Nonlinear Observer-Based Tracking Control Using Piecewise Multi-Linear Models

Tadanari Taniguchi and Michio Sugeno

Abstract—**This paper proposes a nonlinear observer-based tracking controller design using piecewise multi-linear models. The controller is based on feedback and observer linearizations. The piecewise model is a nonlinear approximation and fully parametric. Feedback linearization is an effective method to stabilize nonlinear systems. However the stabilizing conditions are conservative. Further, observer linearization conditions are more conservative than feedback one. There are not many nonlinear systems to which these methods can apply. This paper shows the proposed piecewise multi-linear controller can be applied to a wider class of nonlinear systems. Example is shown to confirm the feasibility of our proposals by computer simulation.**

Index Terms—**observer-based control, nonlinear control, feedback linearization, tracking control, piecewise system.**

I. Introduction

Piecewise linear (PL) systems which are fully parametric have been intensively studied in connection with nonlinear systems [1], [2], [3], [4]. We are interested in the parametric piecewise approximation of nonlinear control systems based on the original idea of PL approximation. The PL approximation has general approximation capability for nonlinear functions with a given precision.

PML approximation [5] also has general approximation capability for nonlinear functions with a given precision. We note that a bilinear function as a basis of PML approximation is, as a nonlinear function, the second simplest one after a linear function. The PML model has the following features. 1) The PML model is derived from fuzzy if-then rules with singleton consequents. 2) It is built on piecewise hyper-cubes partitioned in the state space. 3) It has general approximation capability for nonlinear systems. 4) It is a piecewise nonlinear model, the second simplest after a PL model. 5) It is continuous and fully parametric. So far we have shown the necessary and sufficient conditions for the stability of PML systems with respect to Lyapunov functions in the two dimensional case [6] where membership functions are fully taken into account. However, since the stabilizing conditions are represented by bilinear matrix inequalities (BMIs) [7], it requires a long computing time to obtain a stabilizing controller. To overcome the difficulty, we derived the stabilizing conditions [8] based on a full-state feedback linearization approaches. Although the PML controllers are simpler than the conventional feedback linearization controller, the control performance based on PML model is the same as the conventional one.

This paper deals with an observer-based tracking controller design for nonlinear systems via observer linearization. We proposed some observer design methods for piecewise systems in [9], [10], [11]. The paper [11] dealt with the necessary and sufficient conditions for observer linearization and showed the PML model based linearized observer could be applied to a wider system than the conventional one. However it is difficult to design the observer-based control system via observer linearization because the control system is not robust to modeling errors and perturbations. We proposed a robust observer-based PML controller design [12] for nonlinear systems using a robust PML controller [13]. This paper proposes a nonlinear observer-based tracking controller using piecewise multi-linear models. Further, we apply the proposed methods to TORA (Translational Oscillator with Rotating Actuator) system, which is one of the benchmark problem for nonlinear control. Example is shown to confirm the feasibility of our proposals by computer simulation.

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II. Canonical Forms of Piecewise Multi-Linear Models

A. Open-Loop Systems

In this section, we introduce PML models suggested in [5]. We deal with the two-dimensional case without loss of generality. We consider a two-dimensional nonlinear system:

$$\dot{x} = f(x)$$

Define vector $d(\sigma,\tau)$ and rectangle $R_{\sigma\tau}$ in two-dimensional space as $d(\sigma,\tau) \equiv (d_1(\sigma), d_2(\tau))^T$,

$$R_{\sigma\tau} \equiv [d_1(\sigma), d_1(\sigma+1)] \times [d_2(\tau), d_2(\tau+1)].$$

σ and τ are integers: $-\infty < \sigma, \tau < \infty$ where $d_1(\sigma) < d_1(\sigma+1), d_2(\tau) < d_2(\tau+1)$ and $d(0,0) \equiv (d_1(0), d_2(0))^T$ (see Fig. 1). Superscript ^{T} denotes a *transpose* operation.

For $x = (x_1, x_2) \in R_{\sigma\tau}$, the PML system is expressed as

$$\begin{cases} \dot{x} = f_p(x) = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ x = \displaystyle\sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \\ f(i,j) = f(d_1(i), d_2(j)),\ d(i,j) = [d_1(i), d_2(j)]^T, \end{cases} \tag{1}$$

where $f(i,j)$ is the vertex of nonlinear system $\dot{x} = f(x)$,

$$\begin{cases} \omega_1^\sigma(x_1) = \dfrac{(d_1(\sigma+1) - x_1)}{(d_1(\sigma+1) - d_1(\sigma))}, \\ \omega_1^{\sigma+1}(x_1) = \dfrac{(x_1 - d_1(\sigma))}{(d_1(\sigma+1) - d_1(\sigma))}, \\ \omega_2^\tau(x_2) = \dfrac{(d_2(\tau+1) - x_2)}{(d_2(\tau+1) - d_2(\tau))}, \\ \omega_2^{\tau+1}(x_2) = \dfrac{(x_2 - d_2(\tau))}{(d_2(\tau+1) - d_2(\tau))} \end{cases} \tag{2}$$

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and $\omega_1^i(x_1), \omega_2^j(x_2) \in [0,\ 1]$. In the above, we assume $f(0,0) = 0$ and $d(0,0) = 0$ to guarantee $\dot{x} = 0$ for $x = 0$.

A key point in the system is that state variable x is also expressed by a convex combination of $d(i,j)$ for $\omega_1^i(x_1)$ and $\omega_2^j(x_2)$, just as in the case of $\dot{x}$. As seen in equation (2), x is located inside $R_{\sigma\tau}$ which is a rectangle: a hypercube in general. That is, the expression of x is polytopic with four vertices $d(i,j)$. The model of $\dot{x} = f(x)$ is built on a rectangle including x in state space, it is also polytopic with four vertices $f(i,j)$. We call this form of the canonical model (1) parametric expression.

B. Closed-Loop Systems

We consider a two-dimensional nonlinear control system.

$$\begin{cases} \dot{x} = f(x) + g(x)u(x), \\ y = h(x). \end{cases} \tag{3}$$

For $x \in R_{\sigma\tau}$, the PML model (4) is constructed from a nonlinear system (3).

$$\begin{cases} \dot{x} = f_p(x) + g_p(x)u(x), \\ y = h_p(x), \end{cases} \tag{4}$$

where

$$\begin{cases} f_p(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) f(i,j), \\ g_p(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) g(i,j), \\ h_p(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) h(i,j), \\ x = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1} \omega_1^i(x_1)\omega_2^j(x_2) d(i,j), \end{cases} \tag{5}$$

and $f(i,j)$, $g(i,j)$, $h(i,j)$ and $d(i,j)$ are vertices of the nonlinear system (3). The modeling procedure in region $R_{\sigma\tau}$ is as follows:

1) Assign vertices $d(i,j)$ for $x_1 = d_1(\sigma), d_1(\sigma+1), x_2 = d_2(\tau), d_2(\tau+1)$ of state vector x, then partition state space into piecewise regions (see Fig. 1).
2) Compute vertices $f(i,j)$, $g(i,j)$ and $h(i,j)$ in equation (5) by substituting values of $x_1 = d_1(\sigma), d_1(\sigma+1)$ and $x_2 = d_2(\tau), d_2(\tau+1)$ into original nonlinear functions $f(x)$, $g(x)$ and $h(x)$ in the system (3). Fig. 1 shows the expression of $f_1(x)$ and $x \in R_{\sigma\tau}$.

The overall PML model is obtained automatically when all vertices are assigned. Note that $f(x)$, $g(x)$ and $h(x)$ in the PML model coincide with those in the original system at vertices of all regions. Due to lack of space, $f(x)$, $g(x)$, $u(x)$, and $h(x)$ are represented as f, g, u, and h, respectively.

III. Regulator and Tracking Controller Designs for PML Systems

A. Feedback Linearization

This section deals with the PML controller for nonlinear systems. Since the stabilizing conditions are represented by bilinear matrix inequalities (BMIs) [7], it requires a long computing time to obtain a stabilizing controller. To overcome the difficulty, we derived the stabilizing conditions [14], [8] based on feedback linearization approaches.

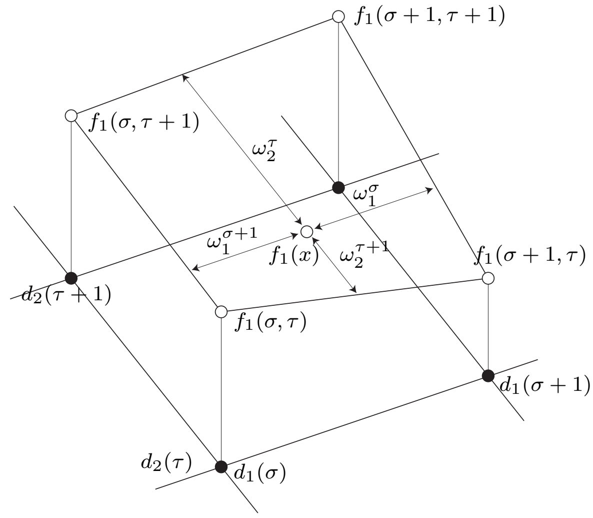

Fig. 1. Piecewise region ($f_1(x) = \sum_{i=\sigma}^{\sigma+1}\sum_{j=\tau}^{\tau+1}\omega_1^i\omega_2^j f_1(i,j)$, $x \in R_{\sigma\tau}$)

We consider the PML system (4), where f_p, g_p and h_p are assumed to be sufficiently smooth in a domain $D \subset R^n$. The mappings $f : D \to R^n$ and $g : D \to R^n$ are called vector fields on D. The time derivative of the output y is calculated until the input u appears. Then the PML controller is obtained as

$$u(x) = \alpha(x) + \beta(x)v, \tag{6}$$

where

$$\alpha(x) = \frac{-L_{f_p}^{\rho} h}{L_{g_p} L_{f_p}^{\rho-1} h_p},\ \beta(x) = \frac{1}{L_{g_p} L_{f_p}^{\rho-1} h_p}.$$

The controller reduces the input-output map to $y^{(\rho)} = v$, which is a chain of ρ integrators. In this case, the integer ρ is called the relative degree of the system.

In Section VI, we show the controller (6) based on PML model is simpler than the conventional feedback linearzing controller. Furthermore, we show the controller (6) can stabilize a wider region than the conventional one.

Definition 3.1: The PML system is said to have relative degree ρ, $1 \le \rho \le n$, in a region $D_0 \subset D$ if

$$L_{g_p} L_{f_p}^{i} h_p = 0,\ i = 0, 1, \cdots, \rho-2$$
$$L_{g_p} L_{f_p}^{\rho-1} h_p \neq 0,$$

for all $x \in D_0$. The feedback linearized system can be formulated as

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ y = C\xi, \end{cases} \tag{7}$$

where $\xi \in \Re^\rho$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix},\ C = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}^T.$$

The stabilizing linear controller $v = -K\xi$ of the linearized system (7) can be obtained so that the transfer function $G = C(sI - A)^{-1}B$ is Hurwitz. Due to lack of space, this paper only deals with the relative degree $\rho = n$. This controller design can be applied to the PML system with the relative degree $\rho \leq n$.

B. Tracking Control for PML Systems

We apply a tracking control [15] to nonlinear systems. Consider the following reference signal model

$$\begin{cases} \dot{x}_r = f_r, \\ y_r = h_r. \end{cases} \tag{8}$$

The controller is designed to make the error signal $e_t = y - y_r = h_p - h_r \to 0$ as $t \to \infty$. The time derivative of the error e is obtained as

$$\dot{e}_t = L_{f_p}h_p - L_{f_r}h_r.$$

The time derivative is calculated until the input u appears. Then the PML controller is obtained as

$$u_t(x) = \alpha_t(x) + \beta_t(x)v_t, \tag{9}$$

where

$$\alpha_t(x) = -\frac{L_{f_p}^{\rho}h_p - L_{f_r}^{\rho}h_r}{L_{g_p}L_{f_p}^{\rho-1}h_p}, \ \beta_t(x) = \frac{1}{L_{g_p}L_{f_p}^{\rho-1}h_p}.$$

The controller reduces the input-output map to $y^{(\rho)} = v$, which is a chain of ρ integrators. In this case, the integer ρ is called the relative degree of the system.

IV. Observer Design for PML Systems

A. Observer Linearization

This subsection deals with the observer linearization problem [16]. If there exists a coordinate transformation $\zeta = \varphi(x)$ such that the system (4) can be transformed into the following system:

$$\begin{aligned} \dot{\zeta} &= A_o\zeta + k(y) + r(y)u \\ y &= C_o\zeta \end{aligned}$$

with (C_o, A_o) observable and $k, r : \Re \to \Re^n$ then it would be possible to build a full order state observer [11]:

$$\begin{aligned} \dot{\hat{\zeta}} &= A_o\hat{\zeta} + k(y) + H(\hat{y} - y) \\ \hat{y} &= C_o\hat{\zeta}, \end{aligned}$$

where

$$A_o = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}, \ C_o = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}^T,$$

and H is the observer gain. The estimation error $e_o = \hat{\zeta} - \zeta$ satisfies the linear differential equation

$$\dot{e}_o = (A_o + HC_o)e.$$

The estimation state is $\hat{x} = \varphi^{-1}(\hat{\zeta})$. This problem is referred to as the observer linearization problem. The following theorem gives a necessary and sufficient condition for the solution of the observer linearization problem.

Theorem 4.1: The observer linearization problem [16] is solvable if and only if there exists the neighborhood V of an initial condition $x(0)$ satisfies the following two conditions.

C1: $\dim\Big(\mathrm{span}\{dh, dL_fh, \ldots, dL_f^{n-1}h\}\Big) = n,$
$\forall x \in V.$

C2: $[ad_f^i\tau, ad_f^j\tau] = 0,$
$0 \leq i \leq n-1,\ 0 \leq j \leq n-1,\ x \in V.$
The vector field τ satisfies

$$\big(dh, dL_fh, \ldots, dL_f^{n-1}h\big)^T \tau = \big(0, \ldots, 1\big)^T.$$

If the nonlinear system (3) is observer linearizable there exists a coordinate transformation $\varphi(x)$ satisfies the following condition.

$$L_{(-1)^{j-1}ad_f^{j-1}\tau}\varphi_i(x) = \begin{cases} 0, \ i \neq j \\ 1, \ i = j \end{cases} \tag{10}$$

A coordinate transformation can be constructed as $\zeta = \varphi(x) = (\varphi_1(x), \varphi_2(x), \ldots, \varphi_n(x))^T$.

B. Observer Based Controller Design

We consider observer-based PML controllers [12]. Substituting the estimation state $\hat{x} = \varphi^{-1}(\hat{\zeta})$ into the controller (6), the observer-based PML controller can be designed as

$$u(\hat{x}) = \alpha(\hat{x}) + \beta(\hat{x})v$$

where $v = -K\hat{\xi}$ and $\hat{\xi} = (h(\hat{x}), L_fh(\hat{x}), \ldots, L_f^{\rho-1}h(\hat{x}))^T$.

We propose an observer-based PML tracking controller. Substituting the estimation state $\hat{x} = \varphi^{-1}(\hat{\zeta})$ into the controller (9), the controller can be designed as

$$u_t(\hat{x}) = \alpha_t(\hat{x}) + \beta_t(\hat{x})v_t$$

where $v_t = -F\hat{\xi}_r$ and $\hat{\xi}_t = (h(\hat{x}) - h_r, L_fh(\hat{x}) - L_{f_r}h_r, \ldots, L_f^{\rho-1}h(\hat{x}) - L_{f_r}^{\rho-1}h_r)^T$.

V. TORA System

The TORA (Translational Oscillator with Rotating Actuator) system [17] has a cart of mass M connected to a wall with a linear spring (constant k). The cart can oscillate without friction in the horizontal plane. A rotating mass m in the cart is actuated by a motor. The mass is eccentric with a radius of eccentricity e and can be imagined to be a point mass mounted on a massless rotor. The rotating motion of the mass m controls the oscillation of the cart. The motor torque is the control variable. The dynamics of TORA system is

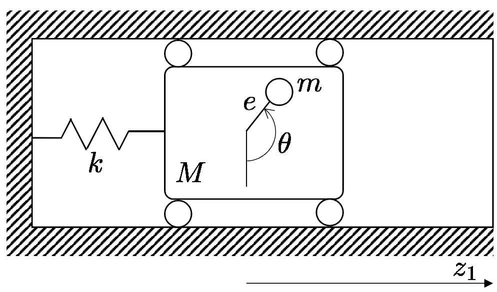

Fig. 2. Kinematic model of TORA system

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$$\begin{cases} \dot{z}_1 = z_2 \\ \dot{z}_2 = \dfrac{-z_1 + \varepsilon z_4^2 \sin z_3}{1-\varepsilon^2 \cos^2 z_3} - \dfrac{-\varepsilon \cos z_3}{1-\varepsilon^2 \cos^2 z_3} v \\ \dot{z}_3 = z_4 \\ \dot{z}_4 = \dfrac{1}{1-\varepsilon^2 \cos^2 z_3} \left\{ \varepsilon \cos z_3 \left(z_1 - \varepsilon z_4^2 \sin z_3 \right) + v \right\} \\ y = z_1, \end{cases} \tag{11}$$

where z_1 and z_2 are the position and velocity of the cart. $z_3 = \theta$ and $z_4 = \dot{\theta}$ are the angle and angular velocity of the rotor. The parameter ε depends on the eccentricity e and the masses M and m. v and y are the control input and output.

The TORA system dynamics has many nonlinear terms. We consider the new variables:

$$\begin{aligned} x_1 &= z_1 + \varepsilon \sin z_3 \\ x_2 &= z_2 + \varepsilon z_4 \cos z_3 \\ u &= \frac{\varepsilon \cos z_3 (x_1 - \varepsilon \sin z_3 (1 + z_4^2)) + v}{1 - \varepsilon^2 \cos^2 z_3} \end{aligned}$$

Substituting the variables x_1, x_2 and u into TORA system (11), we obtain

$$\begin{cases} \dot{x} = f + gu = \begin{pmatrix} x_2 \\ -x_1 + \varepsilon \sin x_3 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u \\ y = h = x_1, \end{cases} \tag{12}$$

where $x \in R^4$, $y \in R$. In the paper, we consider the system (12) as TORA system.

VI. PML Model-Based Controls for TORA System

A. PML Model

We construct the PML model [18] of TORA system (12). The state variable x is divided by $m_1 \times m_2 \times m_3 \times m_4$ vertices,

$$\begin{aligned} x_1 &\in \{d_1(1), \ldots, d_1(m_1)\}, x_2 \in \{d_2(1), \ldots, d_2(m_2)\}, \\ x_3 &\in \{d_3(1), \ldots, d_3(m_3)\}, x_4 \in \{d_4(1), \ldots, d_4(m_4)\}. \end{aligned}$$

The PML model is expressed as

$$\begin{cases} \dot{x} = f_p + g_p u \\ y = h_p = x_1, \end{cases} \tag{13}$$

where $x \in R_{\sigma_1\sigma_2\sigma_3\sigma_4}$,

$$\begin{aligned} f_p &= \sum_{i_1=\sigma_1}^{\sigma_1+1} \sum_{i_2=\sigma_2}^{\sigma_2+1} \sum_{i_3=\sigma_3}^{\sigma_3+1} \sum_{i_4=\sigma_4}^{\sigma_4+1} \omega_1^{i_1}(x_1)\omega_2^{i_2}(x_2)\omega_3^{i_3}(x_3)\omega_4^{i_4}(x_4) \\ &\quad \times \begin{pmatrix} d_2(i_2) & -d_1(i_1) + \varepsilon \sin d_3(i_3) & d_4(i_4) & 0 \end{pmatrix}^T, \\ g_p &= \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, \\ \omega_j^{\sigma_j}(x_j) &= (d_j(\sigma_j+1) - x_j)/(d_j(\sigma_j+1) - d_j(\sigma_j)), \\ \omega_1^{\sigma_j+1}(x_j) &= (x_j - d_j(\sigma_j))/(d_j(\sigma_j+1) - d_j(\sigma_j)), \\ j &= 1, \ldots, 4. \end{aligned}$$

The model is found to be fully parametric and the internal model dynamics is described by multi-linear interpolation of the vertices: $d_1(i_1)$, $d_2(i_2)$, $d_3(i_3)$ and $d_4(i_4)$. The PML model can be represented by a lookup table (LUT).

Note that trigonometric functions of TORA system (12) are smooth functions and are of class C^∞. The PML models are not of class C^∞. In TORA system control, we have to calculate the fourth derivatives of the output y. Therefore the derivative PML models lose some dynamics. However the PML model based control for TORA system can be applied to a wider region than the conventional one.

Note that there are some modeling errors because the PML model is a nonlinear approximation. In proposed method the vertices $d_i(j)$ of an arbitrary number can be set on arbitrary position of the state space. Therefore it is easily possible to adjust the approximated error.

B. Regulators

1) Feedback Linearization of Original Nonlinear System: We design the controller of TORA system (12) via the exact feedback linearization [16]. We calculate the time derivatives of the output y until the input u appears. Then the feedback linearizing controller is obtained as

$$u = \frac{-L_f^4 h}{L_g L_f^3 h} + \frac{1}{L_g L_f^3 h} v. \tag{14}$$

The Lie derivatives are calculated as

$$\begin{aligned} &L_f h = x_2,\ L_f^2 h = -x_1 + \varepsilon \sin x_3, \\ &L_f^3 h = -x_2 + \varepsilon x_4 \cos x_3, \\ &L_f^4 h = x_1 - \varepsilon \sin x_3 - \varepsilon x_4^2 \sin x_3,\ L_g L_f^3 h = \varepsilon \cos x_3. \end{aligned}$$

In equation (14), v is the linear controller for the linearized system:

$$\begin{cases} \dot{\xi} = A\xi + Bv \\ y = C\xi, \\ \xi = (h, L_f h, L_f^2 h, L_f^3 h)^T, \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}^T. \tag{15}$$

However the controller (14) is only well defined at $-\pi/2 < x_3 < \pi/2$ because the denominator of the controller is $L_g L_f^3 h = \varepsilon \cos x_3$. Hence the rotor of TORA system can only be rotated at $-\pi/2 < \theta < \pi/2$.

2) Feedback Linearization of PML System: The time derivative of the output $y = x_1$ has to been calculated until the input u appears. Then the PML controller [18] of (13) is designed as

$$u = \frac{-L_{f_p}^4 h_p}{L_{g_p} L_{f_p}^3 h_p} + \frac{v}{L_{g_p} L_{f_p}^3 h_p} \tag{16}$$

The Lie derivatives are calculated as

$$\begin{aligned} &L_{f_p} h_p = x_2, L_{f_p}^2 h_p = -x_1 + \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3}(x_3) \varepsilon \sin d_3(i_3), \\ &L_{f_p}^3 h_p = -x_2 + \frac{\varepsilon(\sin d_3(\sigma_3+1) - \sin d_3(\sigma_3))}{d_3(\sigma_3+1) - d_3(\sigma_3)} x_4, \\ &L_{f_p}^4 h_p = x_1 - \sum_{i_3=\sigma_3}^{\sigma_3+1} \omega_3^{i_3}(x_3) \varepsilon \sin d_3(i_3), \\ &L_g L_{f_p}^3 h_p = \frac{\varepsilon(\sin d_3(\sigma_3+1) - \sin d_3(\sigma_3))}{d_3(\sigma_3+1) - d_3(\sigma_3)} \end{aligned}$$

In equation (16), $v = -K\xi$ is the linear controller of the linear system:

$$\begin{cases} \dot{\xi} = A\xi + Bv, \\ y = C\xi, \\ \xi = (h_p,\ L_{f_p}h_p,\ L_{f_p}^2 h_p,\ L_{f_p}^3 h_p)^T \end{cases}$$

The matrix A and the vectors B and C are the same as (15).

If $f_s(i) \neq f_s(i+1)$ and $d_3(i) \neq d_3(i+1)$, $i = 1, \ldots, m$, there exists a controller (16) u of TORA system (13) since $\det(L_{g_p}L_{f_p}^3 h_p) \neq 0$. Thus we have to construct the PML model of TORA system such that $f_s(i) \neq f_s(i+1)$ and $d_3(i) \neq d_3(i+1)$, where $i = 1, \ldots, m$. Note that the PML model based controller (16) can be applied to a wider region than the conventional feedback linearized controller.

C. Tracking Control

We design the tracking controller of TORA system using PML model. Consider the following reference signal model (8). The controller is designed to make the error signal $e_t = y - y_r = h_p - h_r \to 0$ as $t \to \infty$. The time derivative of the error e is obtained as

$$\dot{e}_t = L_{f_p}h_p - L_{f_r}h_r.$$

The time derivative is calculated until the input u appears. Then the PML controller is obtained as

$$u_t(x) = \alpha_t(x) + \beta_t(x)v_t, \tag{17}$$

where

$$\alpha_t(x) = -\frac{L_{f_p}^4 h_p - L_{f_r}^4 h_r}{L_{g_p}L_{f_p}^3 h_p},\ \beta_t(x) = \frac{1}{L_{g_p}L_{f_p}^3 h_p}.$$

In equation (16), $v_t = -F\xi_r$ is the linear controller of the linear system:

$$\begin{cases} \dot{\xi}_r = A\xi_r + B\nu, \\ y = C\xi_r, \\ \xi_r = \begin{pmatrix} h_p - h_r \\ L_{f_p}h_p - L_{f_r}h_r \\ L_{f_p}^2 h_p - L_{f_r}^2 h_r \\ L_{f_p}^3 h_p - L_{f_r}^3 h_r \end{pmatrix} \end{cases}$$

The matrix A and the vectors B and C are the same as (15).

D. Observers

1) Observer Design of Original Nonlinear System: C1 of Theorem 4.1 is calculated for the original nonlinear system (12).

$$\det\left(dh^T, dL_f h^T, \ldots, dL_f^{n-1}h^T\right)^T = \varepsilon^2 \cos^2 x_3$$

From this result the above matrix is not linear independence at $x_3 = \pm\pi/2$. One of the condition C2 is calculated for the original nonlinear model as follows:

$$\left[ad_f^0\tau,\ ad_f^3\tau\right] = \frac{2\sin x_3}{\varepsilon^2 \cos^3 x_3}$$

The above equation is equal to 0 at $x_3 = 0$ and the equation cannot be defined at $x_3 = \pm\pi/2$. Therefore the nonlinear system (12) is not observer linearizable.

2) Observer Design [11] of PML System: C1 of Theorem 4.1 is calculated for the PML system (13).

$$\det\left(dh^T, dL_f h^T, \ldots, dL_f^{n-1}h^T\right)^T = \varepsilon \neq 0.$$

C2 of Theorem 4.1 is also calculated for the original nonlinear system (13).

$$[ad_f^i\tau, ad_f^j\tau] = 0,$$

where $0 \leq i \leq 3$, $0 \leq j \leq 3$, and $\tau = \begin{pmatrix} 0 & 0 & 0 & 1/\varepsilon \end{pmatrix}^T$. Therefore the PML system (13) is an observer linearizable. From the condition (10), the coordinate transformation vector is calculated as $\varphi(x) = \begin{pmatrix} \varepsilon x_4 & \varepsilon x_3 & x_2 & x_1 \end{pmatrix}^T$.

E. Observer-Based Tracking Control

We derive an observer-based PML tracking controller for TORA system. Substituting the estimation state $\hat{x} = \varphi^{-1}(\hat{\zeta})$ into the controller (17), the controller can be designed as

$$u_t(\hat{x}) = \alpha_t(\hat{x}) + \beta_t(\hat{x})v_t \tag{18}$$

where $v_t = -F\hat{\xi}_r$ and $\hat{\xi}_t = (h(\hat{x}) - h_r, L_{f_p}h(\hat{x}) - L_{f_r}h_r, \ldots, L_{f_p}^{\rho-1}h(\hat{x}) - L_{f_r}^{\rho-1}h_r)^T$.

VII. Simulation Result

The observer-based PML controller (18) is applied to TORA system (12) in computer simulations. In the simulation, the state variables x_1, x_2, x_3, x_4 of TORA system are divided by the following vertices.

$$\begin{aligned} x_1 \in& \{-2.0,\ 0,\ 2.0\},\ x_2 \in \{-2.0,\ 0,\ 2.0\}, \\ x_3 \in& \{-\pi, -7\pi/8, \ldots, \pi\},\ x_4 \in \{-2.0,\ 0,\ 2.0\} \end{aligned}$$

The parameter ε is 0.5 and the initial condition is $x(0) = (0.5,\ 0,\ 0,\ 0)^T$. We consider the following reference signal model:

$$\begin{cases} \dot{x}_r = a_r \cos t, \\ y_r = h_r = x_r, \end{cases}$$

where $a_r = 0.2$. We use the feedback gain $F = (1.000,\ 3.078,\ 4.236,\ 3.078)$ such that the linearized control system is stable and the observer gain $H = (10.00,\ 25.09,\ 26.47,\ 12.37)^T$ such that the observer system is stable.

Figs. 3, 4, and 5 show the simulation results using the observer-based tracking controller (18). The controller (18) stabilize the TORA system (12) with the estimation error $e_o = \hat{\zeta} - \zeta$ and the tracking error $e_t = y - y_r$. In Fig. 3, the solid line and the dotted line of the upper figure mean the control input y and the reference signal y_r, respectively. The solid line of the lower figure means the error signal $y - y_r$. In Figs. 4 and 5, the solid lines and the dotted lines mean the state responses (ζ_1, ζ_2, ζ_3, and ζ_4) and the estimated states, respectively.

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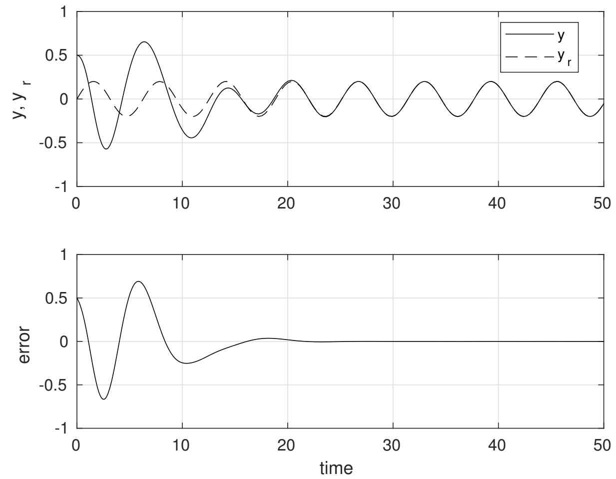

Fig. 3. Control output (y), reference signal (y_r), and the error signal ($y - y_r$)

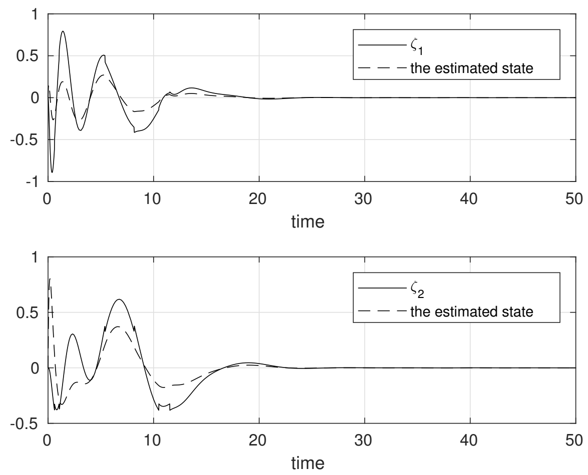

Fig. 4. State responses (ζ_1, ζ_2) and the estimated states

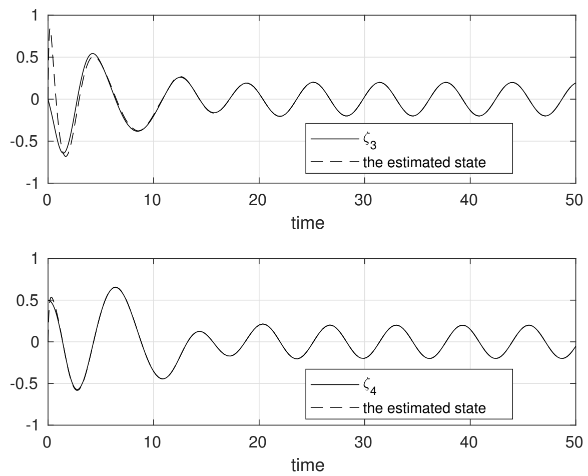

Fig. 5. State responses (ζ_3, ζ_4) and the estimated states

VIII. Conclusions

This paper has proposed a nonlinear observer-based tracking controller design using piecewise multi-linear models. The controller is based on feedback and observer linearizations. The piecewise model is a nonlinear approximation and fully parametric. Feedback linearization is an effective method to stabilize nonlinear systems. However the stabilizing conditions are conservative. Further, observer linearization conditions are more conservative than feedback one. There are not many nonlinear systems to which these methods can apply. This paper has showed the proposed piecewise multi-linear controller can be applied to a wider class of nonlinear systems. Example has been shown to confirm the feasibility of our proposals by computer simulation.

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