A New Analytical Integration Expression Applied To Quadrilateral Finite Elements

D. HAMADI[1] and A. ZATAR[2]

***Abstract*—A new analytical integration expression is presented and used to evaluate the element stiffness matrix for the quadrilateral finite elements. For reasons of applications and particularity of the strain based finite elements (higher order shape functions expressed in terms of independent strains); it is necessary to introduce irregular forms, which require a special integration technique, and a specific classification in programming level for different geometric forms. To overcome this geometrical inconvenience; the paper presents the new integration expression with validation tests, and it is applied to some quadrilateral finite elements, this will help to extend their applications for the distorted forms and irregular structures.**

***Index Terms*— Analytical integration, irregular forms, new expression, quadrilateral element.**

I. Introduction

Many researchers among them, Ashwell, Djoudi, Sabir, Salhi and Belarbi, have developed numerous finite elements; some of them were undertaking their research work at Cardiff University in the U.K. The formulated elements are based on the strain approach, and characterized by a regular form and appropriate coordinates with the form of the element.

The strain based approach was further applied by Sabir [1] to develop a new class of elements for general plane of elasticity problems in Cartesian coordinates. A simple an efficient rectangular element including the in-plane rotation is derived. This element was first applied to the simple problem of cantilevers and simply supported beams, where the results for deflections as well as stresses were satisfactory and converged to the exact solution. With the continuation of the development of the strain based approach many elements for general plane elasticity as well as shells have been derived by Sabir [2] and Belarbi [7].

To model a structure which has irregular geometrical shape in real problem, by a limited number of elements as cited above; is not sufficient at all. To overcome this geometrical inconvenience; the paper presents a new integration expression. The performance of this new expression is tested by applying to the analysis of the problems used in previous publications and to obtain solutions for some practical problems in engineering.

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II. Numerical Integration

The element stiffness matrix [Ke] can be calculated using the well known equation (1).

$$[K_e] = \iint_S [B]^T [D][B] \det J . d\xi . d\eta \tag{1}$$

Where:

[B]: the strain matrix

[D]: the behaviour matrix

det J: the determinant of Jacobian matrix

To carry out the integral, we have to choose either numerical integration (e.g Gauss integration) or analytical integration. One of the disadvantages of the numerical integration is the high order of the monomials after the three multiplications of integral matrices (1), which would signify many integration points.

III. A New Analytical Integration

The evaluation of the element stiffness matrix is summarised with the evaluation of the following expression:

$$[K_e] = \left[A^{-1}\right]^T \left[\iint_S [Q]^T [D].[Q].dx.dy\right] \left[A^{-1}\right] \tag{2a}$$

$$[K_e] = \left[A^{-1}\right]^T [K_0] \left[A^{-1}\right] \tag{2b}$$

With: $$[K_0] = \iint_S [Q]^T [D].[Q].dx\, dy \tag{2c}$$

[A] and its inverse can be evaluated numerically, the evaluation of the integral (2c) becomes the key of the problem. In general, the multiplication $[Q]^T$ [D] [Q] can be done manually, we will end up by calculating the double integrals of the form:

$$I = [K_0] = \iint_S C.x^{\alpha} y^{\beta}\, \mathrm{d}\,x.\mathrm{d}\,y \tag{3}$$

Knowing that, for certain elements, a too great distortion can lead to erroneous numerical results particularly in the calculation of the Jacobien. An expression that is general, and easy to implement numerically is being formulated. It allows the evaluation of the matrix [K_0] in an automatic way whatever the degree of the polynomial of the kinematics field and the distortion of the element (Fig.1.).

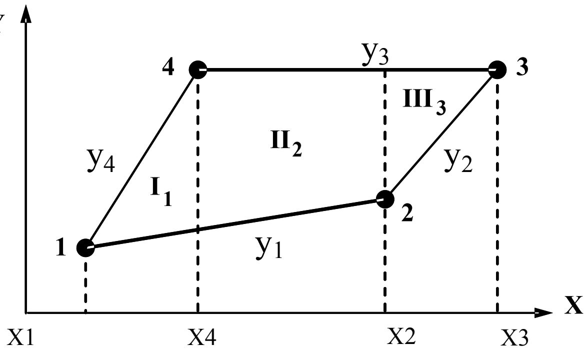


Fig.1. Quadrilateral element

$$I = I_1 + II_2 + III_3 \tag{4}$$

In which

$$\iint x^{\alpha} y^{\beta} dx\,dy = \frac{1}{\beta+1} \sum_{k=1}^{\beta+2} \frac{1}{k+\alpha} C(k) \cdot \left(a_j^{k-1} b_j^{\beta+2-k} - a_i^{k-1} b_i^{\beta+2-k}\right)\left(x_n^{k+\alpha} - x_m^{k+\alpha}\right) \tag{5}$$

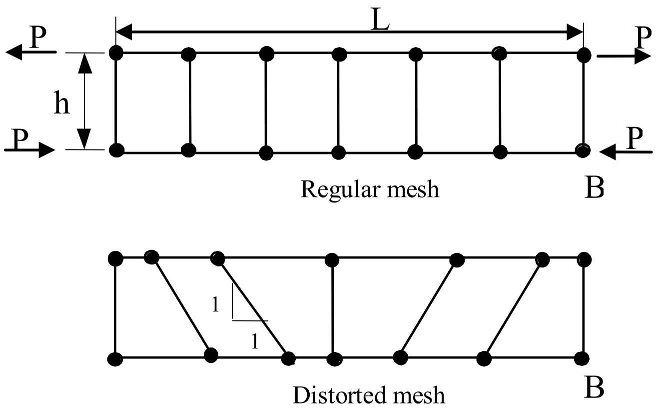
In our case:

$$I = \sum_{P=1}^{3} I_P \tag{6}$$

The general expression of I_P for a quadrilateral would be:

$$I_P = \frac{C}{\beta+1} \sum_{k=1}^{\beta+2} \frac{1}{k+\alpha} C(k) \cdot \left(a_j^{\,k-1} b_j^{\,\beta+2-k} - a_i^{\,k-1} b_i^{\,\beta+2-k}\right)\left(x_n^{\,k+\alpha} - x_m^{\,k+\alpha}\right) \tag{7}$$

The calculation of integral **I** is the principal problem of the calculation of the element stiffness matrix [Ke]. In a very simple and effective manner, the integral is solved by the subroutine "INTEGRATION" [8], [9], see Appendix.



Data:
Young's modulus E = 100, Poisson's ratio $\nu = 0$
P=1, L=10, h=1, Thickness t=1

Fig.2. A simple beam; Data and meshes

IV. Evaluation Of The Element Stiffness Matrix [Ke]

After the programming of the routines which calculate the integral, we can finally carry out the calculation of the element stiffness matrix [Ke].

V. Tests And Applications

In order to investigate the new integration expression is thus developed. We have chosen to examine the membrane element Q4SBE1 [8], [10] through the following tests. These tests are regarded as a tool to validate of the membrane elements.

A. A Simple Beam

A simple beam with a length to height aspect ratio of 10 is subjected to a pure bending state. The beam is modelled by 1x6 meshes with both regular and irregular elements as shown in Fig.2. Only a minimum number of restraints are imposed to eliminate rigid body movement. The load is a unit couple applied at the free end.

This beam is selected as a test problem by Ibrahimbegovic, Taylor and Wilson [11]. The results obtained for both regular and irregular mesh are compared with some of the results available in literature, and the exact solution given by beam's theory. All are presented in Table I.

The results obtained for the distorted element Q4SBE1 are found to be more accurate than the other elements for the same finite element mesh size Table I.

It is observed that the results show very good numerical accuracy obtained for both regular and distorted mesh, and confirm the good performance of the Q4SBE1 element.

Table I: A simple beam under pure bending

Formulation	Mesh	Vertical displacement
Mixte-type [11]	Reg.	1,50000
Mixte-type [11]	Dist.	1,14185
Displ-type [11]	Reg.	1,50000
Displ-type [11]	Dist.	1,14045
Taylor et Simo [12]	Reg.	1,50000
Taylor et Simo [12]	Dist.	1,14195
Q4	Reg.	0,62888
Q4	Dist.	0,26362
Q4SBE1*	Reg.	**1,50000**
Q4SBE1*	Dist.	**1,50000**
Beam's theory		**1,50000**

* Q4SBE1: Strain Based Quadrilateral Element [8], [10]

B. Tapered Panel Under End Shears

This problem, proposed by Cook as a test for the accuracy of quadrilateral elements [13] and Bergan et al. [14], is another popular test problem. A tapered panel of unit thickness with one edge subjected to a distributed shear load and with the other edge fully clamped (u=v=0) is shown in Fig.3. The panel is analyzed by using 2x2 and 4x4 meshes

(Figs. 3a, 3b). The normalized vertical deflection *V*c at point C, maximum principal stress σ_{maxA} at point A and minimum principal stress σ_{minB} at point B are presented in Table II.

The results obtained for the Q4SBE1 element are compared to the other quadrilateral elements. It can be noted that the displacement predictions of the Q4SBE1 are slightly better than the other quadrilateral elements for both meshes (Table II).

The results obtained for the deflection and principal stresses for the refined mesh (4x4) are very good compared to an accurate solution given by Bergan and Felippa using a (32x32) mesh [14] (error 1 %).

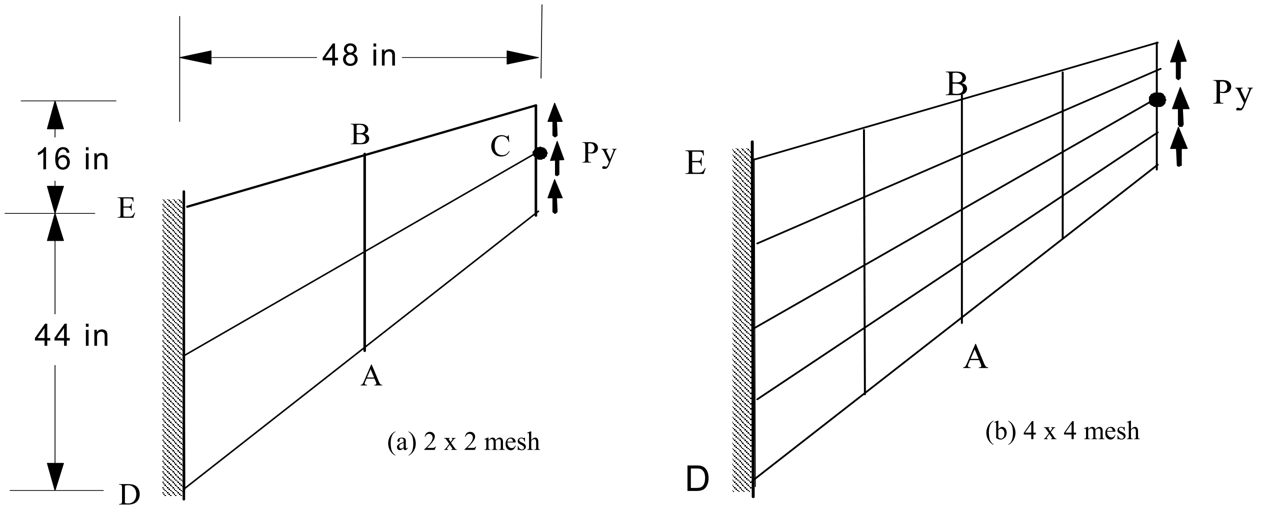


Fig.3: Tapered Panel Subjected to End Shear; Data and Meshes
Py = 1 pi (Uniformly Distributed load)
Young's modulus E = 1 psi
Poisson's ratio ν = 1/3
Thickness t =1 in
Boundary conditions: U=V=0 (DE)

TABLE II: NORMALISED PREDICTION FOR TAPERED PANEL UNDER END SHEAR

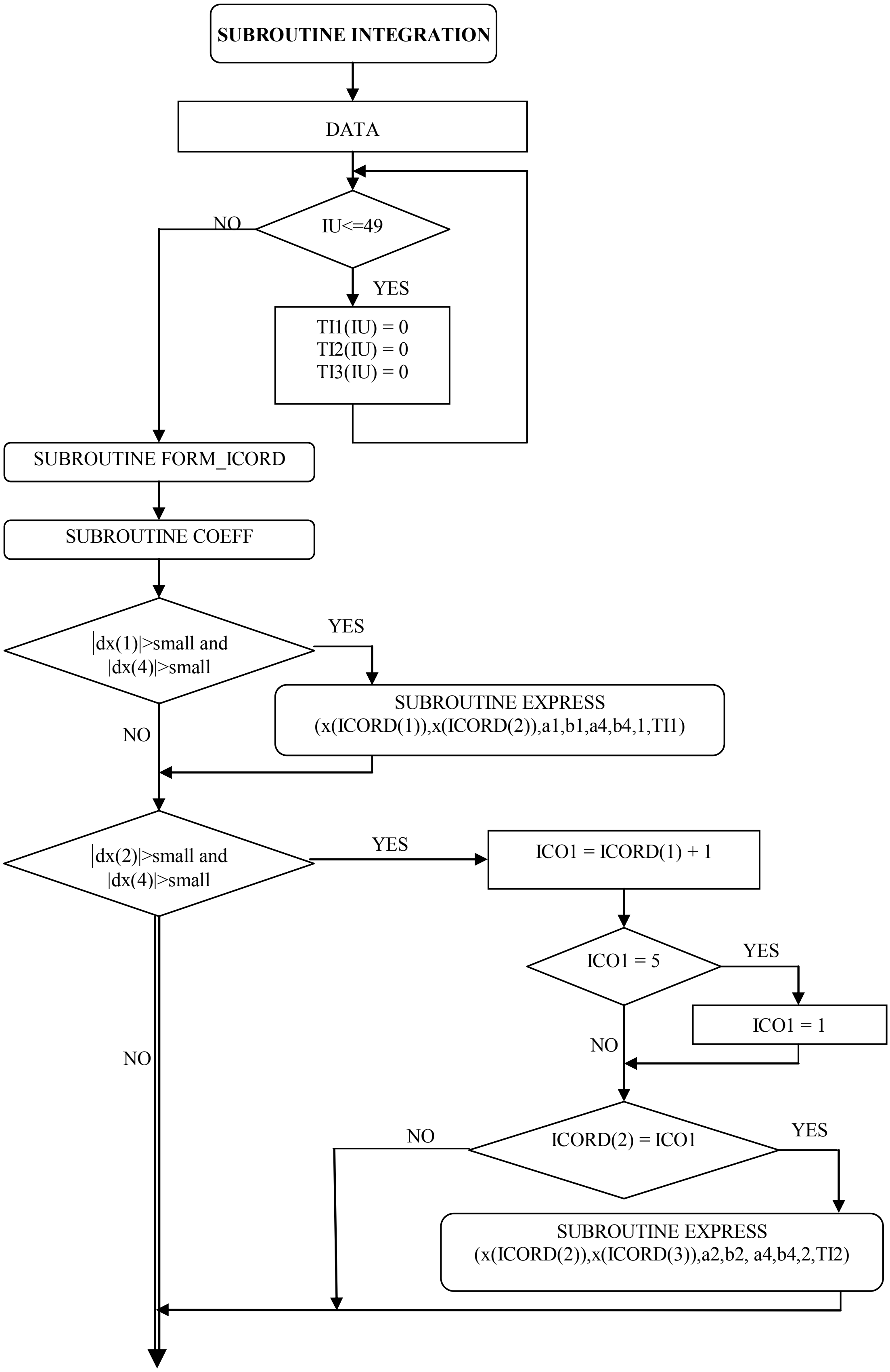
Element model	2 x 2 mesh			4 x 4 mesh		
	V_C	σ_{maxA}	σ_{minB}	V_C	σ_{maxA}	σ_{minB}
Q4	0,496	0,437	0,533	0,766	0,756	0,719
AQ[16]	0,890	0,780	0,900	0,965	0,936	1,010
Ref. [15]	0,848	0,771	0,856	0,953	0,956	0,997
PS5β[17]	0,884	0,786	0,771	0,963	0,950	0,924
MAQ[18)	0,890	0,779	0,886	0,965	0,941	0,967
Ref. [14]	0,852	0,720	0,898	0,938	0,902	0,849
Ref [11]	0,865	-	-	0,962	-	-
07β[19]	0,945	0,835	1,069	0,981	0,982	1,012
Q4SBE1[10]	**1,0652**	**1,508**	**1,171**	**1,011**	**1,004**	**0,992**
32 x 32 mesh **Ref. [14]**	1,000 (23,90)	1,000 (0,236)	1,000 (-0,201)	1,000 (23,90)	1,000 (0,236)	1,000 (-0,201)

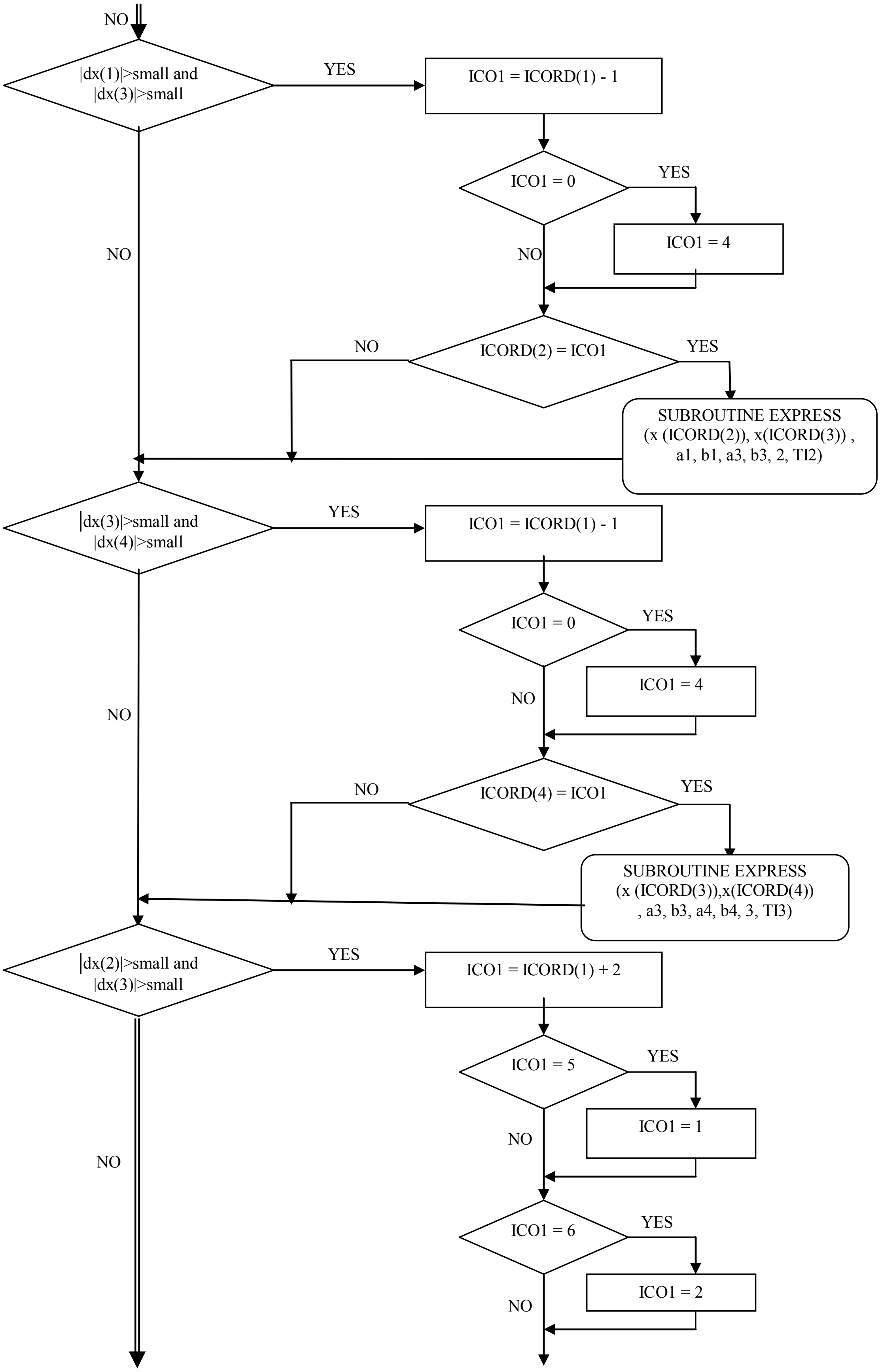
AQ: Cook's quadrilateral counterpart Cook [16].
PS5β*: Pian and Sumihara's four- node five-beta mixed element Pian [17].*
MAQ: a mixed counterpart of AQ using complete linear stress modes (in term of isoparametric coordinates) for all stress components Yunus [18].
07β*: the Size element [19].*

I. CONCLUSION

The efficiency of the new integration expression was shown. The significance of the analytical integration to evaluate the element stiffness matrix for the finite elements with irregular shapes was examined. The good results are partly explained probably by the nature of analytical.

APPENDIX


SUBROUTINE INTEGRATION
DATA
IU<=49
NO
YES
TI1(IU) = 0
TI2(IU) = 0
TI3(IU) = 0
SUBROUTINE FORM_ICORD
SUBROUTINE COEFF
|dx(1)|>small and
|dx(4)|>small
YES
NO
SUBROUTINE EXPRESS
(x(ICORD(1)),x(ICORD(2)),a1,b1,a4,b4,1,TI1)
|dx(2)|>small and
|dx(4)|>small
YES
ICO1 = ICORD(1) + 1
ICO1 = 5
YES
ICO1 = 1
NO
NO
ICORD(2) = ICO1
NO
YES
SUBROUTINE EXPRESS
(x(ICORD(2)),x(ICORD(3)),a2,b2, a4,b4,2,TI2)



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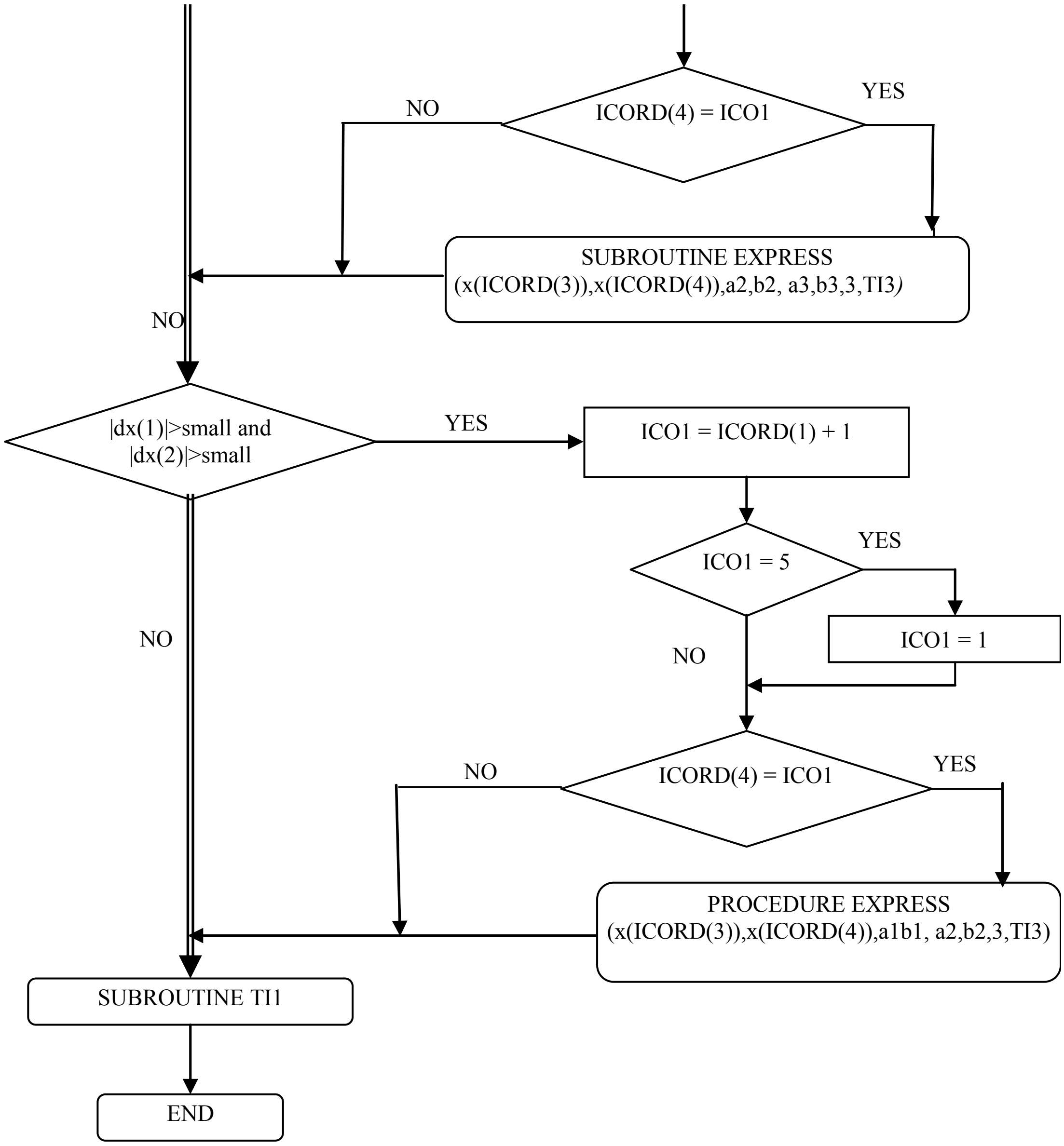

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