Bootstrap Confidence Interval for the Median Failure Time of Three-Parameter Weibull Distribution

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Abstract— In many applications of failure time data analysis, it is important to perform inferences about the median of the distribution function in situations of failure time data modeling with skewed distribution. For failure time distributions where the median of the distribution function can be analytically calculated, its maximum likelihood estimator is easily obtained from the invariance properties of the maximum likelihood estimators. From the asymptotical normality of the maximum likelihood estimators, confidence intervals can be obtained. However, these results might not be very accurate for small sample sizes and/or with large proportion of censored observations. Considering the three-parameter Weibull distribution for the failure time data, we present and compare the accuracy of asymptotical confidence intervals with confidence intervals based on bootstrap simulation. The alternative methodology of confidence intervals for the median of the three-parameter Weibull distribution function is illustrated by using real data from engineering field. The nonparametric bootstrap procedure was implemented in the SAS® system which incorporated proc nlp, proc surveyselect and proc iml in the SAS® macro environment.

Keywords: bootstrap, failure time, three parameter Weibull, skewed,

I. INTRODUCTION

In failure time data analysis, we usually have a skewed distribution function. One of the skewed distributions which play a central role in the analysis of failure time data is Weibull distribution, introduced by Waloddi Weibull, a Swedish physicist, who used it to represent the probability distribution of the breaking strength materials.

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Indeed, this distribution is as central to the parametric analysis of failure time data as the normal distribution is in linear modeling. For the skewed distribution, a more appropriate and more tractable summary of the location of the distribution is the median failure time [1]. Usually, we have interest in the estimation of the median failure time where the central of tendency of the distribution function occurs.

Considering the three-parameter Weibull distribution, we introduce asymptotical based inferences and bootstrap based inferences for the median of the failure time. It is important to note that usually, in the literature of failure time data analysis, confidence intervals for the median of the failure time are based on asymptotic arguments. A recent study about the Weibull distribution, related to this work, is presented in [2]

This paper is organized as follows: in Section II we introduce some characteristics of the three-parameter Weibull distribution; in Section III we introduce the likelihood function in the presence of censored observations; in Section IV we have comparison between asymptotical based inferences and bootstrap simulation based inferences for the median failure time; in Section V we give illustrative example with real data set and section VI is the conclusion..

II. THE THREE-PARAMETER WEIBULL DISTRIBUTION

$$h(t)=\frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1},\ t>\gamma \tag{1}$$

This function depends on three parameters β, η and γ, which are all greater than zero. In the particular case where β = 1, the hazard function takes a constant value η –1, and the failure times have an exponential distribution. For other values of β, the hazard function increases or decreases monotonically, that is, it does not change direction. The shape of the hazard function depends critically on the value of β, and so β is known as the shape parameter, while the parameter η is a scale parameter. Sometimes γ is called a "guarantee parameter" because with γ > 0, failure is impossible before time γ. In some physical applications it makes sense to constraint γ > 0, but there is no mathematical reason to do this. For this particular choice of hazard function, the survivor function according to [3] is given by

$$S(t) = \exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right], \quad t > \gamma \tag{2}$$

The corresponding probability density function is then

$$f(t) = \frac{\beta}{\eta}\left(\frac{t-\gamma}{\eta}\right)^{\beta-1} \exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right], \quad t > \gamma \tag{3}$$

The general form of this density function for $\gamma = 2$ and different combination values of β and η is shown in fig. 1. The right-hand tail of this distribution is longer than the left-hand one, and so the distribution is positively skewed

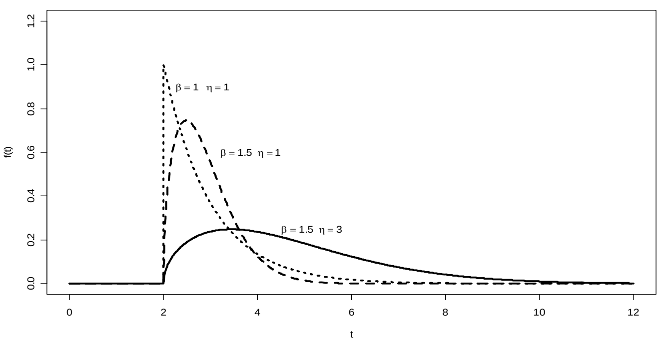

Fig. 1. Weibull probability density function for $\gamma = 2$ and three combinations of β and η

The mean, or expected value, of a random variable T (time to failure) that has three-parameter Weibull distribution is as follows

$$E(T) = \gamma + \eta\Gamma\left(1 + \frac{1}{\beta}\right) \tag{4}$$

where Γ (x) is the gamma function defined by

$$\Gamma(x) = \int_0^{\infty} u^{x-1} e^{-u}\, du \tag{5}$$

However, since the Weibull distribution is skewed, a more appropriate and more tractable summary of the location of the distribution is the median failure time. This is the value t_{50} such that S{ t_{50} } = 0.5, so that

$$\exp\left[-\left(\frac{t_{50}-\gamma}{\eta}\right)^{\beta}\right] = 0.5$$

and

$$t_{50} = \gamma + \eta(\log 2)^{\frac{1}{\beta}} \tag{6}$$

Since the Weibull hazard function can take a variety of forms depending on the value of the shape parameter β, appropriate summary statistics can be easily obtained. This distribution is widely used in the parametric analysis of reliability data.

III. THE LIKELIHOOD FUNCTION IN THE PRESENCE OF RIGHT CENSORED DATA

Let $T_1^0, \ldots, T_n^0$ be the true failure times of a sample of size n, assumed to be independent identically distributed with a three-parameter Weibull distribution with hazard function (1). Assuming that the observations are subject to arbitrary right censoring, the period of follow-up for the ith individual is limited to a value c_i. Then, the observed failure time of the ith individual is given by $t_i = \min\ (T_i^o, c_i)$.

Define δ_i such that $\delta_i = 0$ if $T_i^0 \geq c_i$ (a censored observation) and $\delta_i = 1$ if $T_i^0 < c_i$ (an observed failure of some kind).
The likelihood function for η, β and γ is given by

$$L(\eta,\beta,\gamma|t) = \prod_{i=1}^{n}\left\{\frac{\beta}{\eta}\left(\frac{t_i-\gamma}{\eta}\right)^{\beta-1}\exp\left[-\left(\frac{t_i-\gamma}{\eta}\right)^{\beta}\right]\right\}^{\delta_i}\left\{\exp\left[-\left(\frac{t-\gamma}{\eta}\right)^{\beta}\right]\right\}^{1-\delta_i}$$

The corresponding log-likelihood function is given by

$$l(\eta,\beta,\gamma\,|\,\boldsymbol{t}) = \log(\beta)\sum_{i=1}^{n}\delta_i - \beta\log(\eta)\sum_{i=1}^{n}\delta_i + (\beta-1)\sum_{i=1}^{n}\delta_i\log(t_i-\gamma) - \sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\eta}\right)^{\beta} \tag{7}$$

This function is unbounded, since for any $\beta < 1$, $l(\eta,\beta,\gamma\,|\,t) \rightarrow \infty$ as $\gamma \rightarrow t_{(1)}-$. Consequently, a solution to the likelihood equation $\partial l/\partial\eta = 0$, $\partial l/\partial\beta = 0$, $\partial l/\partial\gamma = 0$, does not produce a global maximum for the likelihood. However, situation in which the three-parameter Weibull distribution is used typically have $\beta \geq 1$, and we restrict attention to this case [3]. With the restriction $\beta \geq 1$, the likelihood function is bounded and it may have a local maximum at a point $(\hat{\eta}, \hat{\beta}, \hat{\gamma})$, with $\hat{\eta} > 0$, $\hat{\beta} > 1$ and $\hat{\gamma} < t_{(1)}$, which are obtained by differentiating the log-likelihood function with respect to η, β and γ, equating the derivative to zero and solve these equations

$$\sum_{i=1}^{n}\delta_i-\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\eta}\right)^{\beta}=0$$
$$\left[\frac{1}{\beta}-\log(\eta)\right]\sum_{i=1}^{n}\delta_i+\sum_{i=1}^{n}\delta_i\log(t_i-\gamma)-\sum_{i=1}^{n}\left(\frac{t_i-\gamma}{\eta}\right)^{\beta}\log\left(\frac{t_i-\gamma}{\eta}\right)=0 \quad (8)$$
$$(1-\beta)\sum_{i=1}^{n}\frac{\delta_i}{t_i-\gamma}+\frac{\beta}{\eta^{\beta}}\sum_{i=1}^{n}(t_i-\gamma)^{\beta-1}=0$$

The maximum likelihood estimator for the median failure time t_{50} is obtained from the maximum likelihood estimators $\hat{\eta}$, $\hat{\beta}$ and $\hat{\gamma}$, that is

$$\hat{t}_{50}=\hat{\gamma}+\hat{\eta}(\log 2)^{\frac{1}{\hat{\beta}}} \quad (9)$$

Asymptotical confidence intervals for $t_{50} = g(\eta,\beta,\gamma)$ are obtained using the delta method, that is, $\hat{t}_{50}\overset{a}{\sim}N[t_{50},\mathrm{var}(t_{50})]$. The asymptotical variance of $\hat{t}_{50}=g(\hat{\eta},\hat{\beta},\hat{\gamma})$ can be obtained by using delta method.

IV. BOOTSTRAP CONFIDENCE INTERVALS FOR t_{50}

In this section we introduce the steps for the construction of bootstrap confidence intervals for t_{50}, the median of the Weibull distribution function. The advantage of the bootstrap is that the joint distribution of the maximum likelihood estimators is not assumed to be normal, unlike in the delta method.

We consider bootstrap method to construct the confidence intervals for t_{50}: the *p*-Bootstrap method suggested by [4], based on the percentiles of the bootstrap distribution. Other existing alternatives for the *p*-Bootstrap, not considered in this paper, could also be used to construct confidence intervals. For a complete review of available approaches to bootstrap confidence intervals, see [5] , [6] and [7].

Let $U=(t,\delta)$ be the observed data where $t = (t_1, ..., t_n)$ is the vector of failure time data and $\boldsymbol{\delta} = (\delta_1, \ldots, \delta_n)$ is the vector of indicators of censored observations

p-Bootstrap

[a] Random select, with replacement from *U*, a bootstrap sample $(t_1^*, \delta_1^*), \ldots, (t_n^*, \delta_n^*)$.
[b] From the bootstrap sample in [a], find the maximum likelihood estimates of t_{50}, denoted by $\hat{t}_{50}^*$.
[c] Repeat steps [a] and [b], *B* times.
[d] From $\hat{t}_{50}^* = (\hat{t}_{50(1)}^* \le \hat{t}_{50(2)}^* \le \ldots \le \hat{t}_{50(B)}^*)$ find a 100×(1 − α)% bootstrap confidence interval given by $(\hat{t}_{50(q_1)}^*, \hat{t}_{50(q_2)}^*)$ where $q_1 = [(\alpha/2)B]$ and $q_2 = B - q_1$.

V. AN ILLUSTRATIVE EXAMPLE

As an illustrative example, consider the sample of failure times (in number of kilometer of use) of vehicle shock absorber (see Table 1), first reported in [8]. Engineer responsible for higher-level automobile system reliability would be interested in the failure-time distribution for this part [9] concluded that the Weibull distribution appears to provide a better description of this data compared to lognormal distribution.

Table 1. Distance to Failure for 38 Vehicle Shock Absorbers

Distance	Failure	Distance	Failure	Distance	Failure
6700	Failure	12870	Censored	20100	Failure
6950	Censored	13150	Failure	20100	Censored
7820	Censored	13330	Censored	20150	Censored
8790	Censored	13470	Censored	20320	Censored
9120	Failure	14040	Censored	20900	Failure
9660	Censored	14300	Failure	22700	Failure
9820	Censored	17520	Failure	23490	Censored
11310	Censored	17540	Censored	26510	Failure
11690	Censored	17890	Censored	27410	Censored
11850	Censored	18450	Censored	27490	Failure
11880	Censored	18960	Censored	27890	Censored
12140	Censored	18980	Censored	28100	Censored
12200	Failure	19410	Censored		

In Tables 2 and 3, we have 95% asymptotical and bootstrap confidence interval for parameters t_{50}. The empirical bootstrap distributions are presented in Fig. 2 (left).

Table 2. Maximum likelihood estimates and asymptotical confidence intervals.

Parameter	MLE	SE	95% Confidence Interval
t_{50}	24721.137	2598.003	(19629.145;29813.128)

To check if the normality of the empirical bootstrap distributions for t_{50} is appropriate, we have in Fig. 2 (right), their normal quantile-quantile plot. If we have normality, then the points in these plots should lie roughly on a straight line. From this plot, we clearly observe that the normality assumption is not appropriate, which justifies the use of bootstrap methods to construct confidence intervals for the parameter. This positively skewed distribution is also supported by the asymmetry index value which is 1.6367 (see Table 4).

Table 3. Bootstrap estimates, *p*-Bootstrap confidence intervals.

Parameter	MLE[a]	SE[a]	95% Confidence Interval *p*-Bootstrap
t_{50}	25082.082	2032.801	(21604.192;29822.508)

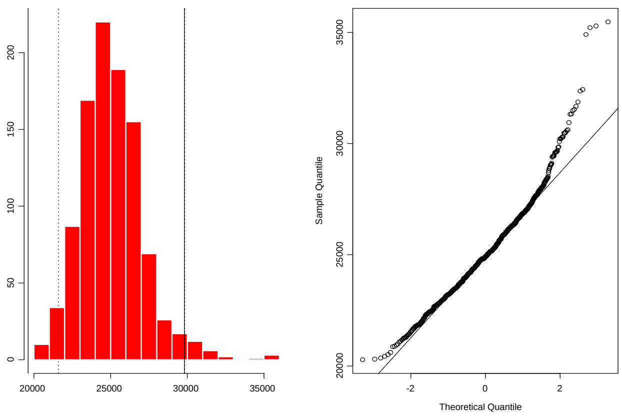

Fig. 2. (Left) Distribution of B = 1000 bootstrap replications for parameter t_{50} along with their 95% confidence intervals, where (—): asymptotic confidence interval and (…): *p*-Bootstrap, (Right) quantile-quantile plots for t_{50}

From the obtained results of Table 4, we observed that the obtained bootstrap confidence intervals are more accurate than the obtained asymptotical intervals. The lengths of interval are 10183.983 and 8218.316 for asymptotic and bootstrap approach, respectively. Beside that the result also provided strong evidence that $\beta > 1$ (not presented here), indicating that the shock absorber population has a hazard function that increases with age. This is consistent with the suggestion that shock absorber tends to wear out

Table 4. Range (R) and asymmetry index (F) for the 95% confidence interval for t_{50}

	t_{50}	
	R	F
Asymptotical	10183.983	1.0000
p-Bootstrap	8218.316	1.6367

VI Conclusion

Considering the three-parameter Weibull distribution, we presented bootstrap based method to construct confidence intervals for the median of failure time. We have showed with numerical example that the nonparametric bootstrap can be quite useful. We observed better inference results considering bootstrap based method in comparison to the usual asymptotical inference based on the normality of the maximum likelihood estimators. As pointed in [2], the difference between asymptotic confidence interval and the bootstrap confidence interval might be due to the fact that the variance obtained by the delta method depends on Taylor series approximation where the error terms are ignored. These terms are part of the variances computed by bootstrapping.

The bootstrap can also be used to obtain confidence intervals for other functions of the parameters. For example, we could obtain confidence intervals for the mean time to failure (MTTF) of the Weibull distribution. The nonparametric bootstrap procedure was implemented using SAS software with macro [10].

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