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Numerical Simulation of a Nonlinear Thin Fluid Film Flow Velocity Model of a Third Grade Fluid on a Moving Belt using Finite Difference Method with Newton Iterative Scheme

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Abstract—**A thin film flow problem on a moving belt has been useful in scientific, engineering, biological and biomedical problems. A thin film flow velocity on a moving belt can be modeled by a nonlinear differential equation. The model is provides the film flow velocity in each thickness layers. In this research, a finite difference method and a Newton iterative method are used to approximate the solutions of the nonlinear thin fluid film flow velocity model. Their numerical simulations of a thin film flow velocity of a third grade fluid in on a moving belt with varied physical parameters are investigated. The proposed numerical techniques give good agreement approximated solutions in several moving belts speed levels.**

Keywords: thin fluid film, flow velocity, numerical simulation, iterative method

1 Introduction

Nowadays, the Non-Newtonian fluid has a key role in the physical system. Physicist use of non-Newtonian fluid to adapt in the several ways such as mechanical engineering and manufacturing process which mean non-Newtonian fluid can lead many benefits in the future and so on. The Non-Newtonian fluid is referring a fluid whose viscosity is variable based on the applied stress or force. People familiar with a non-Newtonian fluid in every day. Behavior of Newtonian fluids like water can be described exclusively by temperature and pressure. The research of Non-Newtonian fluid is bring up some interesting result when combined theory with mechanical engineering. Most of the scientific problems in fluid mechanics are modeled by nonlinear differential equations. It is well known that exact solutions of these nonlinear boundary value problems are difficult to obtain. Therefore, numerical solutions methods and analytical solutions methods are used to handle such type of problems. In [1], the researchers studied thin film flow of a third grade fluid on a moving belt by He's homotopy perturbation method. In [3], the researchers develop a generalized approximation method (GAM) to obtain a solution of a thin film flow of a third grade fluid on a moving belt. The GAM generates a monotone sequence of solutions of linear problems. The sequence of solutions of linear problems converges monotonically and rapidly to a solution of the original nonlinear problem. We present some numerical simulations to illustrate and confirm our results. In [5], the researchers studied a thin film flow of a third grade fluid on a moving belt using a powerful and relatively new approximate analytical technique known as optimal homotopy asymptotic method (OHAM).
In this research, a couple of a finite difference method an Newton iterative method is used to approximate the solutions of a nonlinear governing equation. The proposed numerical techniques give good agreement approximated solutions in several moving belts speed levels.

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2 The thin fluid film flow velocity model of a third grade fluid on a moving belt.

The thin fluid film flow velocity of a third grade fluid on parallel moving belts is governed by the following nonlinear boundary value problem. [1, 2, 4, 5].

$$\frac{d^2v}{dx^2}+\frac{6(\beta_2+\beta_3)}{\mu}\Big(\frac{dv}{dx}\Big)^2\frac{d^2v}{dx^2}-\frac{\rho g}{\mu}=0,\ \text{for all } 0\le x\le\delta, \tag{1}$$

subject to the boundary conditions

$$v(0)=u_l, \tag{2}$$

$$v^{'}(\delta)=u_r, \tag{3}$$

where v is the fluid velocity (m/s), ρ is the density (Kg/m^3), μ is the dynamic viscosity ($Pa{\cdot}s$), β_2,β_3 are the material constants of the third grade fluid (non-units), g is the gravity acceleration (m/s^2), δ is the uniform thickness of the fluid film (m), and u_l,u_r are the belt speed and the rate of change of fluid flow velocity on ended layer.

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We introduce the following dimensionless variables,

$$x^* = \frac{x}{\delta}, \quad (4)$$
$$v^* = \frac{v}{u_l}, \quad (5)$$
$$\beta = \frac{(\beta_2+\beta_3)\ell}{\mu\delta^2}, \quad (6)$$
$$m = \frac{\rho g}{\mu \ell}\delta^2, \quad (7)$$

where $\ell = \max\{u_l, u_r\}$. From Eqs.(4)-(7), we obtain the dimensionless form as (for simplicity we removed *),

$$\frac{d^2v}{dx^2} + 6\beta\Big(\frac{dv}{dx}\Big)^2\frac{d^2v}{dx^2} - m = 0, \text{ for all } 0 \le x \le 1, \quad (8)$$

subject to the boundary conditions

$$v(0) = v_l, \quad (9)$$
$$v^{'}(1) = v_r, \quad (10)$$

where v_l is the nondimensional moving belts speed and v_r is the rate of change of the fluid velocity on the ended layer.

3 Numerical Techniques

3.1 Finite difference method for a thin fluid film flow velocity model of a third grade fluid on moving belt ended layer

We will approximate the solution by using finite difference methods for nonlinear boundary value problem with Dirichlet boundary conditions Eq.(8)-(10) that can be written in a compact form as below,

$$v'' = \frac{m}{1+6\beta(v')^2}, \text{ for all } 0 \le x \le 1, \quad (11)$$

We divide $[0,1]$ into $N+1$ subintervals that endpoints are at $x_i = a + ih$, for all $i = 0, 1, \ldots, N+1$, where $h = 1/(N+1)$. By using the centered finite difference method [6], the finite difference equation can be written as follows,

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = \frac{m}{1+6\beta\Big(\frac{v_{i+1}-v_{i-1}}{2h}\Big)^2}, \quad (12)$$

for all $i = 1, 2, \ldots, N$. It follows that

$$v_{i+1} + \frac{3}{2}\frac{\beta}{h^2}(v_{i+1}^3) - \frac{3}{2}\frac{\beta}{h^2}(v_{i+1}^2 v_{i-1}) - \frac{3}{2}\frac{\beta}{h^2}(v_{i+1}v_{i-1}^2) -2v_i - 3\frac{\beta}{h^2}(v_i v_{i+1}^2) + 6\frac{\beta}{h^2}(v_{i-1}v_i v_{i+1}) - 3\frac{\beta}{h^2}(v_{i-1}^2 v_i) +v_{i-1} + \frac{3}{2}\frac{\beta}{h^2}(v_{i-1}^3) = mh^2, \quad (13)$$

for all $i = 1, 2, \ldots, N$. For $i = 1$, plug the known value of the left boundary $v_0 = v_l$ to Eq.(13) on the left hand side, we obtain

$$(-2 - 3\frac{\beta}{h^2})v_1 + (1 - \frac{3}{2}\frac{\beta}{h^2})v_2 - \frac{3}{2}\frac{\beta}{h^2}v_2^2 - \frac{3}{2}\frac{\beta}{h^2}v_1v_2^2 +6\frac{\beta}{h^2}v_1v_2 + \frac{3}{2}\frac{\beta}{h^2}v_2^3 = mh^2 - v_l - \frac{3}{2}\frac{\beta}{h^2}. \quad (14)$$

For $i = 2, 3, \ldots, N-1$, Eq.(13) becomes

$$v_{i-1} + \frac{3}{2}\frac{\beta}{h^2}v_{i-1}^3 - \frac{3}{2}\frac{\beta}{h^2}v_{i-1}^2 v_{i+1} - \frac{3}{2}\frac{\beta}{h^2}v_{i-1}v_{i+1}^2 - 2v_i -\frac{3}{2}\frac{\beta}{h^2}v_i v_{i+1}^2 - \frac{3}{2}\frac{\beta}{h^2}v_{i-1}^2 v_i + 6\frac{\beta}{h^2}v_{i-1}v_i v_{i+1} +v_{i+1} + \frac{3}{2}\frac{\beta}{h^2}v_{i+1}^3 = mh^2. \quad (15)$$

To modify the finite difference method to a Neumann boundary conditions, $v^{'}(1) = v_r$, Eq.(13). For $i = N$,

$$2hk + v_{N-1} + \frac{3}{2}\frac{\beta}{h^2}(2hk + V_{N-1})^3 -\frac{3}{2}\frac{\beta}{h^2}((2hk + v_{N-1})^2 v_{N-1}) - \frac{3}{2}\frac{\beta}{h^2}((2hk + V_{N-1})V_{N-1}^2) -2v_N - 3\frac{\beta}{h^2}(v_N(2hk + v_{N-1})^2) +6\frac{\beta}{h^2}(v_N V_{N-1}(2hk + V_{N-1})) -3\frac{\beta}{h^2} + V_{N-1} + \frac{3}{2}\frac{\beta}{h^2}(v_{N-1})^3 = mh^2. \quad (16)$$

3.2 Newton Iterative method for nonlinear finite difference equations

Apply the Newton Iterative to the system of simutaneous on nonlinear equation Eqs.(14-16) that can be written in a vector form as
$F(V) = (f_1(v_1, \ldots, v_N), f_2(v_1, \ldots, v_N), \ldots, f_N(v_1, \ldots, v_N))^t$,

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where

$$\begin{aligned} f_1(v_1,\ldots,v_N) &:= (-2-3\frac{\beta}{h^2})v_1+(1-\frac{3}{2}\frac{\beta}{h^2})v_2 \\ &-\frac{3}{2}\frac{\beta}{h^2}v_2^2-\frac{3}{2}\frac{\beta}{h^2}v_1v_2^2+6\frac{\beta}{h^2}v_1v_2 \\ &+\frac{3}{2}\frac{\beta}{h^2}v_2^3-(mh^2-v_l-\frac{3}{2}\frac{\beta}{h^2}), \quad (17) \\ f_i(v_1,\ldots,v_N) &:= v_{i-1}+\frac{3}{2}\frac{\beta}{h^2}v_{i-1}^3-\frac{3}{2}\frac{\beta}{h^2}v_{i-1}^2v_{i+1} \\ &-\frac{3}{2}\frac{\beta}{h^2}v_{i-1}v_{i+1}^2-2v_i-\frac{3}{2}\frac{\beta}{h^2}v_iv_{i+1}^2 \\ &-\frac{3}{2}\frac{\beta}{h^2}v_{i-1}^2v_i+6\frac{\beta}{h^2}v_{i-1}v_iv_{i+1}+v_{i+1} \\ &+\frac{3}{2}\frac{\beta}{h^2}v_{i+1}^3-mh^2, \\ &\text{for all } i=2,3,\ldots,N-1, \quad (18) \\ f_N(v_1,\ldots,v_N) &:= 2hk+v_{N-1}+\frac{3}{2}\frac{\beta}{h^2}(2hk+V_{N-1})^3 \\ &-\frac{3}{2}\frac{\beta}{h^2}((2hk+v_{N-1})^2v_{N-1}) \\ &-\frac{3}{2}\frac{\beta}{h^2}((2hk+V_{N-1})V_{N-1}^2)-2v_N \\ &-3\frac{\beta}{h^2}(v_N(2hk+v_{N-1})^2) \\ &+6\frac{\beta}{h^2}(v_NV_{N-1}(2hk+V_{N-1})) \\ &-3\frac{\beta}{h^2}+V_{N-1}+\frac{3}{2}\frac{\beta}{h^2}(v_{N-1})^3 \\ &-mh^2. \quad (19) \end{aligned}$$

The Jacobian matrix $J(V)$ for the system of Eqs.(17-19) is given by

$$J(V)=\begin{bmatrix} \partial f_1/\partial v_1 & \partial f_1/\partial v_2 & \partial f_1/\partial v_3 & \cdots & \partial f_1/\partial v_m \\ \partial f_2/\partial v_1 & \partial f_2/\partial v_2 & \partial f_2/\partial v_3 & \cdots & \partial f_2/\partial v_m \\ \partial f_3/\partial v_1 & \partial f_3/\partial v_2 & \partial f_3/\partial v_3 & \cdots & \partial f_3/\partial v_m \\ & \vdots & & \vdots & \\ \partial f_m/\partial v_1 & \partial f_m/\partial v_2 & \partial f_m/\partial v_3 & \cdots & \partial f_m/\partial v_m \end{bmatrix}$$

Choosing that $V^0 = (v_1^{(0)},\ldots,v_N^{(0)})^t$, we can obtain $F(V^{(0)})$ and $J(V^{(0)})$. Solving the linear system,

$$J(V^{(0)})U^{(0)} = -F(V^{(0)}), \quad (20)$$

we will obtain $U^{(0)}$. Then

$$V^{(1)} = V^{(0)} + U^{(0)}. \quad (21)$$

Continuing for $k = 2,3,\ldots$, we have

$$V^{(k)} = V^{(k-1)} + U^{(k-1)}, \quad (22)$$

where $U^{(k-1)} = -(J(v_1^{(k-1)},\ldots,v_N^{(k-1)}))^{-1}F(v_1^{(k-1)},\ldots,v_N^{(k-1)})$.

4 Numerical Experiment

Assuming that there is a moving belt in a third grade fluid basin. The basin is filled by the engine oil SAE 15W-40 with the viscosity, 31.350 mm^2/s and the density, 0.8477 g/cm^3 at $70°C$. The material constant of the engine oil is 1.5675×10^{-5}. The moving belt has speed 1 m/s and the uniform thickness of fluid film is 0.001 m. We will consider physical parameters in 4 cases as show in Table 1. By using the finite difference equations Eqs.(14-16) with the Newton iterative techniques Eqs.(17-19), we can obtain the thin fluid film flow velocity in each layers. If the speed of moving belt is uniform and the rates of change of fluid film flow velocity over the ended layer are nonnegative due to external force, the approximated flow velocity in each layers are show in Table 1 and Figure 1. However, the cases of rates of change of fluid film flow velocity over the ended layer are negative due to gravity force and some external force are also investigated as show in Table 2. The comparison of thin film flow velocity when left belt speed are difference is shown in Table 3.

Table 1: Comparison of nondimensional thin film flow velocity in each nonnegative rates of change with $m = 0.265$ and $\beta = 0.5$.

$x \setminus v_r$	0.0000	0.00025	0.0005	0.0010
0.0	1.0000	1.0000	1.0000	1.0000
0.1	0.9762	0.9762	0.9762	0.9763
0.2	0.9546	0.9547	0.9547	0.9548
0.3	0.9355	0.9355	0.9356	0.9357
0.4	0.9187	0.9188	0.9189	0.9191
0.5	0.9044	0.9045	0.9047	0.9049
0.6	0.8927	0.8928	0.8929	0.8932
0.7	0.8835	0.8836	0.8838	0.8841
0.8	0.8769	0.8771	0.8772	0.8776
0.9	0.8729	0.8731	0.8733	0.8737
1.0	0.8716	0.8718	0.8720	0.8725

Table 2: Comparison of nondimensional thin film flow velocity in each negative rates of change with $m = 0.265$ and $\beta = 0.5$.

$x \setminus v_r$	-0.001	-0.0005	-0.00025
0.0	1.0000	1.0000	1.0000
0.1	0.9761	0.9761	0.9762
0.2	0.9545	0.9546	0.9546
0.3	0.9352	0.9353	0.9354
0.4	0.9184	0.9185	0.9186
0.5	0.9040	0.9042	0.9043
0.6	0.8921	0.8924	0.8925
0.7	0.8828	0.8831	0.8833
0.8	0.8761	0.8765	0.8767
0.9	0.8721	0.8725	0.8727
1.0	0.8706	0.8711	0.8713

5 Conclusion

The finite difference method for a nonlinear differential equation with the Newton iterative technique is employed to simulate the thin fluid film flow velocity of a third grade fluid on a moving belt. The results are shown for the thin fluid film flow velocity of motor oil when external forces are positive or negative. If the external force is

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Table 3: Comparison of nondimensional thin film flow velocity in each left speed belt with $m = 0.265$ and $\beta = 0.5$.

$x \setminus v_l$	1.0000	0.7500	0.5000	0.2500
0.0	1.0000	0.7500	0.5000	0.2500
0.1	0.9762	0.7262	0.4762	0.2262
0.2	0.9546	0.7046	0.4546	0.2046
0.3	0.9355	0.6855	0.4355	0.1855
0.4	0.9187	0.6687	0.4187	0.1687
0.5	0.9044	0.6544	0.4044	0.1544
0.6	0.8927	0.6427	0.3927	0.1427
0.7	0.8835	0.6335	0.3835	0.1335
0.8	0.8769	0.6269	0.3769	0.1269
0.9	0.8729	0.6229	0.3729	0.1229
1.0	0.8716	0.6216	0.3716	0.1216

increased, the fluid film flow velocities at all layers are also increased.

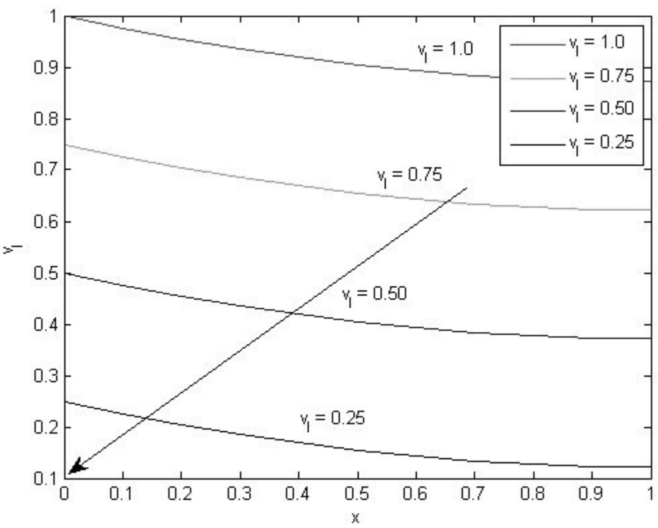

Figure 1: Comparison of thin film flow velocity ($\beta = 0.5, m = 2.65$)

References

[1] Siddiqui,A.M., Mahmood, R. and Ghori,Q.K. "Thin film flow of a third grade fluid on a moving belt by homotopy perturbation method", *International Journal of Nonlinear Sciences and Numerical Simulation*, V7, N1, pp. 7-14, 2006.

[2] Siddiqui, A.M., Mahmood,R. and Ghori,Q.K., "Homotopy perturbation method for thin film flow of a third grade fluid down an inclined plane", *Chaos, Solitons & Fractals*, V35, N1, pp. 140-147, 2008.

[3] Khan,R.A., "Generalized approximation method and a thin film flow of a third grade fluid on a moving belt", *Computational Mathematics and Modeling*, V21, N1, 2010.

[4] Siddiqui,A.M., Farooq, A.A., Haroon, T. and Babcock, B.S., "A comparison of variational iteration and Adomian decomposition methods in solving nonlinear thin film flow problems", *Applied Mathematical Sciences*, V6, N97, pp. 4911-4919, 2012.

[5] Mabood, F. and Pochai, N., "Comparison of Optimal Homotopy Asymptotic and Adomian Decomposition Methods for a Thin Film Flow of a Third Grade Fluid on a Moving Belt", *Advances in Mathematical Physics*, V2015, Article ID 642835, 4 pages, 2015.

[6] Burden, R.L. and Faires, J.D., *Numerical Analysis*, Thomson Brooks/Cole, Singapore, 2005.

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