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A Regularization Approach to Hedging Collateralized Mortgage Obligations

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***Abstract*— The paper presents a novel regularization approach to the hedging of Collateralized Mortgage Obligations (CMO). Our method is related to well known Option-Adjusted Spread (OAS) methodology, but provides a better way to account for embedded optionality. In this method, the construction of the optimal hedging portfolio of liquid benchmark instruments is considered as an essential part of the valuation procedure. To ensure the uniqueness of the solution of the resulting ill-conditioned optimization problem, the standard Tikhonov type regularization technique is applied. The developed numerical optimization technique is based on the combination of the the Broyden-Fletcher-Goldfarb-Shanno (BFGS) and Newton methods. The numerical results on the hedging of the portfolios of CMO and European swaptions based on Monte Carlo simulation are presented.**

***Index Terms*—Collateralized Mortgage Obligations (CMO), Option-Adjusted Spread (OAS), Monte-Carlo simulation, Tikhonov regularization.**

I. INTRODUCTION

Collateralized Mortgage Obligations (CMO) can have a high degree of variability in cash flows. Because of this, it is generally recognized that a yield to maturity of static spread calculation is not a suitable valuation methodology. Since the 1980's Option-Adjusted Spread (OAS) has become a ubiquitous valuation metric in the CMO market. There have been many criticisms of OAS methodology, and some interesting modifications have focused on the prepayment side of the analysis, e.g., [2, 3]. One of the problems with using OAS analysis is the lack of information about the distribution of the individual spreads, which in turn leads to the difficulties in the construction of the hedging portfolio for CMO.

To improve the CMO valuation methodology and to develop a robust procedure for the construction of the optimal hedge for CMO, we introduce a combination of two new metrics. We start with the term-structure/prepayment model approach of OAS and go on to use the path-by-path structure of the cash flows of a CMO in much more detail. Our methodology is to design a portfolio of swaptions which minimizes the variance in the individual spreads as much as possible, *i.e.*, we are minimizing the spread variance. In doing so, we design an optimal hedge; at least it is optimal from the standpoint of the probability distribution more or less implied by swap rates and swaption prices. It should be emphasized that when calculating spreads we are doing so on the portfolio of CMO and swaptions, thus we are including the cost of hedging in our valuation. Our two main outputs are the mean and the standard deviation of the individual spreads.

This new spread variance minimization (SVM) methodology can lead to quite different conclusions about CMO than OAS does. In particular, in comparing two bonds, the more negatively convex bond may look cheaper on an OAS basis, but richer according to our analysis. This is not simply a difference in opinion: In contrast to OAS analysis, we fully value embedded options.

The main difficulty in implementing our new methodology is in the minimization of our spread-variance functional. The difficulty is partly because the optimization problem is ill-conditioned, and in many situations, this can be overcome by introducing a regularization term. Our approach is to use the standard Tikhonov regularization Tikhonov [4], which has the strong intuitive appeal of limiting the sizes used in hedging.

A word about static versus dynamic hedging may be in order. Our methodology is to set up a static hedge. It may be argued that an essentially perfect hedge may be created dynamically. However, even if one is dynamically hedging, then one can't lock in a positive OAS. For a typical CMO dynamic hedging will cost money. Those costs should be discounted flat and thus will decrease a positive spread. . A dynamically-hedged portfolio will not have the OAS as a spread. Moreover, ones hedging costs will be related to the amount of volatility in the future so can be quite uncertain. Our methodology greatly reduces dynamic hedging costs by setting up an optimized static hedge, and thus reduces the uncertainty in dynamic hedging costs.

The rest of the paper is organized as follows. In the second section we briefly review OAS and point out in more detail the problems we see with it. In the third section we explicitly define our hedging methodology. In the forth section we give a brief description of the regularized numerical method. In the fifth section we present some details on the term-structure model used; on our prepayment assumptions and summarize numerical results from our analysis.

II. OPTION-ADJUSTED SPREAD ANALYSIS

Standard OAS analysis is based on finding a spread at which the expected value of the discounted cash flows will be equal to the market value of a CMO. This is encapsulated in eqn. (1):

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$$MV_{CMO} = \tilde{E}\sum_{i=1}^{L} d(t_i, s)\cdot c(t_i)\,. \qquad (1)$$

Here MV_{CMO} is the market value of CMO, $\tilde{E}$ denotes expectation with respect to the risk neutral measure, $d(t_i, s)$ is the discount factors to time t_i, $i = 1,..., L$ with a spread of *s* and $c(t_i)$ is the cash flow at time t_i. Note that in the OAS framework, a single spread term is added to the discounted factors to make this formula a true equality, this spread, *s*, is referred to as *the OAS*.

The goal of OAS analysis is to value a CMO relative to liquid benchmark interest rate derivatives, and, thus, the risk-neutral measure is derived to price those benchmarks accurately. To calculate expected values in practice one uses Monte-Carlo simulation of a stochastic term-structure model. We use the two-factor Gaussian short-rate model G2++ as in Brigo and Mercurio [1] calibrated to U.S. swaption prices, but other choices may be suitable. In terms of numerical approximation, it will give us eqn. (2):

$$MV^k_{bench} = \frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{i=1}^{L} cf^k_{bench}(n,t)\prod_{t=1}^{t}\frac{1}{1+\Delta t_i\cdot r(n,t_i)}\right\} + Err\,. \qquad (2)$$

Here $\Delta t_i = t_i - t_{i-1}$, $i = 1,..., L$, MV^k_{bench}, $k = 1, \ldots, M$ are the market values of the benchmarks, $cf^k_{bench}(n,t_i)$, $n = 1, \ldots, N$, $i = 1, \ldots, L$, $k = 1, \ldots, M$ are the future cash flows of the benchmarks, *N* is the number of generated trajectories, *L* is the number of time intervals until expiration of all benchmarks and CMO, and *M* is the number of benchmarks in the consideration. The last term *Err* represents the error term. Though, the detailed consideration of calibration procedure is outside the scope of this presentation, it is worth to mention that the absolute value of the Err term is bounded in most of our experiments by five basis points.

The second step in the OAS analysis of CMOs is to find the spread term from the eqn. (3):

$$MV_{CMO} = \frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{i=1}^{i} cf(n,t_i)\prod_{j=1}^{i}\frac{1}{1+\Delta t_j\cdot(r(n,t_j)+s)}\right\}. \qquad (3)$$

Here $cf^k(n,t_i)$, $n = 1, \ldots, N$, $i = 1, \ldots, L$, are cash flows of the CMO. These cash flows come from the structure of the CMO, a perfectly known quantity, and a prepayment model, a more subjectively known quantity. The parameter *s*, the OAS, is an indicator as to whether the CMO is underpriced or overpriced: If the OAS is positive then the CMO is underpriced, if it is negative then the CMO is overpriced. Not only its sign, but also the magnitude of the OAS commonly quoted as a measure of cheapness of a CMO.

An important issue is managing a portfolio of CMOs is how to hedge the portfolio by using actively traded benchmarks. We return to this question little bit later but for now let's assume that we somehow have found the list and corresponding weights of benchmarks that would provide a sufficient hedge for our portfolio. To evaluate this portfolio, we will extend the OAS analysis to the valuation of portfolio of CMO and the benchmarks. The straightforward extension of OAS approach will result in eqn. (4):

$$MV_{CMO} + \sum_{k=1}^{M} w^k MV^k_{bench} =$$

$$\frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{i=1}^{L}\left[cf(n,t_i) + \sum_{k=1}^{M} w^k\cdot cf^k_{bench}\right]\prod_{j=1}^{i}\frac{1}{1+\Delta t_j\cdot(r(n,t_j)+s)}\right\}. \qquad (4)$$

This consideration makes some serious drawbacks and flaws of OAS analysis apparent. Some of them are:

- It matches a mean to MV, but provides no information on the distribution of the individual discounted values.
- One can use it to calculate some standard risk metrics, but it gives no way to design a refined hedge.
- It is sensitive to positions in your benchmarks. Suppose the OAS on a CMO is positive. Now suppose you have a short position in group of benchmarks. Then necessarily the OAS of your portfolio increases. This is simply because of eqn. (5):

$$\begin{aligned} MV_{bench} &= \frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{i=1}^{L} cf_{bench}(n,t_i)\prod_{j=1}^{i}\frac{1}{1+\Delta t_j\cdot r(n,t_j)}\right\} \\ &> \frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{i=1}^{L} cf_{bench}(n,t_i)\prod_{j=1}^{i}\frac{1}{1+\Delta t_j\cdot(r(n,t_j)+s_{OAS})}\right\} \end{aligned} \qquad (5)$$

and so

$$MV_{CMO} - MV_{bench} <$$

$$\frac{1}{N}\sum_{n=1}^{N}\left\{\sum_{t=1}^{T}\left[cf(n,t) - cf_{bench}(n,t)\right]\prod_{j=1}^{i}\frac{1}{1+\Delta t_j\cdot(r(n,t_j)+s_{OAS})}\right\}. \qquad (6)$$

This shows that OAS is sensitive to leveraging by shorting a combination of swaptions to increase the s_{OAS}.

III. The Spread Variance Minimization(SVM) Approach

In our approach, instead of using the same spread for all paths, we are looking at the individual spreads for every path of the portfolio of CMO and benchmarks. We try to find a portfolio of benchmarks so that we minimize the variation in the individual spreads. The spread for path *n* is the value *s(n)* such that eqn. (7) satisfied:

$$\begin{aligned} &MV_{CMO} + \sum_{k=1}^{M} w^k MV^k_{bench} = \\ &\sum_{i=1}^{L}\left[cf(n,t_i) + \sum_{k=1}^{M} w^k\cdot cf^k_{bench}(n,t_i)\right]\prod_{j=1}^{i}\frac{1}{1+\Delta t_i\cdot(r(n,t_j)+s(n))} \end{aligned}, \qquad (7)$$

where the w^k, $k=1, \ldots, M$ are the weights of the individual benchmarks (a negative indicates a short position). The goal is to apply an optimization algorithm to find weights so that the variance of the *s(n)* is a minimum.

In this form the problem is not well defined; the weights may exhibit some instability and tend to drift to infinity: Buy larger and larger amounts of the entire portfolio of benchmarks and your individual spreads will all approach zero, so your spread variance will approach zero. A common approach to the solution of this type of problem is to add regularization term to the target functional. Instead of minimizing *var(s(n))*, minimize *var(s(n))*+ $\alpha\sum_k [w^k]^2$ Tikhonov [4], where α is small, on the order of 10^{-9}. Tikhonov regularization makes the problem well defined and a solution method stable. From the hedging point of view, such regularization provides the bounded vector of optimal weights for the benchmarks and so introduces practicalities in the implementation of an actual hedge.

IV. Numerical method

As it was mentioned before, we are considering that for every interest rate trajectory the individual spread depends on the weights for benchmarks in the portfolio. Then the target functional is defined in eqn. (8):

$$f(w_1,...,w_n) = \frac{1}{N}\sum_{n=1}^{N} s(n,w_1,...,w_n)^2 - \mu^2, \qquad (8)$$

where $\mu = \frac{1}{N}\sum_{n=1}^{N} s(n,w_1,...,w_n)$. The Jacobian and Hessian of the function are given by eqn. (9):

$$\frac{\partial f}{\partial w_i} = \frac{2}{N}\sum_{n=1}^{N}(s(n,w_1,...,w_n)-\mu)\frac{\partial s(n,w_1,...,w_n)}{\partial w_i}, \; i = 1,...,M, \qquad (9)$$

$$\frac{\partial^2 f}{\partial w_i \partial w_l} = \frac{2}{N}\sum_{n=1}^{N}\left\{\frac{\partial s}{\partial w_i}\cdot\frac{\partial s}{\partial w_l} + (s-\mu)\frac{\partial^2 s}{\partial w_i \partial w_l}\right\} - 2\cdot\left\{\frac{1}{N}\sum_{n=1}^{N}\frac{\partial s}{\partial w_i}\right\}\cdot\left\{\frac{1}{N}\sum_{n=1}^{N}\frac{\partial s}{\partial w_l}\right\}, \; i,l = 1,...,M. \qquad (10)$$

Using implicit differentiation one can find $\frac{\partial s}{\partial w_i}, \frac{\partial^2 s}{\partial w_i \partial w_l}, \quad i,l = 1,...,M$. But this functional in general is ill-conditioned. To ensure the convergence of the optimization method, we introduce standard Tikhonov regularization. The modified target functional could be presented as in eqn. (11):

$$\tilde{f}(w_1,...,w_n) = f(w_1,...,w_n) + \alpha\|w\|_2^2. \qquad (11)$$

In most situations this guaranties the convergence of the numerical method to the unique solution. In our approach we are using the optimization technique based on the combination of the Broyden-Fletcher-Goldfarb-Shanno (BFGS) Vogel [5] and Newton methods. The BFGS approach is applied on the early iterations. When the $l_2 - norm$ of the gradient of the target function becomes less then 10^{-10} we apply the Newton method assuming that the approximation is already close enough to the solution and the quadratic convergence rate of the Newton method can be achieved.

As we already mentioned, this regularization keeps the value of the target functional small and stabilizes the optimization method by preventing the weights of the benchmarks in the portfolio w_i from becoming large through the penalty term $\alpha\|w\|_2^2$. To keep the condition number of the Hessian bounded, one has to use fairly large regularization parameter. On the other hand, in order to keep the regularized problem reasonably close the original one, we expect that α needs to be small. In addition to these pure mathematical requirements, in the case of managing a portfolio, one has to take into consideration the cost of the hedging. Since regularization term prevent the optimal weights drift to infinity, the regularization parameter becomes a desirable tool in keeping hedging cost under control. In our experiments we found that the parameter $\alpha = 10^{-9}$ represents a good choice for the regularization in most numerical experiments.

V. Numerical results

To illustrate our proposed methodology we consider hedging an unstructured trust IO (interest only) strip with a basket of European swaptions of different strikes and expiration dates, all with ten-year tenor. To generate the trajectories of the short-rate, we use the two-additive-factor Gaussian model G2++ Brigo and Mercurio [1]. The dynamics of the instantaneous-short-rate process under the risk-neutral measure are given by $r(t) = x(t) + y(t) + \varphi(t), \; r(0) = r_0$, where the processes $\{x(t): t \geq 0\}$ and $\{y(t): t \geq 0\}$ satisfy (12):

$$dx(t) = -ax(t)dt + \sigma dW_1(t), \; x(0) = 0,$$
$$dy(t) = -by(t)dt + \eta dW_2(t), \; y(0) = 0. \qquad (12)$$

Here (W_1, W_2) is a two-dimensional Brownian motion with instantaneous correlation ρ : $dW_1(t)dW_2(t) = \rho dt$, . The parameters a, b, σ, η, ρ are defined in the calibration procedure to match the prices of a set of actively traded European swaptions. The $l_2 - norm$ of the difference vector of the model swaption prices and the market prices in our experiments was bounded by five basis points.

In our experiments, we will use the standard deviation of the spread as an indicator of the quality of the hedge. The bond under consideration has the following parameters: WALA = 60 months, WAC = 6%, coupon = 5.5%, and a price of $22.519. We are using very simple prepayment model defined by linear interpolation of the data from Table 1.

Table 1: Prepayment rates.

Interest rates(%)	3.0	3.5	4.0	4.5	5.0
CPR(%)	70	50	30	15	10
Interest rates(%)	5.5	6.0	6.5	7.0	7.5
CPR(%)	8	4	3	3	3

For optimization we use a basket of 64 payer and receiver European swaptions with yearly expiration dates from years 1 to 16, including at the money (ATM) and +/- 50 out of the money. The regularization parameter used is 10^{-9}, and the number of trajectories in all experiments was 500. The SVM numerical method was implemented in Matlab with using the BFGS standard implementation available in the Matlab optimization toolbox. The computer time for the Matlab optimization routine was 42 sec on the standard PC with 2Hz processor frequency.

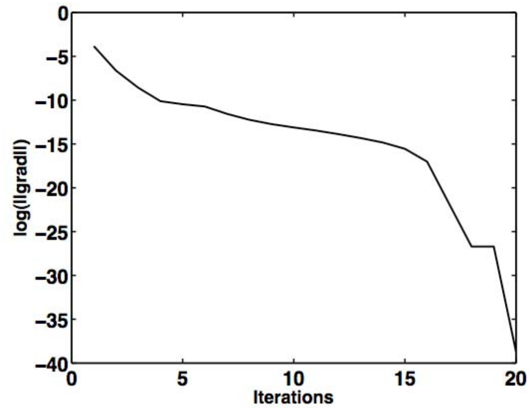

Figure 1: Convergence history.

The Fig. 1 represents the convergence history of the iterations in the optimization method. Fig. 2 shows the square root of the target functional, which is approximately the standard deviation of the spread distribution. One can see that as a result of the application of the new methodology, the standard deviation of the spread distribution reduced significantly, which in turn could be considered as reduction in risk of the portfolio of CMO and benchmarks.

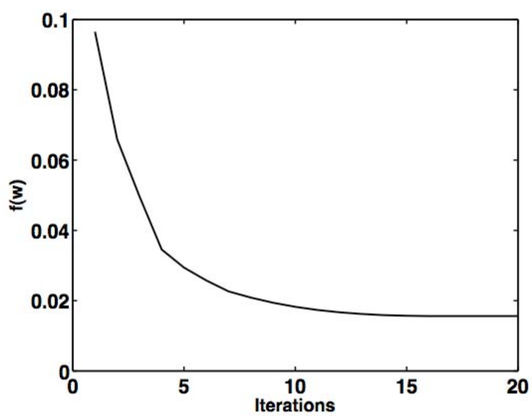

Figure 2: Convergence of the target functional.

Fig. 3 presents the evolution of weights in our portfolio.

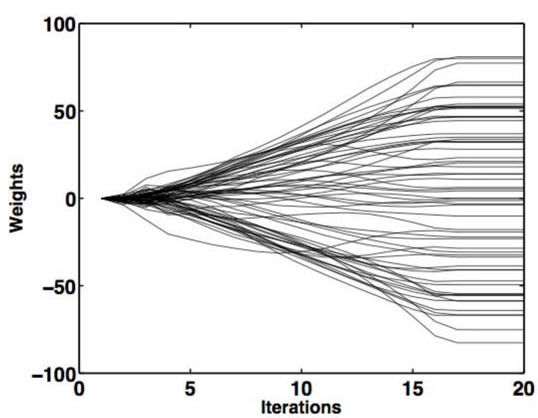

Figure 3: Evolution of weights.

In the next series of experiments, we illustrate the quality of the hedge constructed using different numbers of swaptions. Table 2 presents the results of these experiments. In our experiments we used the IO strip in combination with different numbers of swaptions. For reference we include results with no hedge at all; this is the common OAS analysis (but we also include the mean spread). For our SVM approach, we first use just two ATM swaptions, one payer and one receiver, both with expiration date in one year. Then we use eight swaptions with expiration dates of 1 and 2 years. For each of these expirations we include an ATM payer, an ATM receiver, an ATM+50 bps payer, and an ATM-50 bps receiver. The test with twenty options uses expiration dates 1 to 5 years, again with four swaptions per expiration date. In the last experiment we use swaptions with expiration dates 1 to 16. In all of these experiments we used one unit of the trust IO with the cost of $22.519.

As we can see from the first line of the table, the cost of the hedge is an increasing function of the number of options. But the most importantly, we manage to significantly decrease the standard deviation of the spread distribution by using spread variance minimization methodology.

Table 2: Hedging results.

Number of Swaptions	0	2	8	20	64
Cost of hedge($)	0	-.0020	5.6284	6.0144	11.3618
σ (basis points(bps))	870	614	407	224	133
Mean spread (bps)	-71	-18	37	49	33
OAS (bps)	127	101	95	68	40

Notice that, when using just two options, the cost of our hedging portfolio is essentially zero, we are buying the ATM receiver swaption and selling an equal amount of the ATM payer swaption. This is effectively entering into a forward rate agreement, and so this case could be considered as a hedging strategy based on only the duration of portfolio. The experiments with the larger set of swaptions is a refinement this strategy and take into account more detail about the structure of the cash flows of the bond. It proves to be a very successful approach to the hedging of the path-dependent bond.

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The last two rows of the table present two different metrics: The mean spread and the OAS of the portfolio of bond and hedges. We can see that they become close as standard deviation decreases. In fact, they would be the same if the standard deviation becomes zero. With no swaptions, or with very few of them, these metrics can result in drastically different conclusions about the cheapness of the bond. Contrary to a common interpretation of OAS, it does not represent the mean spread of an unhedged bond. In fact, it can be expected to be close to the mean spread only when considering a refined hedging portfolio of swaptions. Though the discussion of the advantages of the SVM analysis is outside of the scope of this paper, it is worth mentioning that it caries significant information about the path-dependent bond and is worth taking into account alongside the standard OAS parameter.

VI. CONCLUSION

In this paper we presented a regularization approach to the construction of an optimal portfolio of CMO and swaptions. The standard Tikhonov regularization term serves as an important tool for preventing the weights of the benchmarks in the optimal portfolio drift to the infinity and so keeps the hedging procedure practical. The optimization method demonstrates excellent convergent properties and could be used in practical applications for hedging a portfolio of CMO. The numerical results demonstrate the effectiveness of the proposed methodology. The future development may include the construction of new target functional based on the combination of the spread variance and cost function. This modification might improve the efficiency of the developed hedging strategy.

There appears to be a wide variety of application of our SVM approach, and in general of systematic path-by-path analysis of securities with stochastic cash flows. We plan to apply our analysis to other CMO structures directly. In addition, our analysis lends itself to comparing structured CMO with liquid strip interest only and principal only bonds.

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