $\frac{|a_{jj}|}{C_j} > (\frac{R_j}{C_j})^{\alpha}$, that is,

$$| a_{jj} | > R_j^{\alpha} C_j^{1-\alpha}$$

and for each $i \in \Re_0 \setminus \{l : C_l = 0\}$, $\frac{|a_{ii}|}{C_i} > (\frac{R_i}{C_i})^{\alpha}$, that is,

$$| a_{ii} | > R_i^{\alpha} C_i^{1-\alpha}.$$

Case 2. Without loss of generality, we assume that there exist some $i_0 \in \Re_0 \setminus \{l : C_l = 0\}$, $j_0 \in \mathcal{C}_0 \setminus \{l : R_l = 0\}$, such that

$$\log_{\frac{R_{i_0}}{C_{i_0}}} \frac{| a_{i_0 i_0} |}{C_{i_0}} = \log_{\frac{C_{j_0}}{R_{j_0}}} \frac{C_{j_0}}{| a_{j_0 j_0} |},$$

and for any $i \in \Re_0 \setminus (\{l : C_l = 0\} \cup \{i_0\})$, $j \in \mathcal{C}_0 \setminus (\{l : R_l = 0\} \cup \{j_0\})$,

$$\log_{\frac{R_i}{C_i}} \frac{| a_{ii} |}{C_i} > \log_{\frac{C_j}{R_j}} \frac{C_j}{| a_{jj} |}.$$

Let

$$\alpha_0 = \log_{\frac{R_{i_0}}{C_{i_0}}} \frac{|\ a_{i_0 i_0}\ |}{C_{i_0}} = \log_{\frac{C_{j_0}}{R_{j_0}}} \frac{C_{j_0}}{|\ a_{j_0 j_0}\ |},$$

and hence,

$$|\ a_{i_0 i_0}\ | = R_{i_0}^{\alpha_0} C_{i_0}^{1-\alpha_0};$$

$$|\ a_{j_0 j_0}\ | = R_{j_0}^{\alpha_0} C_{j_0}^{1-\alpha_0},$$

From condition (ii), it is obvious to get for all $i \in \Re_0 \backslash (\{l : C_l = 0\} \cup \{i_0\})$, $j \in \mathcal{C}_0 \setminus (\{l : R_l = 0\} \cup \{j_0\})$,

$$\log_{\frac{C_j}{R_j}} \frac{C_j}{|\ a_{jj}\ |} < \alpha_0 < \log_{\frac{R_i}{C_i}} \frac{|\ a_{ii}\ |}{C_i},$$

implying in *Case* 1 that

$$|\ a_{ii}\ | > R_i^{\alpha_0} C_i^{1-\alpha_0};$$

$$|\ a_{jj}\ | > R_j^{\alpha_0} C_j^{1-\alpha_0}.$$

To sum up, for any $i \in \Re_0 \cup \mathcal{C}_0 \cup \varepsilon_0 = \mathbf{N}$, there exists some $\alpha \in [0, 1]$, such that

$$|\ a_{ii}\ | \geq R_i^{\alpha} C_i^{1-\alpha},$$

with equality holding only for vertex $s \in \mathbf{N}$ satisfying $|\ a_{ss}\ | = R_s = C_s$, and vertex $i \in \Re_0$, $j \in \mathcal{C}_0$ satisfying $\log_{\frac{R_i}{C_i}} \frac{|a_{ii}|}{C_i} = \log_{\frac{C_j}{R_j}} \frac{C_j}{|a_{jj}|}$.

Further by condition (iii), for each vertex i of A with $|\ a_{ii}\ | = R_i^{\alpha} C_i^{1-\alpha}$, there exists a nonzero elements chain $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_l j}$, such that

$$j \in J' = \{i \in \mathbf{N}| :|\ a_{ii}\ | > R_i^{\alpha} C_i^{1-\alpha}\} \neq \emptyset.$$

Since A is a non-strict α_2-matrices with nonzero elements chain, then we can complete the proof. ∎

Next, we will state and prove the similar result, precisely, as follows.

Theorem 2.2. A matrix $A = [a_{ij}] \in \mathbf{C}^{n\times n}$, $n \geq 2$, satisfies the following three conditions:

(i) $|\ a_{ii}\ | \geq \min\{R_i,\ C_i\}$ for all $i \in \mathbf{N}$;

(ii)

$$\frac{|\ a_{ii}\ | - C_i}{R_i - C_i} \geq \frac{C_j - |\ a_{jj}\ |}{C_j - R_j}, \tag{9}$$

for all $i \in \Re_0$, and for all $j \in \mathcal{C}_0$;

(iii) for each vertex $s \in \mathbf{N}$ with $|\ a_{ss}\ | = R_s = C_s$, and each vertex $i \in \Re_0$, $j \in \mathcal{C}_0$ with

$$\frac{|\ a_{ii}\ | - C_i}{R_i - C_i} = \frac{C_j - |\ a_{jj}\ |}{C_j - R_j}, \tag{10}$$

if they exist, there are three nonzero elements chains $a_{ss_1}, a_{s_1 s_2}, \ldots, a_{s_k p}$; $a_{ii_1}, a_{i_1 i_2}, \ldots, a_{i_l q}$ and a_{jj_1}, $a_{j_1 j_2}, \ldots, a_{j_m t}$, such that

Then A is nonsingular, more over A is an H-matrix.

Proof: Similar discussion as in the proof of the Theorem 2.1, based we obtain A is a non-strictly α_1-matrices with nonzero elements chain. It follows that A is nonsingular, more over A is an H-matrix. ∎

III. Example

Example Let

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 3 & -0.5 & 0 \\ 0 & -1 & 4 & 3.1 \\ 0 & 0 & 1.2 & 1.5 \end{pmatrix}.$$

We have

$$|\ a_{11}\ |= 1,\ \ |\ a_{22}\ |= 3,\ \ |\ a_{33}\ |= 4,\ \ |\ a_{44}\ |= 1.5;$$

$$R_1 = 1,\ R_2 = 1.5,\ R_3 = 4.1,\ R_4 = 1.2;$$

$$C_1 = 1,\ C_2 = 2,\ C_3 = 1.7,\ C_4 = 3.1.$$

These conditions do not satisfy the theorem 4 or theorem 5 in [5], and it is hard and complicated to determine the value of α if we want to check if A is a non-strictly α-matrix with nonzero elements chain or not.

Nevertheless, by this paper, we know $|\ a_{ii}\ | \geq \min\{R_i, C_i\}$ for all $i \in \mathbf{N}$ with $|\ a_{11}\ |= R_1 = C_1$, and

$$\Re_0 = \{3\},\ \mathcal{C}_0 = \{2, 4\},\ \varepsilon_0 = \{1\},$$

An easy calculating yields

$$0.9720 \approx \log_{\frac{R_3}{C_3}} \frac{|\ a_{33}\ |}{C_3} > \log_{\frac{C_2}{R_2}} \frac{C_2}{|\ a_{22}\ |} \approx -1.4094;$$

$$0.9720 \approx \log_{\frac{R_3}{C_3}} \frac{|\ a_{33}\ |}{C_3} > \log_{\frac{C_4}{R_4}} \frac{C_4}{|\ a_{44}\ |} \approx 0.7649,$$

that is, for all $i \in \Re_0, j \in \mathcal{C}_0$

$$\log_{\frac{R_i}{C_i}} \frac{|\ a_{ii}\ |}{C_i} > \log_{\frac{C_j}{R_j}} \frac{C_j}{|\ a_{jj}\ |},$$

and for the vertex 1 with $|\ a_{11}\ |= R_1 = C_1$, there exists a nonzero elements chain a_{12}, a_{23} such that

$$3 \in J_0' = \{3 \in \Re_0| : \log_{\frac{R_3}{C_3}} \frac{|\ a_{33}\ |}{C_3} > \log_{\frac{C_j}{R_j}} \frac{C_j}{|\ a_{jj}\ |},\ j \in \mathcal{C}_0\} \neq \emptyset.$$

So, the matrix A satisfies conditions of the Theorem 2.1 in this section, and then A is an H-matrix.

IV. Conclusion

In conclusion, we extend previous generalizations of α-matrices, provide some criteria for nonsingular H-matrices by the theorems of non-strict α-matrices with nonzero elements chain theorem and illustrate the effectiveness and advantage of the new result by one numerical example.

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ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

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ISBN: 978-988-14048-8-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)