Average Run Length of CUSUM Chart Based on SARX(P,r)$_L$ Model

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Abstract—**The objectives of this research are to derive an explicit analytical formula and develop numerical integration for evaluating the Average Run Length (ARL) of Cumulative Sum chart for a seasonal autoregressive with exogenous variable; SARX(P,r)$_L$ model with exponential distribution white noise. The integral equations for solving the close form for ARL and use the midpoint rule to approximate the numerical integration. The numerical results obtained from the proposed explicit formulas are compared with results obtained from numerical integration technique.**

Index Terms—**Cumulative Sum chart, average run length, explicit formula, numerical integration, SARX(P,r)$_L$ model.**

I. Introduction

THE main objective of Statistical Process Control (SPC) is to improve the capacity of the process. One of the instruments on the quality tool set is the control chart (Vargas et al., [1]). Control charts are an essential statistical tool of continuous quality control to monitor and improve quality characteristics of production processes, which are widely used for detecting changes in the mean or variance of the processes. Generally, the manufacturers wish to control the variability as an aid to effective problem solving. Control charts are widely used in many areas of applications such as economics, finance, medicine and engineering. It is used in monitoring and controlling the quality of processes.

Control charts such as Shewhart, Exponentially Weighted Moving Average (EWMA), and Cumulative Sum (CUSUM) charts have been developed for detecting changes in the process means. The Shewhart chart was first introduced by Walter Shewhart in 1924 [2], it is effective when detecting large changes in process means ($\delta > 1.5\sigma$). The effective control chart is used to detect a small shift, namely the Cumulative Sum (CUSUM) chart, which is proposed by Page [3]. The literatures concerning control of small shifts)$\delta < 1.5\sigma$) are recommended such as Hawkins and Olwell [4] and Lucas [5] to show that the CUSUM chart is more efficient than the Shewhart chart in term of detection of small changes in the process means.

Time series are defined as observations of a variable, often equally spaced time points such as the hourly air temperature, the daily closing price of a stock, or the monthly revenue of a grocery store. In this research, we will focus on time dependent time series, i.e. observations at time *t* are correlated with the previous observations. Many time series display seasonality. In economics, time-series data tends to be seasonal. If seasonality is present, it must be incorporated into the time series model such as the monthly data, the period is 12 since there are 12 months in a year.

Etuk [6] uses the seasonal autoregressive integrated moving average (SARIMA) process to model gross domestic product. Similarly, Eni et al. [7] use the SARIMA model to study the patterns of temperature. Ayinde and Abdulwahab [8] identify a time series model for crude oil exports through the SARIMA model. Doguwa and Alade [9] use SARIMA and SARIMAX models to model Nigeria's headline, core and food inflation. An autoregressive integrated moving average with exogenous variables (ARIMAX) model, which has the capacity to identify the underlying patterns in time series data and to quantify the impact of environmental influences.

The Average Run Length (ARL) is a measurement of the control chart's performance. The ARL is the expected number of samples that should occur before a sample shows the out-of-control condition. There are two characteristics of ARL: 1) the average number of samples taken from an in-control process until the control chart falsely signals out-of-control is denoted by ARL_0. An ARL_0 will be regarded as acceptance if it is large enough to keep the level of false alarms at an acceptable level and 2) the average number of observations that fall within the control limits before giving an alarm that the process is out-of-control is denoted by ARL_1. There are several methods that can be utilized to find the ARL_0 and ARL_1 of EWMA and CUSUM charts have been discussed in the literatures, e.g., Monte Carlo simulations (MC) method, Markov Chain approach (MCA) and Integral Equation approach (IE). For example, Monte Carlo simulation is a simple method used for checking the accuracy of analytical results, but it is very time consuming to run. Roberts [10], who first introduced the ARL for EWMA chart by using simulation to estimate the ARL. Crowder [11] used Integral Equations to find the ARL for Gaussian distribution. Lucas and Saccucci [5] have evaluated the ARL by using a finite-state Markov Chain approximation. However, the limitations of the MCA, IE and MC methods provide the motivation for finding explicit formulas for evaluating the ARL. Recently, Sukparungsee and Novikov [12] used the Martingale approach to derive approximate analytical formulas of Average Run Length (ARL) and Average Delay (AD) in case of Gaussian and Non-Gaussian distributions. Later, Areepong and Novikov [13] derived the explicit formulas of ARL and AD for EWMA chart with Exponential distribution. Recently, Mititelu et al. [14] presented the exact solution of ARL by Fredholm Integral

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Equation for one-sided EWMA chart with Laplace distribution and the CUSUM chart with Hyperexponential distribution.

The choice of control charts to be used depends on the characteristics to be measured in the process, as well as the way that these samples are taken. Control charts are usually designed under the assumption that the observations from a process are independent and identically distributed. However, there are many situations in which the process is autocorrelated such as in chemical process, so it needs to be monitored by appropriate control charts. Some authors evaluate the ARL when the process is serially correlation, such as Mastrangelo and Montgomery [15] have been evaluated the performance of EWMA chart for serially-correlated process based on Monte Carlo simulation technique. Vanbrackle and Reynold [16] were estimated the ARL by using an Integral Equation technique and Markov Chain Approach to evaluate EWMA and CUSUM charts in case of the first order autoregressive; AR(1) process with additional random error. Consequently, Busaba et al. [17] was proposed the close form of ARL for CUSUM chart in situation of the observations are a stationary first order autoregressive; AR(1) model. Later, Petcharat et at. [18] derived explicit formulas of ARL for EWMA and CUSUM charts when observations are the q order moving average; MA(q) with exponential white noise by using the Integral Equation which based on the Fredholm Integral Equation of the second type technique. Recently, Phanyaem et al. [19] presented explicit formulas of the ARL of CUSUM chart for autoregressive and moving average; ARMA(1,1) model. Later, Phanyaem [20] proposed explicit formulas for ARL of CUSUM chart when observations are $SARMA(1,1)_L$ model. Recently, Piyapatr and Lili [20] presented explicit formulas of the ARL of CUSUM chart for an autoregressive integrated moving average; ARIMA(p,d,q) model with an exponential white noise and compare it with the numerical integration.

The objectives of research are to derive a closed form expression and to approximate the numerical integration for evaluating the Average Run Length of CUSUM chart for the seasonal autoregressive with an exogenous variable; $SARX(P,r)_L$ model with exponential distribution white noise. The rest of the paper is organized as follows: the CUSUM chart for $SARX(P,r)_L$ model is presented in Section 2. The numerical method of ARL for CUSUM chart proposed in Section 3. In Section 4, we compared the results from close form expression with the results from numerical solution of an integral equation. Finally, Section 5 provides a conclusion.

II. CUSUM Chart for $SARX(P,R)_L$ Model

This section we present the characteristics of CUSUM charts, which were first introduced by Page [3] in 1954, are widely used and powerful tools for monitoring and detecting the mean in the process. Suppose that we have $Y_1, Y_2, \ldots$ be a sequential observation based on a seasonal autoregressive with an exogenous variable; $SARX(P,r)_L$ model. The CUSUM statistics at the time t denoted by C_t is defined as follows

$$C_t = \max(C_{t-1} + Y_t - a,\ 0);\ t = 1,2,.... \quad (1)$$

where Y_t is a sequence of $SARX(P,r)_L$ model, $C_0 = u$ is an initial value, a is a reference value of CUSUM chart.

The general form of $SARX(P,r)_L$ model described by the following recursion

$$Y_t = \sum_{i=1}^{r} \beta_i X_{it} + \mu + \phi_1 Y_{t-L} + ... + \phi_P Y_{t-PL} + \varepsilon_t \quad (2)$$

where X_{it} is an exogenous variable,

β_i is a coefficient of X_{it},

ϕ_i is an autoregressive coefficient, $i = 1, 2,\ldots, P$

ε_t is an exponential white noise.

Let Y_{t-L}, Y_{t-2L}, …, Y_{t-PL} be an initial value of $SARX(P,r)_L$ model.

The stopping time of CUSUM control chart is given by

$$\tau_h = \inf\{t > 0;\ C_t > h\}, \quad (3)$$

where τ_h is the stopping time

h is the constant parameter as upper control limit.

Suppose $H(u)$ denote the ARL for the $SARX(P,r)_L$ model with an initial value $C_0 = u$. To define $H(u)$ as follows

$$ARL = H(u) = \mathbb{E}_\infty(\tau_h) < \infty. \quad (4)$$

where $\mathbb{E}_\infty$ (.) is the expectation under density function $f(x,\alpha)$

III. Explicit Formula of ARL for $SARX(P,R)_L$ Model

In this section, we present the methodology for evaluating the ARL for CUSUM chart for the $SARX(P,r)_L$ model with an exponential distribution white noise. We find analytical explicit formula of ARL for CUSUM chart by using Integral Equation and compare the results obtained from Numerical Integral Equation (NIE) method. The steps of the study are given as

Step 1. To define the function $ARL = H(u) = \mathbb{E}_\infty(\tau_h) < \infty$ as follows

$$H(u) = 1 + \mathbb{E}_C\left[I\{0 < C_1 < h\}H(C_1)\right] + \mathbb{P}_C\{C_1 = 0\}H(0).$$

Step 2. To extend the function into the Fredholm Integral Equations of the second kind.

$$H(u) = 1 + H(0)F(a - u - Y_t) + \int_0^h L(y) f(y + a - u - Y_t)dy.$$

Step 3. To check the uniqueness of solution by using the Banach's Fixed Point Theorem.

Banach's Fixed Point Theorem

Let $(\mathbb{M}, d)$ be a complete metric space and let the mapping $T : \mathbb{M} \to \mathbb{M}$ be a *contraction*, then T is unique on Fixed Point. In other words, the Banach's Fixed Point theorem states that for a contractive mapping T on a complete metric space, there exists a unique solution to the fixed point equation $T(u) = u$.

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Proof. The mapping $T(u)=u,$ we need to prove the followings:

1) The uniqueness of the fixed point when it exists.

2) The existence of the fixed point by show first that sequence of successive approximations $u_{n+1} = T(u_n)$ is Cauchy convergent, hence convergent since it is in a complete metric space. More important is show that the limit point for this convergent sequence $u=\lim_{n\to\infty} u_n$ is indeed the fixed point for the equation $T(u)=u.$

Step 4. To derive the closed form expression of ARL for the CUSUM control chart when the observation is SARX(P,r)$_L$ model with exponential white noise.

$$H(u)=1+\alpha e^{\alpha\left(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+\ldots+\phi_P Y_{t-PL}\right)}\int_0^h H(y)e^{-\alpha y}dy + \left(1-e^{-\alpha\left(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}\right)}\right)H(0) \quad (5)$$

Let d is a constant as $d=\int_0^h H(y)e^{-\alpha y}dy.$ Thus, the function $H(u)$ can be written as

$$H(u)=1+\alpha e^{\alpha\left(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+\ldots+\phi_P Y_{t-PL}\right)}d + \left(1-e^{-\alpha\left(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}\right)}\right)H(0) \quad (6)$$

For $u=0$, thus we have $H(0)$ as following form:

$$H(0)=1+\alpha e^{\alpha\left(-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+\ldots+\phi_P Y_{t-PL}\right)}d + \left(1-e^{-\alpha\left(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}\right)}\right)H(0)$$

Hence, substituting $H(0)$ into equation (6) as following form:

$$H(u)=1+\alpha d+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})}-e^{\alpha u}. \quad (7)$$

To find the constant d as following form:

$$d=\int_0^h H(y)e^{-\alpha y}dy.$$
$$=\frac{e^{\alpha h}}{\alpha}(1-e^{-\alpha h})\left(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})}\right)-he^{\alpha h}$$

Consequently, the explicit formulas obtained by substituting the constant d into equation (7) as following form:

$$H(u)= e^{\alpha h}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})}-\alpha h)-e^{\alpha u}.$$

As mentioned above, the value of parameter α is equal to α_0 when the process is in-control. Hence, the explicit formula for ARL$_0$ is

$$ARL_0 = e^{\alpha_0 h}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})}-\alpha_0 h)-e^{\alpha_0 u}.$$

On the other hand, the process is out-of-control, the value of parameter α is equal to α_1; where $\alpha_1=\alpha_0(1+\delta).$ The explicit formula for ARL$_1$ is

$$ARL_1 = e^{\alpha_1 h}(1+e^{\alpha(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})}-\alpha_1 h)-e^{\alpha_1 u}.$$

where $0\le\phi_i\le 1$ is autoregressive coefficient; i = 1, 2,…, P and α is a parameter of the exponential distribution, X_{it} is an exogenous variable, β_i is a coefficient of X_{it} and h is the constant parameter as upper control limit.

Step 5. To evaluate the ARL$_0$ from the closed form expression in step 4, which given ARL$_0$ is equal to 370.

Step 6. To evaluate the ARL$_1$ where shift sizes (δ) are equal to 0.01, 0.02, 0.03, 0.04, 0.05, 0.10, 0.20, 0.30, 0.40 and 0.50 respectively.

Step 7. To calculate the ARL$_0$ and ARL$_1$ from the closed form expression are developed using Mathematical program.

IV. Numerical Integration of ARL for CUSUM Chart Base on SARX(P,r)$_L$ Model

In this section, the numerical integral equation method called the NIE method is introduced [22]. Let $y\sim Exp(\alpha)$, and $f(y)$ be the probability density function of exponential distribution and $F(y)$ be the cumulative density function of the exponential distribution.

$$F(y) = 1-e^{-\alpha y} \quad \text{and} \quad f(y) = \frac{dF(y)}{du} = \alpha e^{-\alpha y}.$$

The midpoint rule for estimating an integral equation uses a finite sum with subintervals of equal width and the midpoints of each subinterval in place of a_j. Let $W(y)$ and $F(y)$ are given functions. The corresponding approximate integral equation is given as:

$$\int_0^h W(y)F(y)dy \approx \sum_{j=1}^{m} w_j F(a_j), \quad (8)$$

where a_j is a set of point, $0\le a_1\le a_2\le\ldots\le a_m\le h$ and w_j is a set of constant weight, $w_j = h/m\ge 0.$

Let $\mathbb{P}_c$ be the probability measure and $\mathbb{E}_c$ be the expectation corresponding to initial value $C_0 = u$. Then the solution of integral equation can be written as:

$$H(u) = 1+\mathbb{E}_c\left[I\{0<C_1<h\}H(C_1)\right]+\mathbb{P}_c\{C_1=0\}H(0). \quad (9)$$

The integral equation of CUSUM chart as follows:

$$H(u)=1+\alpha e^{\alpha\left(u-a+\mu+\sum_{i=1}^{r}\beta_i X_{it}+\phi_1 Y_{t-L}+\ldots+\phi_P Y_{t-PL}\right)}\int_0^h H(y)e^{-\alpha y}dy + \left(1-e^{-\alpha\left(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}\right)}\right)H(0). \qquad (10)$$

Firstly, the integral equation (5) can be written as:

$$\tilde{H}(u)=1+H(0)\,\mathrm{F}(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + \int_0^h H(y)f(y+a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})dy.$$

where $F(u) = 1-e^{-\alpha u}$ and $f(u) = \dfrac{dF(u)}{du} = \alpha e^{-\alpha u}$.

Let $\tilde{H}(u)$ denote the approximated solution of $H(u)$ by using the midpoint rule, then the integral equation can be approximated by

$$\tilde{H}(a_i)=1+\tilde{H}(0)\,\mathrm{F}(a-a_i-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + \sum_{j=1}^{m}w_j\tilde{H}(a_j)f(a_j+a-a_i-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})$$

A system of m linear equations $\tilde{H}(a_1),\tilde{H}(a_2),..,\tilde{H}(a_m)$, can be written as

$$\tilde{H}(a_1)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_1-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + w_1 f(a-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})] + \sum_{j=2}^{m}w_j\tilde{H}(a_j)f(a_j+a-a_1-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})$$

$$\tilde{H}(a_2)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + w_1 f(a_1+a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})] + \sum_{j=2}^{m}w_j\tilde{H}(a_j)f(a_j+a-a_2-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})$$

$$\vdots$$

$$\tilde{H}(a_m)=1+\tilde{H}(a_1)[\mathrm{F}(a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + w_1 f(a_1+a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})] + \sum_{j=2}^{m}w_j\tilde{H}(a_j)f(a_j+a-a_m-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})$$

It can be rewritten in matrix form as follows:

$$\mathbf{H}_{m\times1}=\mathbf{1}_{m\times1}+\mathbf{R}_{m\times m}\mathbf{H}_{m\times1}$$

where $\mathbf{H}_{m\times1}=\begin{pmatrix}\tilde{H}(a_1)\\ \tilde{H}(a_2)\\ \vdots\\ \tilde{H}(a_m)\end{pmatrix}$, $\mathbf{1}_{m\times1}=\begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix}$

and $\mathbf{I}_m = diag(1,1,\ldots,1)$ is the unit matrix of order m. If there exists $(\mathbf{I}_m - \mathbf{R}_{mxm})^{-1}$, then the solution of matrix equation as follows

$$\mathbf{H}_{m\times1}=(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times1}.$$

Therefore, the numerical integration of ARL for CUSUM control chart based on SARX(P,r)$_L$ model as follows:

$$\tilde{H}(u)=1+\tilde{H}(a_1)[\mathrm{F}(a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) + w_1 f(a_1+a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL})] + \sum_{j=2}^{m}w_j\tilde{H}(a_j)f(a_j+a-u-\mu-\sum_{i=1}^{r}\beta_i X_{it}-\phi_1 Y_{t-L}-\ldots-\phi_P Y_{t-PL}) \qquad (10)$$

where $w_j=\dfrac{h}{m}$ and $a_j=\dfrac{h}{m}\left(j-\dfrac{1}{2}\right); j=1,2,...,m.$

V. Numerical Analysis

In this section, we present the results obtained from the explicit formulas of ARL for CUSUM control chart when observations are the seasonal autoregressive with exogenous variables; SARX(P,r)$_L$ model. The explicit formula is compared with numerical integral equation (NIE) method using the midpoint rule with 500 nodes and to verify which method is better with the absolute percentage error and the computation (CPU) times are used.

Table 1-3 presents the ARL of explicit formula and NIE methods for SARX(1,1)$_{12}$, SARX(2,1)$_{12}$ and SARX(3,1)$_{12}$ models, respectively. Given the in-control parameter $\alpha_0 = 1$ and out-of-control parameter $\alpha_1 = \alpha_0\,(1+\delta)$ where $\delta = 0.00$, 0.01, 0.03, 0.05, 0.10, 0.20, 0.30, 0.40, 0.50, 1.00 and 1.50 respectively. The CUSUM chart was set the reference values (a) is 2.50 and an initial value (u) is 1. The value of upper control limit (h) for the CUSUM chart was chosen by giving desired ARL$_0$ is 370.

Table 1, we set the parameter values for SARX(1,1)$_{12}$ model with $\phi_1 = 0.10$, $\beta_1 = 0.1$ then the parameter values of CUSUM chart are $a = 2.50$ and $h = 3.976$. Similarity, Table 2, we set the parameter values of SARX(2,1)$_{12}$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$ and $\beta_1 = 0.1$, then the parameter values of CUSUM chart are $a = 2.50$ and $h = 4.151$. Finally, Table 3, we set the parameter values of SARX(3,1)$_{12}$ model with $\phi_1 = 0.10$, $\phi_2 = 0.10$, $\phi_3 = 0.10$ and $\beta_1 = 0.1$ then the parameter of CUSUM chart are $a = 2.50$ and $h = 4.349$.

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TABLE I
THE ARL VALUES FOR SARX(1,1)$_{12}$ MODEL USING EXPLICIT FORMULAS AGAINST NIE METHOD GIVEN a = 2.5 AND h = 3.976

Shift size δ	Explicit Formulas Method	NIE Method	Absolute Percentage Difference
0.00	370.309	369.445 (11.68)	0.354
0.01	346.445	346.240 (11.95)	0.059
0.03	304.411	303.386 (12.33)	0.337
0.05	268.803	267.926 (12.51)	0.326
0.10	200.930	200.323 (12.89)	0.302
0.20	120.929	120.614 (12.48)	0.260
0.30	78.930	78.751 (12.39)	0.227
0.40	54.951	54.843 (12.22)	0.197
0.50	40.301	40.231 (13.09)	0.174
1.00	14.196	14.182 (11.80)	0.099
1.50	7.922	7.917 (12.66)	0.063

a. The values in parentheses are CPU times in numerical integration methods (Minutes)

TABLE II
THE ARL VALUES FOR SARX(2,1)$_{12}$ MODEL USING EXPLICIT FORMULAS AGAINST NIE METHOD GIVEN a = 2.5 AND h = 4.151

Shift size δ	Explicit Formulas Method	NIE Method	Absolute Percentage Difference
0.00	370.267	368.929 (12.51)	0.361
0.01	345.929	345.175 (12.80)	0.355
0.03	303.154	302.115 (11.48)	0.343
0.05	267.021	266.137 (13.81)	0.331
0.10	198.465	197.861 (13.03)	0.304
0.20	118.361	118.055 (12.89)	0.259
0.30	76.753	76.583 (12.93)	0.221
0.40	53.203	53.102 (13.11)	0.190
0.50	38.916	38.852 (13.33)	0.164
1.00	13.720	13.708 (13.60)	0.087
1.50	7.718	7.714 (14.08)	0.049

TABLE III
THE ARL VALUES FOR SARX(3,1)$_{12}$ MODEL USING EXPLICIT FORMULAS AGAINST NIE METHOD GIVEN a = 2.5 AND h = 4.349

Shift size δ	Explicit Formulas Method	NIE Method	Absolute Percentage Difference
0.00	370.136	368.778 (11.71)	0.367
0.01	345.182	343.940 (11.49)	0.360
0.03	301.451	300.408 (14.53)	0.346
0.05	264.650	263.769 (14.19)	0.333
0.10	195.243	194.652 (13.33)	0.303
0.20	115.070	114.780 (13.65)	0.252
0.30	74.003	73.847 (13.61)	0.211
0.40	51.023	50.933 (13.32)	0.176
0.50	37.207	37.151 (13.67)	0.151
1.00	13.159	13.150 (14.36)	0.068
1.50	7.488	7.485 (14.81)	0.040

The results from Table 1 to Table 3, show that the absolute percentage difference are less than 1.0% by the numerical integration for the case of division points m = 500, and the CPU times of approximately 11-15 minutes. While, the CPU times from the proposed explicit formulas are less than 1 second.

VI. CONCLUSION

The paper has successfully proposed the explicit formula of ARL for a seasonal autoregressive with exogenous variables; SARX(P,r)$_L$ model on CUSUM chart and approximated the ARL using the numerical integral equation (NIE) method. The ARL that was computed from the explicit formula was in excellent agreement with the ARL obtained from the NIE method with the absolute percentage difference are less than 1.0%. In addition, the CPU time of the NIE method has between 11-15 minutes, whereas that of the explicit formula was less than 1 second. This means that the proposed explicit formula would be very useful to find the ARL of CUSUM chart and it can be applied in real applications for different process data, for example in network traffic and chemical process. Further studies can be carried out on some of control chart not derived in this paper or extended to other serially correlated observations.

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