A Hierarchy of Diagonal Base Hypergraphs

Stefan Porschen *

Abstract— **Investigating the structure of base hypergraphs of CNF classes we specifically prove the existence of connected strictly diagonal and simple base hypergraphs with loops. Further a hierarchy of diagonal base hypergraphs of non-decreasing complexity regarding their fibre-transversal orbit spaces w.r.t. the flipping action is identified. Several existence results are provided, some remain open.**

Keywords: minimal-unsatisfiable-CNF, hypergraph, orbit, bifurcation

1 Introduction

A fundamental open question in mathematics is the NP versus P problem which is attacked within the theory of NP-completeness. The genuine and one of the most important NP-complete problems is the propositional satisfiability problem (SAT) for conjunctive normal form (CNF) formulas [6] lying at the heart of computational complexity theory. Specifically one is interested in subclasses for which SAT can be solved in polynomial time. There are known several such classes, e.g., quadratic formulas, (extended and q-)Horn formulas, matching formulas, nested and co-nested formulas etc. [2, 4, 5, 7, 8, 9, 10, 16, 18]. Moreover, it turns out that a useful tool in revealing the structure of CNF-SAT is provided by linear CNF formulas [15] and linear hypergraphs. Introducing a bifurcation concept, in this paper we further investigate the structure of base hypergraphs of CNF formula classes and the orbit spaces of their fibre-transversals with respect to the flipping action. Specifically we prove the existence of strictly diagonal base hypergraphs [13] admitting loops. On that basis we propose a hierarchy of base hypergraphs with non-decreasing complexity within the orbit space of the corresponding sets of diagonal fibre-transversals. Also the class of simple diagonal base hypergraphs [13] is investigated further. So we prove the existence of simple and loopless diagonal base hypergraphs. Here also certain connections to the strict diagonal case and also to minimal unsatisfiable formulas appear.

2 Preliminaries

A Boolean or propositional variable x taking values from $\{0,1\}$ can appear as a positive literal which is x or as a negative literal which is the negated variable $\overline{x}$ also called the flipped or complemented variable. For a variable x, let $l(x) \in \{x, \bar{x}\}$ denote a literal over x which is not specified. Setting a literal to 1 means to set the corresponding variable accordingly. A clause c is a finite non-empty disjunction of different literals and it is represented as a set $c = \{l_1, \ldots, l_k\}$. A unit clause contains exactly one literal. A conjunctive normal form formula C, for short formula, is a finite conjunction of different clauses and is considered as a set of these clauses $C = \{c_1, \ldots, c_m\}$. A formula can also be empty which is denoted as $\varnothing$. Let CNF be the collection of all formulas. For a formula C (clause c), by $V(C)$ ($V(c)$) denote the set of variables occurring in C (c). Let CNF_+ denote that part of CNF without occurrences of negated variables. A formula $C \in \mathrm{CNF}$ is called linear if each pair $c_i, c_j \in C$, $i \neq j$, satisfies $|V(c_i) \cap V(c_j)| \leq 1$. By LCNF the class of linear formulas is denoted. For a finite set M, let 2^M denote its powerset. For a positive integer n, let $[n] := \{1, \ldots, n\}$. As usual iff means if and only if. Given $C \in \mathrm{CNF}$, SAT asks whether there is a truth assignment $t : V(C) \to \{0,1\}$ such that there is no $c \in C$ all literals of which are set to 0. Such an assignment is called a model of C. Let $\mathrm{SAT} \subseteq \mathrm{CNF}$ denote the collection of all formulas admitting a model, and $\mathrm{UNSAT} := \mathrm{CNF} \setminus \mathrm{SAT}$. Clauses containing a complemented pair of literals are always satisfied. Hence, it is assumed throughout that clauses only contain literals over different variables. By $\mathcal{I} \subset \mathrm{UNSAT}$ we denote the class of minimal unsatisfiable formulas [1]. For a set V of propositional variables V, let c^X be the clause obtained from c by complementing all variables in $X \cap V(c)$, where X is an arbitrary subset of V, for short we set $c^\gamma := c^{V(c)}$, and further $c^\emptyset := c$. This flipping operation induces an action on CNF by observing that $\{c\} \in \mathrm{CNF}$: For $C = \{c_1, \ldots, c_m\} \in \mathrm{CNF}$ and $X \in 2^V$ let $C^X := \{c_1^X, \ldots, c_m^X\} =: C^X \in \mathrm{CNF}$. Again set $C^\gamma := C^{V(C)}$ in case that all variables in C are flipped, and $C^\emptyset := C$. Thus formally we obtain the G_V-action of the abelian group $G_V := (2^V, \oplus)$ with neutral element $\emptyset$ on CNF [13]. By $\mathcal{O}(C) := \{C^X : X \in G_V\}$ denote the (G_V-)orbit of C in CNF. The hyperedge set $B(C)$ of the base hypergraph $\mathcal{H}(C) = (V(C), B(C))$ assigned to a formula $C \in \mathrm{CNF}$ is defined as $B(C) := \{V(c) : c \in C\} \in \mathrm{CNF}_+$. As introduced in [11] the collection of all clauses c such that $V(c) = b$, for a fixed $b \in B(C)$, is the fibre C_b of C over b yielding the fibre-decomposition $\bigcup_{b \in B(C)} C_b$ of C. Conversely, a hypergraph $\mathcal{H} = (V, B)$ appears as a base hypergraph if its vertices $x \in V$ are regarded as

*Mathematics Group, Department 4, HTW Berlin, D-10313 Berlin, Germany, Email: porschen@htw-berlin.de.

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

Boolean variables such that for every $x \in V$ there is a (hyper)edge $b \in B$ containing x. By $W_b := \{c : V(c) = b\}$ denote the collection of all possible clauses over a fixed $b \in B$. By definition, a hypergraph $\mathcal{H} = (V, B)$ is linear if $|b \cap b'| \leq 1$, for all distinct $b, b' \in B$, and $\mathcal{H}$ is exact linear if $\leq$ above is replaced with $=$. A hypergraph $\mathcal{H} = (V, B)$ is called loopless if $|b| \geq 2$, for all $b \in B$ [3]. Given a hypergraph $\mathcal{H}$, denote its vertex set by $V(\mathcal{H})$ and its edge set by $B(\mathcal{H})$, if $b \in B(\mathcal{H})$ then $\mathcal{H} \setminus \{b\}$ means the hypergraph obtained from $\mathcal{H}$ by deleting edge b from $B(\mathcal{H})$. Observe that the base hypergraph $\mathcal{H}(C)$ is linear if the formula C is linear. Moreover $\mathcal{H}(C)$ is loopless if C is free of unit clauses. The intersection graph of $\mathcal{H} = (V, B)$ gets a vertex for each $b \in B$ and there is exactly one edge joining a pair of vertices $b \neq b'$ iff $b \cap b' \neq \emptyset$. A hypergraph $\mathcal{H}$ is called connected if its intersection graph is connected in the usual sense. A hypergraph is called Sperner if no hyperedge is contained in any other hyperedge of $\mathcal{H}$ [3]. Note that every loopless linear hypergraph is Sperner. The set of all clauses over $\mathcal{H}$ is $K_{\mathcal{H}} := \bigcup_{b \in B} W_b$. A $\mathcal{H}$-based formula is a subset $C \subseteq K_{\mathcal{H}}$ such that $C_b := C \cap W_b \neq \emptyset$, for every $b \in B$. For a $\mathcal{H}$-based $C \subseteq K_{\mathcal{H}}$, let $\bar{C} := K_{\mathcal{H}} \setminus C$ be its complement formula. A fibre-transversal of $K_{\mathcal{H}}$ is a $\mathcal{H}$-based formula $F \subset K_{\mathcal{H}}$ such that $|F \cap W_b| = 1$, for every $b \in B$, this clause is denoted as $F(b)$. By $\mathcal{F}(K_{\mathcal{H}})$ denote the set of all fibre-transversals of $K_{\mathcal{H}}$. Observe that, given a linear base hypergraph $\mathcal{H}$ then every fibre-transversal $F \in \mathcal{F}(K_{\mathcal{H}})$ is linear. However, a linear formula with complementary unit clauses is no fibre-transversal over its base hypergraph. A compatible fibre-transversal is defined by the property that $\bigcup_{b \in B} F(b) \in W_V$. $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ is the set of all compatible fibre-transversals of $K_{\mathcal{H}}$. A fibre-transversal F is diagonal if $F \cap F' \neq \emptyset$, for all $F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. Let $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ be the set of all diagonal fibre-transversals of $K_{\mathcal{H}}$. A fibre-transversal F of $C \subseteq K_{\mathcal{H}}$ contains exactly one clause of each fibre C_b of C. The collection of all fibre-transversals of C is denoted as $\mathcal{F}(C)$, and $\mathcal{F}_{\text{diag}}(C) := \mathcal{F}(C) \cap \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. A base hypergraph $\mathcal{H} = (V, B)$ is called diagonal iff $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \emptyset$, and it is called strictly diagonal if for every $C \subset K_{\mathcal{H}}$ with $B(C) = B = B(\bar{C})$ one has the equivalence $C \in \text{UNSAT} \Leftrightarrow \mathcal{F}_{\text{diag}}(C) \neq \emptyset$ [12].

3 A Bifurcation Concept

For a base hypergraph $\mathcal{H} = (V, B)$, let the integer $\delta(\mathcal{H}) \geq 0$ denote the cardinality of the orbit space $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})/G_V$. Denoting the number of orbits in $\mathcal{F}(K_{\mathcal{H}})$ by $\omega(\mathcal{H})$, and $\beta(\mathcal{H}) := \sum_{b \in B} |b| - |V| \geq 0$ one has $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})} \geq 1$, $|\mathcal{F}(K_{\mathcal{H}})| = \omega(\mathcal{H}) 2^{|V|}$, $|\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})| = \delta(\mathcal{H}) 2^{|V|}$, and that G_V acts transitively on $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ which all is shown in [13]. Further, let $\rho(\mathcal{H})$ denote the number of orbits of all fibre-transversals in $\mathcal{F}(K_{\mathcal{H}})$ which are neither compatible nor diagonal.

Lemma 1 *For integer $r \geq 2$, let $\mathcal{H}_i = (V_i, B_i)$, $i \in [r]$, be base hypergraphs with $V_i \cap V_j = \emptyset$, $i, j \in [r], i \neq j$. Let $\mathcal{H} = \bigcup_{i=1}^{r} \mathcal{H}_i$ then:*
(i) $\omega(\mathcal{H}) = \prod_{i=1}^{r} \omega(\mathcal{H}_i)$,
(ii) $\delta(\mathcal{H}) = \prod_{i=1}^{r} \omega(\mathcal{H}_i) - \prod_{i=1}^{r} (\omega(\mathcal{H}_i) - \delta(\mathcal{H}_i))$,
(iii) $\rho(\mathcal{H}) = -1 + \prod_{i=1}^{r} (1 + \rho(\mathcal{H}_i))$.

Proof. Clearly $V(\mathcal{H}) = \bigcup_{i=1}^{r} V_i$ as disjoint union and by assumption one has $B_i \cap B_j = \emptyset$, $i, j \in [r], i \neq j$. Thus also $B(\mathcal{H}) = \bigcup_{i=1}^{r} B_i$ as disjoint union implying $\beta(\mathcal{H}) = \sum_{b \in B(\mathcal{H})} |b| - |V(\mathcal{H})| = \sum_{i=1}^{r} [\sum_{b \in B_i} |b|] - \sum_{i=1}^{r} |V_i| = \sum_{i=1}^{r} \beta(\mathcal{H}_i)$ yielding (i) because of $\omega(\mathcal{H}) = 2^{\beta(\mathcal{H})}$. As G_V acts transitively on $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}_0})$ it follows that $\omega(\mathcal{H}_0) = 1 + \delta(\mathcal{H}_0) + \rho(\mathcal{H}_0)$, for every base hypergraph $\mathcal{H}_0$. Let us verify the remaining assertions by induction on r. Using (i) for $r = 2$ and setting $\omega(\mathcal{H}_i) =: \omega_i$, $\delta(\mathcal{H}_i) =: \delta_i$, $\rho(\mathcal{H}_i) =: \rho_i$, $i \in [r]$, we obtain

$$\begin{aligned} \omega(\mathcal{H}) &= 1 + \delta(\mathcal{H}) + \rho(\mathcal{H}) \\ &= (1 + \delta_1 + \rho_1)(1 + \delta_2 + \rho_2) \\ &= 1 + [\delta_1 \omega_2 + \delta_2 \omega_1 - \delta_1 \delta_2] + [\rho_1 + \rho_2 + \rho_1 \rho_2] \end{aligned}$$

As $\mathcal{H}_1$ is disconnected from $\mathcal{H}_2$, every satisfiable fibre-transversal over B_1 that is continuated over B_2 by any satisfiable fibre-transversal yields a satisfiable fibre-transversal over $B := B_1 \cup B_2$, and vice versa. Only, if taking an unsatisfiable fibre-transversal over B_1, respectively B_2, it yields an unsatisfiable fibre-transversal over B if it is continuated over B_2, respectively B_1. According to a result in [14] a fibre-transversal is unsatisfiable iff it is diagonal. Hence, for $r = 2$, from the first term in brackets in the third equation above it follows (ii), and from the second term in rectangular brackets it follows (iii) each by comparison with the corresponding terms in the first equation above. Now assume that (ii), (iii) are true for any fixed $r \geq 2$, and assume that $\mathcal{H}' = \mathcal{H} \cup \mathcal{H}_{r+1}$, where $\mathcal{H} = \bigcup_{i=1}^{r} \mathcal{H}_i$ of pairwise disjoint components $\mathcal{H}_i$, $i \in [r]$, and $\mathcal{H} \cup \mathcal{H}_{r+1} = \emptyset$. Let $\omega(\mathcal{H}_{r+1}) =: \omega_{r+1}$, $\delta(\mathcal{H}_{r+1}) =: \delta_{r+1}$, $\rho(\mathcal{H}_{r+1}) =: \rho_{r+1}$, then by the induction base $\delta(\mathcal{H}') = \delta(\mathcal{H})(\omega_{r+1} - \delta_{r+1}) + \delta_{r+1} \omega(\mathcal{H})$, and $\rho(\mathcal{H}') = \rho(\mathcal{H})(1 + \rho_{r+1}) + \rho_{r+1}$. From these equations (ii) and (iii) can be derived straightforwardly, for $r + 1$, by the corresponding induction hypotheses. □

For fixed $b \in B$ and $c, c' \in W_b$ let $Y(c, c') \in G_b$ be the unique transition member satisfying $c^{Y(c,c')} = c'$. There is a local criterion for disjoint orbits of fibre-transversals:

Lemma 2 *Let $\mathcal{H} = (V, B)$, $F, F' \in \mathcal{F}(K_{\mathcal{H}})$. Then $\mathcal{O}(F) \neq \mathcal{O}(F')$ iff there are $b, \tilde{b} \in B$ with $x \in b \cap \tilde{b}$ such that $x \in Y(F(b), F'(b)) \oplus Y(F(\tilde{b}), F'(\tilde{b}))$.*

Proof. For the if-part w.l.o.g. we may assume that $x \in F(b)$, $\bar{x} \in F'(b)$, $x \in F(\tilde{b})$, and $x \in F'(\tilde{b})$, which directly implies $\mathcal{O}(F) \neq \mathcal{O}(F')$. Regarding the only-if-part we assume that for all $b_1, b_2 \in B$ and all $x \in b_1 \cap b_2$ holds $x \notin Y_{b_1} \oplus Y_{b_2}$, where for any fixed

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$b \in B$ we set $Y_b := Y(F(b), F'(b))$. We claim that then for $X := \bigcup_{b \in B} Y_b$ the relation $F^X = F'$ is valid yielding the assertion by contraposition. To verify the claim observe that obviously one has $Y_b \subseteq X \cap b$, for all $b \in B$. Assume there is an $x \in X \cap b$ with $x \notin Y_b$ then there must exist a $b' \in B$ such that $x \in b'$ and $x \in Y_{b'}$ implying $x \in b \cap b'$ and $x \in Y_b \oplus Y_{b'}$ yielding a contradiction to the assumption. Thus for every $b \in B$ it holds that $Y_b = X \cap b$. Therefore using the fibre-decomposition one obtains

$$F^X = \bigcup_{b \in B} \{F(b)^{X \cap b}\} = \bigcup_{b \in B} \{F(b)^{Y_b}\} = \bigcup_{b \in B} \{F'(b)\} = F'$$

finishing the proof. □

On that basis the following notion turns out to be useful.

Definition 1 *Given not necessarily connected base hypergraphs $\mathcal{H}_0 = (V_0, B_0), \mathcal{H} = (V, B)$ then a triple (b_1, x, b_2), where $x \in V_0$, $b_1 \in B_0$, $b_2 \notin B_0$, $V = V_0 \cup b_2$, $B = B_0 \cup \{b_2\}$, and such that $x \in b_1 \cap b_2$, is called a* bifurcation. *Moreover $\mathcal{H}$ then is called a* bifurcation augmentation *of $\mathcal{H}_0$ at (b_1, x, b_2).*

A base hypergraph admitting compatible fibre-transversals only, hence consisting of pairwise disjoint edges, cannot be a bifurcation augmentation of any subhypergraph, more generally, one has:

Lemma 3 *Given a base hypergraph $\mathcal{H} = (V, B)$ then there exist $F_1, F_2 \in \mathcal{F}(K_\mathcal{H})$ at least one non-compatible such that $\mathcal{O}(F_1) \neq \mathcal{O}(F_2)$ iff there is $\mathcal{H}_0 \subsetneq \mathcal{H}$ such that $\mathcal{H}$ is a bifurcation augmentation of $\mathcal{H}_0$.*

Proof. Let $\mathcal{H}$ be a bifurcation augmentation of some proper subhypergraph $\mathcal{H}_0 = (V_0, B_0)$ at (b_1, x, b_2). Let $F_0 \in \mathcal{F}(K_{\mathcal{H}_0}) \neq \emptyset$ be chosen arbitrarily and define distinct fibre-transversals $F_1, F_2 \in \mathcal{F}(K_\mathcal{H})$ by $F_1 := F_0 \cup \{b_2\}$ and $F_2 := F_0 \cup \{b_2^{\{x\}}\}$. Observe that at least one of F_1, F_2 is non-compatible by construction. As $x \in b_1 \cap b_2$ and $b_1 \in B_0$ we have $l(x) \in F_1(b_1) \cap F_2(b_1)$ hence $x \notin Y(F_1(b_1), F_2(b_1))$ but by definition $x \in Y(F_1(b_2), F_2(b_2))$. Thus $x \in Y(F_1(b_1), F_2(b_1)) \oplus Y(F_1(b_2), F_2(b_2))$ implying the assertion due to Lemma 2. Conversely, assume by contraposition that there is no proper subhypergraph $\mathcal{H}_0$ of which $\mathcal{H}$ is a bifurcation augmentation. Then every distinct $b, b' \in B$ must be disjoint. Hence $\beta(\mathcal{H}) = 0$ implying $\omega(\mathcal{H}) = |\mathcal{F}(K_\mathcal{H})/G_V| = 1$ and $\mathcal{F}(K_\mathcal{H}) = \mathcal{F}_{\text{comp}}(K_\mathcal{H})$ yielding the assertion. □

As a direct consequence from the previous proof one has:

Corollary 1 *If a base hypergraph $\mathcal{H} = (V, B)$ is no bifurcation augmentation of a subhypergraph then $\delta(\mathcal{H}) = 0$.*

Lemma 4 *Given a base hypergraph $\mathcal{H} = (V, B)$ that is a bifurcation augmentation of a diagonal base hypergraph $\mathcal{H}_0$ then $\delta(\mathcal{H}) \geq 2\delta(\mathcal{H}_0)$.*

Proof. Let $F_0 \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$ and assume $\mathcal{H}$ is a bifurcation augmentation of $\mathcal{H}_0 = (V_0, B_0)$ at (b_1, x, b_2) where $b_1 \in B_0, b_2 \in B \setminus B_0$. Then $F_0' := F_0 \cup \{b_2\}$ and $F_0'' := F_0 \cup \{b_2^{\{x\}}\}$ are distinct diagonal fibre-transversals in $\mathcal{F}_{\text{diag}}(K_\mathcal{H})$ such that $\mathcal{O}(F_0') \neq \mathcal{O}(F_0'')$ according to the proof of Lemma 3. Hence every orbit $\mathcal{O}(F_0)$ in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})/G_{V_0}$ yields the distinct orbits $\mathcal{O}(F_0'), \mathcal{O}(F_0'') \in \mathcal{F}_{\text{diag}}(K_\mathcal{H})/G_V$ from which the assertion follows. □

For positive integer n, let $\mathcal{H}_n$ denote a base hypergraph such that $|V(\mathcal{H}_n)| = n$. Regarding the existence of base hypergraphs with small non-trivial fibre-transversal orbit spaces one has:

Theorem 1 *There is no $\mathcal{H}_1$ such that $\omega(\mathcal{H}_1) = 2$. There is a connected and linear $\mathcal{H}_2$ such that $\omega(\mathcal{H}_2) = 2$, but there neither is a loopless nor a Spernerian instance. For each positive integer $n \geq 3$ there is a connected, loopless, and linear (hence Spernerian) $\mathcal{H}_n$ with $\omega(\mathcal{H}_n) = 2$.*

Proof. Obviously there is no $\mathcal{H}_1$ admitting a bifurcation augmentation hence $\omega(\mathcal{H}_1) = 1$ by Lemma 3. Any $\mathcal{H}_2$ obviously satisfies $|B(\mathcal{H}_2)| \leq 3$. By Lemma 3 it admits a bifurcation augmentation only if $|B(\mathcal{H}_2)| \geq 2$ and there is a b with $|b| = 2$ and a loop $b' \subset b$. In case $|B| = 3$ one obtains $\beta(\mathcal{H}_2) = 2$ thus $\omega(\mathcal{H}_2) > 2$. Hence setting $V = \{x_1, x_2\}$ and $B = \{b_1, b_2\}$ with $b_1 = \{x_1\}$,$b_2 = \{x_1, x_2\}$ yields a connected linear instance $\mathcal{H} = (V, B)$ with $\omega(\mathcal{H}) = 2$. Now let $n \geq 3$, set $V = \{x_1, \ldots, x_n\}$, $B = \{b_1, b_2\}$, and let n be even. Setting $b_1 = \{x_1, \ldots, x_{n/2}\}$,$b_2 = \{x_{n/2}, x_{n/2+1}, \ldots, x_n\}$, one has $|b_1| + |b_2| = n/2 + (n/2 + 1)$ yielding the assertion. For n odd, we set $b_1 = \{x_1, \ldots, x_{\frac{n+1}{2}}\}$, $b_2 = \{x_{\frac{n+1}{2}}, x_{\frac{n+1}{2}+1}, \ldots, x_n\}$, and obtain uniform $|b_1| = |b_2| = (n+1)/2$ again yielding the assertion in this case. □

Observe that the hypergraphs constructed in the previous proof are smallest possible instances regarding the size of $B(\mathcal{H}_n)$ with the required properties.

4 A Hierarchy of Diagonal Hypergraphs

As shown in [13] there exist connected, loopless and linear diagonal base hypergraphs $\mathcal{H} = (V, B)$ such that $\delta(\mathcal{H}) > 1$. And due to Lemma 4 it inductively follows that there are also base hypergraphs with arbitrary large orbit spaces regarding their diagonal fibre-transversal sets. Therefore, one obtains a hierarchy in the class $\mathfrak{H}$ of connected base hypergraphs, namely for non-negative integer d, let $\mathfrak{H}_d := \{\mathcal{H} = (V, B) \in \mathfrak{H} : \delta(\mathcal{H}) \leq d\}$, then

$$\mathfrak{H}_0 \subseteq \mathfrak{H}_1 \subseteq \mathfrak{H}_2 \subseteq \mathfrak{H}_3 \subseteq \ldots$$

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

and $\mathfrak{H} = \bigcup_{d\geq 0} \hat{\mathfrak{H}}_d$ as a disjoint union where $\hat{\mathfrak{H}}_d := \mathfrak{H}_{d+1} \setminus \mathfrak{H}_d$, for every integer $d \geq 0$. A loopless exact linear hypergraph $\mathcal{H}$ does not admit diagonal fibre-transversals at all [15], hence it belongs to the class $\mathfrak{H}_0$ verifying that $\mathfrak{H}_d \neq \emptyset$, for every integer $d \geq 0$. The existence of strictly diagonal base hypergraphs needs to be proven. Slightly adapting the proof of Theorem 7 in [13] regarding the connectedness one has:

Theorem 2 $\mathfrak{H}_{\text{sdiag}} \neq \mathfrak{H}_{\text{diag}}$.

PROOF. One takes a loopless exact linear, hence obviously connected base hypergraph $\mathcal{H}_1 = (V_1, B_1)$ such that there is a $\mathcal{H}_1$-based unsatisfiable formula $C_1 \subset K_{\mathcal{H}_1}$ such that $\bar{C}_1$ also is $\mathcal{H}_1$-based. Such an instance exists on behalf of a result in [14]. Next note that a formula $F' \in \text{LCNF} \cap \text{UNSAT}$ free of unit clauses and such that $\mathcal{H}(F')$ is connected exists due to a result in [15]. Further one can assume that $V(F') \cap V_1 = \emptyset$. Take any fixed $x \in V_1$ and exchange all occurrences of a fixed $y \in V(F')$ in F' by x so obtaining F_2 from F', and $\mathcal{H}_2 := \mathcal{H}(F_2) =: (V_2, B_2)$ from $\mathcal{H}(F')$. Then specifically $F_2 \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_2})$ is ensured. Moreover $B_2 \cap B_1 = \varnothing$, but $\mathcal{H} := (V_1 \cup V_2, B_1 \cup B_2)$ is connected. The rest of the proof proceeds as outlined in [13] establishing that $\mathcal{H} \in \mathfrak{H}_{\text{diag}} \setminus \mathfrak{H}_{\text{sdiag}}$. □

Note that from the previous proof one can derive that the assertion of the theorem also is true for the loopless and linear subclasses. In [13] a connected diagonal base hypergraph is defined as *simple* if it belongs to $\hat{\mathfrak{H}}_1$. The next result already is stated in [13]; here it comes with the proof:

Lemma 5 *$\mathcal{H} = (V, B)$ is simple iff there is a G_V-equivariant bijection between $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ and $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$.*

PROOF. Assume that there is an equivariant bijection [17] with respect to the flipping operation between $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$ and $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. Then there is a bijection at all which is equivalent with $|\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})| = |\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})| = 2^{|V|}$, equivalent with $\delta(\mathcal{H}) = 1$, and equivalent with $\mathcal{H} \in \hat{\mathfrak{H}}_1$. Conversely, let $\mathcal{H}$ be simple then as above one has $|\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})| = |\mathcal{F}_{\text{comp}}(K_{\mathcal{H}})|$ corresponding to a bijection $f : \mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) \to \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. Now f can be ensured to be G_V-equivariant as follows. Fix an arbitrary pair $F_c \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$, and $F_d \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ setting $F_d =: f(F_c)$. As shown in [12], $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = \mathcal{O}(F)$, for any fixed $F \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}})$. So we can assume $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = \{F_c^X : X \in G_V\}$. Hence defining $f(F_c^X) := F_d^X = f(F_c)^X$, for every $X \in G_V$ means equivariance of f. □

In view of Corollary 1 and Lemma 4 a simple base hypergraph might be the bifurcation augmentation of a non-diagonal connected base hypergraph. Such an instance, connected and loopless, indeed is constructed below as stated in Theorem 8. However it appears not to be strictly diagonal. As a next result the existence of simple and also those of connected strictly diagonal base hypergraphs can be established at least in the case that loops are allowed.

Theorem 3 *For every integer $n \geq 2$, there is a connected $\mathcal{H} = (V, B) \in \hat{\mathfrak{H}}_1 \cap \mathfrak{H}_{\text{sdiag}} \neq \emptyset$ such that $|B| = n+1$, and $\omega(\mathcal{H}) = 2^n$.*

PROOF. Given $n \geq 2$ define the connected base hypergraph $\mathcal{H} = (V, B)$ through $V = \{x_1, x_2, \ldots, x_n\}$ and $B = \{b_1, b_2, \ldots, b_{n+1}\}$ where $b_i = \{x_i\}$, $i \in [n]$, and $b_{n+1} := V$. Then one has $\beta(\mathcal{H}) = n$ thus $\omega(\mathcal{H}) = 2^n$. Next we claim that $\mathcal{H}$ is simple. Indeed, one can set $\mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) = \{F_c^X : X \in G_V\}$ where $F_c(b_i) := b_i$, $i \in [n+1]$ using the clause-wise notation for a fibre-transversal. We define $F_d := f(F_c) := \{\{\bar{x}_i\} : i \in [n]\} \cup \{\{x_1, x_2, \ldots, x_n\}\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$, and claim that setting $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) = \{F_d^X : X \in G_V\}$ one directly obtains an equivariant bijection $f : \mathcal{F}_{\text{comp}}(K_{\mathcal{H}}) \to \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ where $F_d^X =: f(F_c^X) = [f(F_c)]^X$, for every $X \in G_V$. From this claim the assertion can be concluded using Lemma 5. To prove the claim, observe that there are exactly 2^n unsatisfiable fibre-transversals in $\mathcal{F}(K_{\mathcal{H}})$, namely by selecting all variables in X as negative literals in the unit clauses over b_i, $i \in [n]$, and simultaneously negating exactly those variables in $V \setminus X$ in the clause over b_{n+1}, for every $X \in G_V$. Indeed, for any different negation structure in the clauses of a fibre-transversal F there is a $j \in [n]$ such that the literal $l(x_j)$ is contained in the clause over b_j as well as in the clause over b_{n+1}. Hence $F \in \text{SAT}$ and therefore $F \notin \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$. Next observe that every $\mathcal{H}$-based formula $C \subseteq K_{\mathcal{H}}$ must contain exactly one clause from W_{b_i}, for every $i \in [n]$, otherwise $\bar{C}$ cannot be $\mathcal{H}$-based. So, as outlined previously, if $C \in \text{UNSAT}$ it also must contain a clause of $W_{b_{n+1}}$ completing these unit clauses to a member of $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ being a subset of C implying $\mathcal{H} \in \mathfrak{H}_{\text{sdiag}}$ and finishing the proof. □

However the next result tells us that the properties simple and strictly diagonal are not always valid together, at least in the case of disconnected base hypergraphs.

Theorem 4 *There are, not necessarily connected, base hypergraphs which are strictly diagonal and non-simple.*

PROOF. Let $n \geq 2$ and $\mathcal{H}_i = (V_i, B_i) \in \hat{\mathfrak{H}}_1 \cap \mathfrak{H}_{\text{sdiag}}$, $|V_i| = n$, $i \in [2]$, according to Theorem 3 and such that $V_1 \cap V_2 = \emptyset$. For $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2 =: (V, B)$ by Lemma 1 one has $\delta(\mathcal{H}) = 2^{n+1} - 1 > 1$ hence $\mathcal{H}$ is not simple. Suppose $\mathcal{H}$ was not strictly diagonal, then there is an unsatisfiable $C \subset K_{\mathcal{H}}$ with $B(C) = B = B(\bar{C})$ and such that $\mathcal{F}_{\text{diag}}(C) = \emptyset$. Clearly then there are $C_i \subseteq K_{\mathcal{H}_i}$, with $B(C_i) = B_i = B(\bar{C}_i)$, for $i \in [2]$, and such that

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$C = C_1 \cup C_2$ holds as disjoint union. Hence at least one of them, say C_1, must be unsatisfiable. And $\mathcal{F}_{\text{diag}}(C) = \emptyset$ implies $\mathcal{F}_{\text{diag}}(C_1) = \emptyset$ because any continuation of a diagonal fibre-transversal over B_1 to the whole of B remained diagonal. Therefore $\mathcal{H}_1$ cannot be strictly diagonal yielding a contradiction. □

Theorem 5 *For every positive integer d which is a power of 2, there is a not necessarily connected base hypergraph $\mathcal{H}$ such that $\delta(\mathcal{H}) = d$. Moreover, there is a not necessarily connected $\mathcal{H}$ such that for every fixed positive integer d there is an odd $\delta(\mathcal{H}) \geq d$.*

Proof. Let $d = 2^k$ then we prove the assertion by induction on k. For $k = 0$ take any member of $\hat{\mathfrak{H}}_1$ providing the induction base. Now let $\mathcal{H}_2 = (V, B)$ such that $\delta(\mathcal{H}_2) = 2^k$, for any fixed $k \geq 0$. On behalf of Theorem 1 there is $\mathcal{H}_1 = (V_1, B_1)$ such that $\omega(\mathcal{H}_1) = 2$, and $|V_1| \geq 3$. Moreover observe that $\mathcal{H}_1$ by construction is exact linear and loopless, hence $\delta(\mathcal{H}_1) = 0$. Assuming that $V \cap V_1 = \emptyset$ it is implied that $\mathcal{H}_1 \cup \mathcal{H}_2$ cannot be the bifurcation augmentation of a diagonal subhypergraph. Thus according to Lemma 1, (ii), for $r = 2$, it directly follows that $\delta(\mathcal{H}_1 \cup \mathcal{H}_2) = \delta(\mathcal{H}_1)\omega(\mathcal{H}_2) + \delta(\mathcal{H}_2)\omega(\mathcal{H}_1) - \delta(\mathcal{H}_1)\delta(\mathcal{H}_2)$ $= \delta(\mathcal{H}_2)\omega(\mathcal{H}_1) = 2^{k+1}$ yielding the first assertion. Let d be arbitrarily fixed, choose an integer $n \geq \log_2(d+1) - 1$ and let $\mathcal{H}_i = (V_i, B_i) \in \hat{\mathfrak{H}}_1 \cap \mathfrak{H}_{\text{sdiag}}$, $|V_i| = n$, $i \in [2]$, with $V_1 \cap V_2 = \emptyset$. Using Theorem 3 and Lemma 1 yields an odd $\delta(\mathcal{H}_1 \cup \mathcal{H}_2) = 2^{n+1} - 1 \geq d$. □

From Theorem 3 one can deduce directly:

Corollary 2 *There is a connected diagonal base hypergraph which is the bifurcation augmentation of a non-diagonal subhypergraph.*

Observe that a minimal diagonal base hypergraph always is connected. Moreover as proven in [13], for a strictly diagonal base hypergraph $\mathcal{H}$ which is minimal diagonal it holds that a $\mathcal{H}$-based formula $C \subset K_{\mathcal{H}}$ is minimal unsatisfiable iff it is a diagonal fibre-transversal of $K_{\mathcal{H}}$. Further one has:

Theorem 6 *Let $\mathcal{H}$ be a diagonal base hypergraph. Then $\mathcal{H}$ is minimal diagonal iff $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \subset \mathcal{I}$.*

Proof. For the only-if-part, using the assumption we can suppose, by contraposition, that there is $F \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \neq \emptyset$ which is not minimal unsatisfiable then there is $b \in B(\mathcal{H})$ such that $F \setminus \{F(b)\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$. Hence $\mathcal{H}_0 := \mathcal{H} \setminus \{b\}$ is a diagonal subhypergraph. Regarding the if-part observe that then $\mathcal{H}$ clearly is connected. Suppose there is $b \in B(\mathcal{H})$ such that $\mathcal{H}_0 := \mathcal{H} \setminus \{b\}$ is diagonal and let $F_0 \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$. Take any $c \in W_b$ then clearly $F' := F_0 \cup \{c\} \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \setminus \mathcal{I}$ providing a contradiction. □

In the simple case at least one has:

Theorem 7 *If $\mathcal{H} \in \hat{\mathfrak{H}}_1$ then $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}}) \subset \mathcal{I}$.*

Proof. Let $\mathcal{H} = (V, B)$ be simple and assume there is $F_1 \in \mathcal{F}_{\text{diag}}(K_{\mathcal{H}})$ that is non-minimal unsatisfiable then there is $b_2 \in B$ such that $F_1' := F_1 \setminus \{F_1(b_2)\} \in$ UNSAT and $\mathcal{H}_0 := \mathcal{H}(F_1')$ is diagonal. As $\mathcal{H}$ is connected there is $b_1 \in B \setminus \{b_2\}$ such that $b_1 \cap b_2 \neq \emptyset$. Let $x \in b_1 \cap b_2$ then $\mathcal{H}$ is a bifurcation augmentation of $\mathcal{H}_0$ at (b_1, x, b_2) establishing $\delta(\mathcal{H}) > 1$ according to Lemma 4 yielding a contradiction. □

As a direct consequence of Theorems 6, 7 it follows:

Corollary 3 *Every $\mathcal{H} \in \hat{\mathfrak{H}}_1$ is minimal diagonal.*

Lemma 6 *For $\mathcal{H} = (V, B)$ and arbitrary $b \in B$, let $F \in \mathcal{F}(K_{\mathcal{H}}) \cap \mathcal{I}$ then $(F \setminus \{F(b)\}) \cup \{c\} \in$ SAT for all $c \in W_b \setminus \{F(b)\}$.*

Proof. Let t be a model of $F \setminus \{F(b)\}$ which exists for each fixed $b \in B$ as $F \in \mathcal{I}$. Then $|Y(c, F(b))| \geq 1$ for every $c \in W_b \setminus \{F(b)\}$. Therefore as t necessarily sets all literals in $F(b)$ to 0, one has that t satisfies c via any literal in $Y(c, F(b))$ yielding $(F \setminus \{F(b)\}) \cup \{c\} \in$ SAT. □

Lemma 7 *Let $\mathcal{H} = (V, B)$ be a diagonal base hypergraph. If there is a loopless exact linear subhypergraph $\mathcal{H}_0 \subseteq \mathcal{H}$ such that every $b \in B(\mathcal{H}')$ has a variable not in $V(\mathcal{H}_0)$, where $\mathcal{H}' := \mathcal{H} \setminus \mathcal{H}_0$ then $\mathcal{H}$ cannot be strictly diagonal.*

Proof. Let $\mathcal{H}_0 = (V_0, B_0)$ be a loopless, exact linear subhypergraph of $\mathcal{H}$, then $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0}) = \emptyset$ according to [15]. Let $F_0 \in \mathcal{F}(K_{\mathcal{H}_0})$ be any non-compatible fibre-transversal then due to [11] $C_0 := K_{\mathcal{H}_0} \setminus F_0 \in$ UNSAT. Moreover C_0 and $\bar{C}_0$ are $\mathcal{H}_0$-based, so taking $F' \in \mathcal{F}_{\text{comp}}(K_{\mathcal{H}'})$ where $\mathcal{H}' := \mathcal{H} \setminus \mathcal{H}_0$ yields an unsatisfiable $\mathcal{H}$-based formula $C := C_0 \cup F' \in K_{\mathcal{H}}$ such that also $\bar{C}$ is $\mathcal{H}$-based. Suppose there is $F \in \mathcal{F}(C_0)$ such that $F \cup F' \in \mathcal{F}_{\text{diag}}(C)$. By assumption every $F(b)$ has a literal over a variable not in V_0 therefore F' can be satisfied as a compatible fibre-transversal only by variables not in F. Therefore F must be in $\mathcal{F}_{\text{diag}}(K_{\mathcal{H}_0})$ which is impossible. □

The next result shows that there are loopless members in $\hat{\mathfrak{H}}_1$, and that this class does not coincide with that of connected strictly diagonal base hypergraphs.

Theorem 8 *There are connected, loopless and linear minimal diagonal base hypergraphs which are simple, but not strictly diagonal.*

Proof. Let $\mathcal{H} = (V, B)$ be such that $B = \{b_i : i \in [6]\}$, $V = \bigcup_{i \in [6]} b_i$, and $b_i = \{x, y_i\}$, $i \in [2]$, $b_i = \{x, y_i\}$,

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$i \in \{3,4\}$, and $b_5 = \{y_1, y_2\}$, $b_6 = \{y_3, y_4\}$. Clearly, $\mathcal{H}$ is loopless and linear, and also connected. It suffices to show that $\delta(\mathcal{H}) = 1$ meaning $\mathcal{H} \in \hat{\mathfrak{H}}_1$ and implying $\mathcal{H}$ is minimal diagonal according to Corollary 3. Observe that $\delta(\mathcal{H}) \geq 1$ because $F = \{c_i : i \in [6]\} \in \mathcal{F}_{\mathrm{diag}}(K_{\mathcal{H}})$ with $c_i = \{x, y_i\}$, $i \in [2]$, $c_i = \{\bar{x}, y_i\}$, $i \in \{3,4\}$, and $c_5 = \{\bar{y}_1, \bar{y}_2\}$, $c_6 = \{\bar{y}_3, \bar{y}_4\}$. Obviously $F \in \mathcal{I}$. On behalf of Lemma 2 it is clear that an orbit distinct from $\mathcal{O}(F)$ can be obtained only by a local transformation as stated in that Lemma. As F is minimal unsatisfiable, Lemma 6 implies that at least two such local transformations must be applied to F in order to obtain an unsatisfiable formula. Relying on two bifurcations with respect to one of the variables $\{y_i : i \in [4]\}$ only an unsatisfiable formula in the orbit of F can be produced, as these variables occur only twice and as mutually complemented literals. If one tries x only satisfiable formulas appear if they are members of distinct orbits. Also combining bifurcations using x and one of y_i, $i \in [4]$. Thus $\delta(\mathcal{H}) = 1$. For the second assertion observe that $\mathcal{H}_0 = (V_0, B_0)$ with $V_0 = \{x, y_1, y_2\}$, $B_0 = \{b_1, b_2, b_5\}$ is a loopless, exact linear subhypergraph of $\mathcal{H}$. Further for $\mathcal{H}' = \mathcal{H} \setminus \mathcal{H}_0$ one has $B(\mathcal{H}') = \{b_3, b_4, b_6\}$ and $V(\mathcal{H}') = \{x, y_3, y_4\}$. Thus each $b \in B(\mathcal{H}')$ contains a member not in V_0, so by Lemma 7 it follows that $\mathcal{H}$ cannot be strictly diagonal. □

5 Open Problems

There occurs an existence problem whether $\hat{\mathfrak{H}}_d \neq \emptyset$, for every $d > 1$. Specifically, in the case $d \geq 2$ the existence of members that are loopless and linear or at least Sperner needs to be clarified. In that context one has to reveal the specific structure of simple hypergraphs. Finally, the existence problem for loopless strictly diagonal hypergraphs is still open, specifically concerning those that are Sperner or even linear. Moreover, so far we know only to construct simple instances with loops meeting the property in Corollary 2, and the conjecture here is that there are no base hypergraphs in $\hat{\mathfrak{H}}_d$, for any $d > 1$, which appear as a bifurcation augmentation of a non-diagonal subhypergraph. In view of Theorem 5 it would be desirable to guarantee the existence also of connected base hypergraphs admitting the specified properties. Further it is open whether the classes of minimal diagonal and that of connected simple base hypergraphs coincide. Also it remains open whether a loopless and Sperner or even linear base hypergraph $\mathcal{H} = (V, B)$ exists being simple or connected strictly diagonal such that there are $b \in B$ with $|b| \geq 3$.

References

[1] Aharoni, R., Linial, N., "Minimal non two-colorable hypergraphs and minimal unsatisfiable formulas," *J. Combin. Theory A* pp. 196-204, 43/1986.

[2] Aspvall, B., Plass, M.R., Tarjan, R.E., "A linear-time algorithm for testing the truth of certain quantified Boolean formulas," *Information Process. Lett.* pp. 121-123, 8/1979.

[3] Berge, C., *Hypergraphs*, North-Holland, Amsterdam, 1989.

[4] Boros, E., Crama, Y., Hammer, P.L., "Polynomial-time inference of all valid implications for Horn and related formulae," *Annals of Math. Artif. Intellig.* pp. 21-32, 1/1990.

[5] Boros, E., Crama, Y., Hammer, P.L., Sun, X., "Recognition of q-Horn formulae in linear time," *Discrete Appl. Math.* pp. 1-13, 55/1994

[6] Cook, S.A., "The Complexity of Theorem Proving Procedures," *3rd ACM Symposium on Theory of Computing* pp. 151-158, 1971.

[7] Franco, J., VanGelder, A., "A perspective on certain polynomial-time solvable classes of satisfiability," *Discrete Appl. Math.* pp. 177-214, 125/2003.

[8] Knuth, D.E., "Nested satisfiability," *Acta Informatica* pp. 1-6, 28/1990.

[9] Kratochvil, J., Krivanek, M., "Satisfiability of co-nested formulas," *Acta Informatica* pp. 397-403, 30/1993.

[10] Lewis, H.R., "Renaming a Set of Clauses as a Horn Set," *J. ACM* pp. 134-135, 25/1978.

[11] Porschen, S., "A CNF Formula Hierarchy over the Hypercube," Proc. AI 2007, *LNAI* pp. 234-243, 4830/2007.

[12] Porschen, S., "Structural Aspects of Propositional SAT," Proc. IMECS 2016, Hong Kong, pp. 126-131.

[13] Porschen, S., "Base Hypergraphs and Orbits of CNF Formulas," Proc. IMECS 2018, Hong Kong, pp. 106-111.

[14] Porschen, S., Speckenmeyer, E., "A CNF class generalizing Exact Linear Formulas," Proc. SAT 2008, *LNCS* pp. 231-245, 2008.

[15] Porschen, S., Speckenmeyer, E., Zhao, X., "Linear CNF formulas and satisfiability," *Discrete Appl. Math.* pp. 1046-1068, 157/2009.

[16] Schlipf, J., Annexstein, F.S., Franco, J., Swaminathan, R.P., "On finding solutions for extended Horn formulas," *Information Process. Lett.* pp. 133-137, 54/1995.

[17] Spannier, E.H., *Algebraic Topology*, McGraw-Hill, New York, 1966.

[18] Tovey, C.A., "A Simplified NP-Complete Satisfiability Problem", *Discrete Appl. Math.* pp. 85-89, 8/1984.

ISBN: 978-988-14048-5-5
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)