The Performance of EWMA Control Chart for MAX(1,r) Process

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Abstract—**The paper proposes the method to derive the explicit formula for Average Run Length (ARL) of moving average process with explanatory variable (MAX(1,r)) on Exponentially Weighted Moving Average (EWMA) control chart. In addition, proposes the numerical results from the explicit formula, Gaussian rule, and midpoint rule. The results show that ARL values from the three methods are in good agreement and useful to detect the change in the process.**

Index Terms— **EWMA, Moving Average, Average run length, Explanatory variable, Exponential white noise.**

I. INTRODUCTION

Quality control in the production process is one of the powerful tools to reduce various deficiencies in the production process. The statistical process control (SPC) is applying for the development and improvement of the process. The effectiveness tool for controlling the production process in real-time is the control chart. Walter A. Shewhart [1] gave the idea of the control chart in 1924 to reduce waste and improve quality in manufacturing processes. Control charts apply in detecting and monitoring changes in the quality of production processes in many applications, for example, industrial manufacturing, public health, computer network and telecommunication, financial and economic, environment science, and other areas. Nowadays, popular control charts are Shewhart, cumulative sum (CUSUM), and exponentially weighted moving average (EWMA) control charts. Robert [3] developed the EWMA control chart that is popular for detecting small changes in the mean of the production process. Usually, the production processes are normally and independently distributed data.

The Average Run Length (ARL) is a famous criterion for evaluating the performance of a control chart, which is the signal's expectation of change in parameter distribution. The ARL_0 represents the time between processes going out of control and should be large enough, whereas the ARL_1 represents the time between processes going out of control and should be smaller. The ARL has been examined using a variety of methodologies, including Monte Carlo simulations (MC), Markov Chain Approach (MCA) [4]-[5], Martingale Approach (MA) [6], and Numerical Integral Equation Approach (NIE) [7]-[9].

In reality, data may contain serially correlated Lu and Reynolds [10] used the integral equation approach to determine the ARL when the data are AR(1) and ARMA(1,1) processes. The time series models are autoregressive (AR), moving average (MA) and autoregressive moving average model (ARMA) models. For this reason, some authors evaluate ARL value when the process is a serial correlation, Petcharat [9] derived the explicit formula for ARL of seasonal $AR(p)_L$ with exponential white noise for EWMA control chart and compared to CUSUM control chart. The results found that the EWMA control chart was more sensitive than the CUSUM control chart. Next, Petcharat [11] derived the explicit formula of ARL for EWMA control chart when observations are seasonal $MA(Q)_L$ process with exponential white noise. The comparison results between EWMA and CUSUM control charts observed that the EWMA control chart detected process changes faster than the CUSUM control chart. In addition, Paichit [12] presented the exact solution of ARL for the EWMA control chart when processes are autoregressive with an explanatory variable model. Later, Sunthornwa et al. [13] derived analytical ARL and approximately the numerical ARL for EWMA control charts using the long-memory ARFIMA with exponential white noise. Recently, Sunthornwat and Areepong [14] proposed an exact solution of ARL on the CUSUM control chart for both seasonal and non-seasonal moving averages processes by using integral equations and numerical integral equations.

Consequently, the objective of this paper is to prove the explicit formula of the average run length (ARL) of the EWMA control chart for MAX(1,r) with exponential white noise and compare it with the numerical integration method. This article could organize in the following manner. The MAX(1,r) process describes in section 2. Section 3 contains the explicit formula of ARL based on the EWMA control chart when the observations are MAX(1,r) model with exponential white noise. Section 4 presents the numerical integration of ARL for the EWMA control chart. Whereas Section 5 explains the simulation study. Finally, the conclusion is in Section 6.

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II. The EWMA control chart for MAX $(1,r)$ process

This section presents the characteristics of the EWMA control chart for MAX(1,r). The EWMA control chart is a dominant tool in detecting a mean shift. Let be the sequence of a moving average process with explanatory variable, MAX(1,r). Then the MAX(1,r) with exponential white noise is defined as,

$$Z_t = \mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}, \qquad (1)$$

where ε_t is exponential white noise : $\varepsilon_t \sim Exp(\alpha)$,

μ is process mean,

θ is moving average coefficient , $-1 \le \theta_1 \le 1$,

The recursive equation of EWMA statistic based on MAX(1,r) process is defined by

$$X_t = (1-\lambda) X_{t-1} + \lambda Z_t \ , \quad t = 1,2,... \qquad (2)$$

where Z_t is sequence of MAX(1,r) process,

λ is a exponential smoothing parameter, $0 < \lambda < 1$.

The corresponding stopping time for (2) define as

$$\tau = \inf\{t > 0; X_t > b\} \ , C_0 = u, \ b > x. \qquad (3)$$

where b denote control limit.

Let $\mathbf{E}_\omega(.)$ denote the expectation under probability density function $f(x,\alpha)$ that the change-point occurs at point ω, where $\omega < \infty$. Thus by definition, the ARL for MAX(1,r) process with an initial value $C_0 = u$ is as follow

$$ARL = H(u) = \mathbf{E}_\infty(\tau_b) < \infty. \qquad (4)$$

III. The Average Length (ARL) for MAX $(1,r)$ Process based on EWMA Control Chart

In this section, the explicit formulas of average run length of EWMA control chart for MAX(1,r) process with exponential white noise is presented. Let $j(u)$ denote the average run length for EWMA chart. We assume that, the process initially in-control $C_0 = u$. The integral equation defines in $j(u)$ as follow;

$$j(u) = 1 + \frac{1}{\lambda} \int_0^b j(y) f\left(\frac{y-(1-\lambda)u}{\lambda} + \left(\mu + \varepsilon_t - \theta_1 \varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it} \right) \right) d(y),$$

Such that

$$j(u) = 1 + \frac{1}{\lambda\alpha} \int_0^b j(y) e^{\frac{y}{\lambda\alpha}} e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} d(\mathrm{y}), \qquad (5)$$

Let $C(u) = \exp\left[\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}\right)}{\alpha} \right]$,

then the function $j(u)$ in (5) can written as

$$j(u) = 1 + \frac{C(u)}{\lambda\alpha} \int_0^b L(y) e^{\frac{1}{\lambda\alpha}} d(y), \ 0 \le u \le b. \qquad (6)$$

The right-hand side of (5) is continuous such that the solution of the integral equations (5) is continuous function.

Considering the complete metric space $\left(C(I), \|\ \|_\infty\right)$ where $C(I)$ denote the space of all continuous function on I , where I is a compact interval, with the norm $\|j\|_\infty = \sup_{u \in I} |j(u)|$. Then operator T is named as a contraction, if there exist a real constant $0 \le q < 1$ such that $\|j(L_1) - j(L_2)\| \le q \|j_1 - j_2\|$ for $\forall j_1, j_2 \in C(I)$. In this case let T be an operation in the class of all continuous function $C(I)$ where $I = [0,b]$ defined by

$$T(j(u)) = j(u) = 1 + \frac{1}{\lambda\alpha} \int_0^b j(y) e^{\frac{y}{\lambda\alpha}} e^{\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}}{\alpha}} d(\mathrm{y}). \qquad (7)$$

According to Banach's fixed point theorem, if an operator T is a contraction, then the fixed point equation $T(j(u)) = j(u)$ has a unique solution. To prove the uniqueness of solution of (7), then prove Theorem 1 that T is contraction.

Theorem 1. *On the metric space* $\left(C(I), \|\ \|_\infty\right)$ *with the norm* $\|j\|_\infty = \sup_{u \in I} |j(u)|$ *the operator T is a contraction*

Proof

First, showing T is a contraction for any $u \in I$, and $j_1, j_2 \in C(I)$ .The inequality $\|T(j_1) - T(j_2)\| \le q \|j_1 - j_2\|$ for $\forall j_1, j_2 \in C(I)$ with $0 \le q < 1$. According to (7), then $\|T(j_1) - T(j_2)\|_\infty$

$$\le \sup_{u \in [0,b)} \left| j_1(0) - j_2(0) \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} \int_0^b j(y) e^{-\frac{y}{\lambda\alpha}} d(y) \right|$$

$$\le \sup_{u \in [0,b)} \left| \|j_1 - j_2\| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} (-\lambda\alpha) \right|$$

$$= \|j_1 - j_2\|_\infty \left| 1 - e^{-\frac{b}{\lambda\alpha}} \right| \sup_{u \in [0,b)} \left| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha} + \frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} \right|$$

$$\leq q\left\| j_1 - j_2 \right\|_\infty ,$$

where

$$0 \leq q = \left|1 - e^{-\frac{b}{\lambda\alpha}}\right| \sup_{u\in[0,b)} \left| \frac{1}{\lambda\alpha} e^{-\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\left(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}\right)}{\alpha}} \right| < 1 ,$$

$0 \leq \lambda < 1$, $\alpha > 0$ and $\varepsilon_1 = 1$.

Triangular inequality has been used and the fact that is

$$\left|j_1(0) - j_2(0)\right| \leq \sup_{u\in[0,b)} \left|j_1(u) - j_2(u)\right| = \left\|j_1 - j_2\right\|_\infty$$

Therefore, the uniqueness of solution is guaranteed via *Theorem 1* and the Banach fixed point theorem. Then, using the Fredholm integral equation of second kind to derive the ARL for MAX(1,r) process.

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The explicit formula of ARL of EWMA chart defined as follows:

$$ARL_0 = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\lambda\alpha_0}\right)\exp\left(-\frac{b}{\lambda\alpha_0}\right) - 1}{\lambda \exp\left(\frac{\mu+\varepsilon_t-\theta\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}}{\alpha_0}\right) + \exp\left(-\frac{b}{\alpha_0}\right) - 1}, \quad (8)$$

$$ARL_1 = 1 - \frac{\lambda \exp\left(\frac{(1-\lambda)u}{\lambda\alpha_1}\right)\exp\left(-\frac{b}{\lambda\alpha_1}\right) - 1}{\lambda \exp\left(\frac{\mu+\varepsilon_t-\theta\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}}{\alpha_1}\right) + \exp\left(-\frac{b}{\alpha_1}\right) - 1}, \quad (9)$$

where process in-control parameter $\alpha = \alpha_0$ and process out-of-control parameter $\alpha = \alpha_1$, moving average coefficient $-1 \leq \theta_1 \leq 1$, λ is the smoothing parameter and b is control limit.

IV. NUMERICAL INTEGRAL EQUATION (NIE) OF ARL FOR MAX (1,r) ON EWMA CONTROL CHART

In this section we present numerical method to approximate the ARL value of EWMA control chart for MAX (1,r) with exponential white noise process satisfying (1) and according (2) can be written as

$$X_t = (1-\lambda)X_{t-1} + \lambda(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}) , \quad t = 1,2,...$$

$$X_t = (1-\lambda)X_{t-1} + \lambda\mu + \lambda\varepsilon_t - \lambda\theta_1\varepsilon_{t-1} + \lambda\sum_{i=1}^{r} \beta_i X_{it} ,$$

where $t = 1,2,...$, $X_0 = Z_0 = u$. Let $\tilde{j}(u)$ denote ARL for MAX(1,r), if $\varepsilon_t \geq 0$ LCL=0 and UCL=b, respectively. Then function $\tilde{j}(u)$ can be derived by Fredholm integral equation of the second kind. The $\tilde{j}(u)$ define as follows:

$$\tilde{j}(u) = 1 + \int_0^b j(y) f\left(\frac{y-(1-\lambda)u}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}\right)\right) dy . \quad (10)$$

Equation (10) can use numerical quadrature rules for approximation [15]. In this research, there are two methods, namely Gaussian Rule and Midpoint Rule.

A. Gaussian Rule

The approximation for an integral is evaluated by the quadrature rule as follow the numerical integral equation method can evaluate the solution by the Gauss–Legendre quadrature rule. Let set of point $\{a_k, k = 1,...,m\}$ on the interval $[0,b]$ and set of weights $\{w_k$,k=1,...,m$\}$ as $w_k = b/m \geq 0$; $k = 1,2,...,m$. The approximation of an integral is evaluated by the quadrature rule as follows:

$$\int_0^b W(y) f(y) dy \approx \sum_{k=1}^{m} w_k f(a_k) ,$$

where $a_k = \frac{b}{m}\left(k - \frac{1}{2}\right)$; $k = 1,2,...,m$.

It can be written $\mathbb{R}_{m\times m}$ be matrix form as

$$[R]_{jk} = \frac{1}{\lambda}\sum_{k=1}^{m} w_k f\left(\frac{a_k - (1-\lambda)a_m}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}\right)\right), \; j,k = 1,2,...,m.$$

Let $\tilde{j}(u)$ be a numerical approximation for an integral equation solution in (10) which can be found as the linear equations as follows:

$$\tilde{j}(u) = 1 + \frac{1}{\lambda}\sum_{k=1}^{m} w_k j(a_k) f(\mu+\varepsilon_t-\theta_1\varepsilon_{t-1}+\sum_{i=1}^{r}\beta_i X_{it}) , \quad (11)$$

where $a_k = \frac{b}{m}\left(k - \frac{1}{2}\right)$; $k = 1,2,...,m$.

B. Midpoint Rule

By using the midpoint rule, Let m be subinterval on $[0,b]$ and let $f(A_k) = f\left(\frac{a_k - (1-\lambda)u}{\lambda} + \left(\mu + \varepsilon_t - \theta_1\varepsilon_{t-1} + \sum_{i=1}^{r} \beta_i X_{it}\right)\right)$,

The approximation for the integral is given by

$$\tilde{j}_M(u) = 1 + \frac{1}{\lambda}\sum_{k=1}^{m} w_k j(a_k) f(A_k) , \quad (12)$$

where $a_k = \frac{b}{m}\left(k - \frac{1}{2}\right)$ and $w_k = \frac{b}{m}$; $k = 1, 2, ..., m$.

V. NUMERICAL RESULTS

In this section, we compare the ARL obtained from the explicit formulas according to (8) and (9) for EWMA control chart on MAX(1,r) process with exponential white noise and compare to the ARL from numerical integral equation (NIE) method using the Gaussian Rule and midpoint rule according to (11) and (12) on m = 500 subintervals. We set $J(u)$ is ARL from explicit formula, $\tilde{J}(u)$ is ARL from NIE method using the Gaussian rule and $\tilde{J}_M(u)$ is ARL from NIE method using the midpoint rule. We also compare computational time between three methods. The computational time of three methods are approximated by central processing unit (CPU) time (Operating system: Window 8 OEM, intel(R) core(TM)i-5 8265U CPU@1.60GHz 1.80GHz Ram 8.00 GB (7.89 GB usable)) in seconds.

In Table I, the parameters value b for EWMA control chart was selected by setting λ = (0.01, 0.05, 0.15), ARL_0=370 and α_0 = 1 in the case of MAX(1,1) with parameter $\beta = 1.0$ and θ=(0.25, 0.35, 0.45), respectively.

Table I show that ARL_0 from explicit solution is very close to NIE on m=500 subintervals. Nevertheless, the CPU time from the explicit formula is much less than the CPU time from the numerical method. Besides, the numerical method using the midpoint rule takes fewer CPU times than the Gaussian rule.

Table II and Table III, ARL values show the performance for detection change in processes between explicit formula and numerical integration method on m=500 subintervals with process mean are shifting. In in-control state, the value of parameter $\alpha_0 = 1$ and out of control state parameter values $\alpha_1 = \alpha_0(1+\delta)$ where shift size (δ) =0.005, 0.01, 0.03, 0.05, 0.07, 0.09, 0.11, 0.13 and 0.15. In Table II, the initial ARL_0= 370 for MAX(1,1) with parameters θ = 0.45, with β = 2.0, λ = 0.15 and b= 0.032273923. In Table III, the initial ARL_0= 370 for MAX(1,2) with parameters θ= 0.45, with $\beta_1 = 0.75$, $\beta_2 = 0.5$, λ =0.1 and $b = 0.0563214$.

The results from Tables II and III show that ARL_0 from explicit solution and numerical integration method on m=500 subinterval are good agreement to detect changes in the process. Nevertheless, the CPU time of the explicit formula is much less than the CPU time from the numerical method. Besides, the numerical method using the midpoint rule takes fewer CPU times than the Gaussian rule, see Fig. 1 and Fig. 2, respectively.

TABLE I
ARL VALUES FOR IN CONTROL PROCESS FOR MAX (1, 1) USING EXPLICIT FORMULA AGAINST NUMERICAL INTEGRAL EQUATION METHOD GIVEN $\beta = 1.0$ FOR ARL_0=370.

λ	θ	b	$J(u)$ (Time: seconds)	$\tilde{J}(u)$ (Time: seconds)	$\tilde{J}_M(u)$ (Time: seconds)
0.01	0.25	0.00472462	370.5505 (0.01)	370.5505 (9.437)	370.5505 (1.61)
	0.35	0.00522307	370.7192 (0.01)	370.7192 (9.453)	370.7192 (1.704)
	0.45	0.00577427	370.6413 (0.01)	370.6413 (9.703)	370.6412 (1.61)
0.05	0.25	0.02384914	370.5107 (0.01)	370.5107 (9.219)	370.5106 (1.72)
	0.35	0.02639200	370.3867 (0.01)	370.3867 (9.328)	370.3867 (1.75)
	0.45	0.02921009	370.3755 (0.01)	370.3755 (9.281)	370.3754 (1.72)
0.15	0.25	0.07332148	370.3748 (0.01)	370.3748 (9.226)	370.3747 (1.69)
	0.35	0.08135870	370.7690 (0.01)	370.7690 (9.766)	370.7688 (1.74)
	0.45	0.09031750	370.7075 (0.01)	370.7075 (9.500)	370.7072 (1.70)

TABLE II
ARL VALUES FOR IN CONTROL PROCESS FOR MAX (1,1) USING EXPLICIT FORMULA AGAINST NUMERICAL INTEGRAL EQUATION METHOD GIVEN $\lambda = 0.15, \theta = 0.45,\ \beta = 2.0$ AND $b = 0.032273923$ FOR ARL_0=370.

shift (δ)	$J(u)$ (Time: seconds)	$\tilde{J}(u)$ (Time: seconds)	$\tilde{J}_M(u)$ (Time: seconds)
0	370.685533 (0.01)	370.68553 (9.375)	370.68550 (1.812)
0.005	61.164939 (0.01)	61.164938 (9.375)	61.164937 (1.703)
0.01	33.691589 (0.01)	33.691589 (9.453)	33.691589 (1.718)
0.03	12.497838 (0.01)	12.497838 (10.359)	12.497838 (1.734)
0.05	7.936779 (0.01)	7.936779 (9.578)	7.936779 (1.72)
0.07	5.947433 (0.01)	5.947433 (9.438)	5.947433 (1.766)
0.09	4.833643 (0.01)	4.833643 (9.281)	4.833643 (1.750)
0.11	4.121993 (0.01)	4.121993 (9.226)	4.121993 (1.766)
0.13	3.628237 (0.01)	3.628237 (9.375)	3.628237 (1.703)
0.15	3.265758 (0.01)	3.265758 (9.549)	3.265758 (1.766)

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TABLE III
ARL VALUES FOR IN CONTROL PROCESS FOR MAX$(1,2)$ USING EXPLICIT FORMULA AGAINST NUMERICAL INTEGRAL EQUATION METHOD GIVEN $\lambda=0.1, \theta=0.45, \beta_1=0.75$ $\beta_2=0.5$ AND $b=0.0563214$ FOR $ARL_0=370$.

shift (δ)	$j(u)$ (Time: seconds)	$\tilde{j}(u)$ (Time: seconds)	$\tilde{j}_M(u)$ (Time: seconds)
0	370.78640 (0.01)	370.78640 (9.375)	370.78629 (1.765)
0.005	79.65995 (0.01)	79.65995 (9.734)	79.65994 (1.859)
0.01	45.03443 (0.01)	45.03443 (9.672)	45.03443 (1.859)
0.03	16.98001 (0.01)	16.98001 (9.703)	16.98001 (1.891)
0.05	10.77829 (0.01)	10.77829 (9.750)	10.77829 (1.891)
0.07	8.05279 (0.01)	8.05279 (9.609)	8.05279 (1.953)
0.09	6.52029 (0.01)	6.52029 (9.470)	6.52029 (1.875)
0.11	5.53799 (0.01)	5.53799 (9.672)	5.53799 (1.906)
0.13	4.85456 (0.01)	4.85456 (9.719)	4.85456 (1.992)
0.15	4.35153 (0.01)	4.35153 (9.813)	4.35153 (1.859)

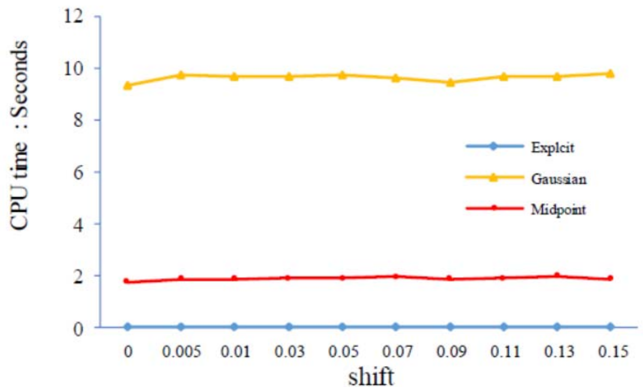

Fig. 1. The CPU times for evaluating ARL values of MAX(1,1) process on EWMA control chart.

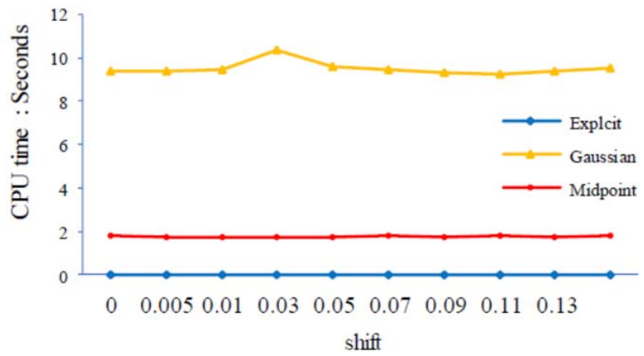

Fig. 2. The CPU times for evaluating ARL values of MAX(1,2) process on EWMA control chart.

VI. Conclusion

This paper proposes an exact solution and numerical integration method using Gaussian and midpoint rules of ARL for moving average process with explanatory variable (MAX(1,r)) on EWMA control chart. Further, the existence and uniqueness of the explicit ARL have been proving. The results found that the ARL value from the proposed exact solution is much closed to the numerical integration method. The computational times for computing explicit formula take less than 1 second as well the numerical integration method takes no more than 11 seconds in the case of MAX(1,r). Besides, the numerical integration method using the midpoint rule takes fewer CPU times than the Gaussian rule. Therefore, the explicit formulas can decrease the computational times better than the numerical integration method.

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