

# Method of Transformation from Inertial to Reference Frame Formulation of Flexible Multibody Systems

Dmitry Vlasenko, Roland Kasper

**Abstract:** This paper presents a method of the reformulation of equations in the inertial frame formulation to the floating frame formulation. Results of a simulation example, performed using the modal transformation, show the good agreement with the standard inertial frame methods.

**Index Terms:** Flexible multibody systems, floating frame of reference formulation, inertial frame formulation, ANSYS.

## I. INTRODUCTION

In recent years, considerable effort has been devoted to modeling, design and control of flexible multibody systems (FMS).

The *inertial frame formulation (IFF)* and the *floating frame of reference formulation (FFRF)* are nowadays two popular techniques for the simulation of dynamics of FMS, which are widely implemented in many multibody simulation software (e.g. ABAQUS/explicit, Simpack, etc.) and in engineering simulation software (e.g. ANSYS) [1], [2].

Nowadays the FFRF offers the most efficient method for the simulation of FMS undergoing small elastic deformations and slow elastic speed. In the FFRF the motion of elements is considered as a combination of the rigid body motion and deformations. This approach allows using the modal reduction technique, which greatly decreases the simulation time.

Because of the wide use of both simulation approaches it seems very promising to develop a method of the transformation of equations of motion, obtained using IFF, to the equations in the FFRF. This method can be used as a basement for an interface between various simulation tools, using different formulations.

Let us consider a finite-element ANSYS model of a flexible body, consisting on continuum elements. Let  $n$  denote the number of nodes in the simulated body. Nowadays the nodes in all continuum elements in ANSYS have no rotation degrees freedom (DOFs), but only three translation DOFs (e.g. SOLID45, SOLID95, SOLID46, etc) [4]. Using IFF, implemented in ANSYS, we can write the following equations of body motion in the absence of constraints:

$$\mathbf{M}_A \ddot{\mathbf{u}} + \mathbf{K}_A \mathbf{u} = \boldsymbol{\tau} \quad (1)$$

where  $\mathbf{u}$  denote the  $3 \cdot n$  vector of nodal displacements,  $\boldsymbol{\tau}$  is the  $3 \cdot n$  length vector of nodal point loads,  $\mathbf{M}_A$  is a mass matrix, and  $\mathbf{K}_A$  is a stiffness matrix.

Below we show how we can reformulate the equations of motion of the model in the FFRF, using the initial values  $\mathbf{M}_A|_{t=0}$ ,  $\mathbf{K}_A|_{t=0}$  generated by ANSYS. The method was implemented for the simulation of the dynamics of a flexible pendulum example.

## II. KINEMATICS OF A MODEL

Let us consider an orthonormal body coordinate system  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ . Let  $\mathbf{x}_0$  denote the absolute coordinates of the body reference with respect to the global coordinate system  $\{O \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3\}$ , as it is shown in Fig. 1.

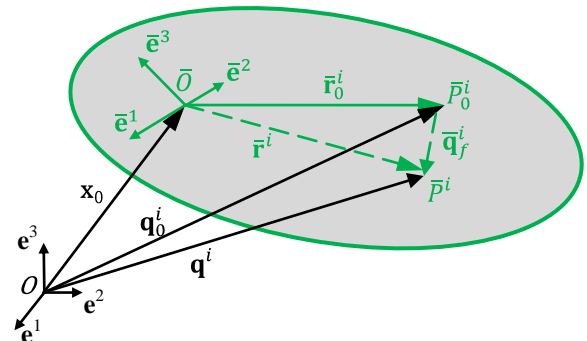


Fig. 1 General model of a flexible body

Let  $\mathbf{q}_0^i$  be the absolute position of the  $i$ -th node in the undeformed state:

$$\mathbf{q}_0^i = \mathbf{x}_0 + \mathbf{A}(\boldsymbol{\theta}_0) \cdot \bar{\mathbf{r}}_0^i \quad i = 1..n \quad (2)$$

where  $\bar{\mathbf{r}}_0^i$  is position of the  $i$ -th node in the undeformed state in body frame,  $\mathbf{A}(\boldsymbol{\theta}_0)$  is the transformation matrix that defines the orientation of  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$  with respect to  $\{O \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3\}$ ,  $\boldsymbol{\theta}_0$  is the vector of Euler parameters.

In vector form we can write (2) as

$$\mathbf{q}_0 = \mathbf{x}_0^n + \mathbf{A}^n \bar{\mathbf{r}}_0 \quad (3)$$

where,  $\mathbf{q}_0 = (\mathbf{q}_0^1^T \dots \mathbf{q}_0^n^T)^T$ ,  $\bar{\mathbf{r}}_0 = (\bar{\mathbf{r}}_0^1^T \dots \bar{\mathbf{r}}_0^n^T)^T$ ,  $\mathbf{x}_0^n$  is a  $3n$ -length vector, consisting of  $\mathbf{x}_0$ :

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$$\mathbf{x}_0^n = (\mathbf{x}_0^T \quad \dots \quad \mathbf{x}_0^T)^T \quad (4)$$

$\mathbf{A}^n$  is a  $(3n, 3n)$  block diagonal matrix, consisting of  $\mathbf{A}$ :

$$\mathbf{A}^n = \text{diag}_i(\mathbf{A}) \quad i = 1..n \quad (5)$$

Let  $\mathbf{q}^i$  denote the vector of absolute position of the  $i$ -th node. Then the vector of the deformation of the  $i$ -th node in inertial frame can be defined as:

$$\mathbf{q}_f^i = \mathbf{q}^i - \mathbf{q}_0^i \quad i = 1..n \quad (6)$$

Using (2), we get the system of equations:

$$\mathbf{q}^i = \mathbf{x}_0 + \mathbf{A}(\bar{\mathbf{r}}_0^i + \bar{\mathbf{q}}_f^i) \quad i = 1..n \quad (7)$$

where  $\bar{\mathbf{q}}_f^i$  is the deformation vector of the  $i$ -th node in the body frame  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ . Let  $\bar{\mathbf{r}}^i$  denote the position of the  $i$ -th node in the deformed state in  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ .

$$\bar{\mathbf{r}}^i = \bar{\mathbf{r}}_0^i + \bar{\mathbf{q}}_f^i \quad i = 1..n \quad (8)$$

Substituting  $\bar{\mathbf{r}}^i$  in (7), we get

$$\mathbf{q}^i = \mathbf{x}_0 + \mathbf{A}\bar{\mathbf{r}}^i \quad i = 1..n \quad (9)$$

This system of equations can be rewritten as

$$\mathbf{q} = \mathbf{x}_0^n + \mathbf{A}^n \bar{\mathbf{r}} \quad (10)$$

where  $\mathbf{q} = (\mathbf{q}^1^T \quad \dots \quad \mathbf{q}^n^T)^T$  and  $\bar{\mathbf{r}} = (\bar{\mathbf{r}}^1^T \quad \dots \quad \bar{\mathbf{r}}^n^T)^T$

### III. RELATION BETWEEN INERTIA SHAPE INTEGRAL $\bar{\mathbf{S}}_{ff}$ AND $\mathbf{M}_A$

The mass matrix  $\mathbf{M}_A$  in the IFF is a symmetric matrix defined as [4]

$$\mathbf{M}_A = \int_V \rho \mathbf{N}^T \mathbf{N} dV \quad (11)$$

where  $\mathbf{N}$  is the matrix of shape functions, setting the relation between the displacements within the body  $\mathbf{w}$  and the vector of nodal displacements  $\mathbf{u}$  in  $\{O \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3\}$ :

$$\mathbf{w} = \mathbf{N} \mathbf{u} \quad (12)$$

Let  $\bar{\mathbf{N}}$  denote a constant matrix of shape functions in the body frame  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ :

$$\bar{\mathbf{w}} = \bar{\mathbf{N}} \bar{\mathbf{u}} \quad (13)$$

where  $\bar{\mathbf{w}}$  is the vector of displacements, expressed in  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ ,  $\bar{\mathbf{u}}$  is the vector of nodal displacements, expressed in  $\{\bar{O} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ .

Our goal is to find the relation between  $\mathbf{M}_A$  and the time-constant inertia shape integral  $\bar{\mathbf{S}}_{ff}$ , defined as [3]:

$$\bar{\mathbf{S}}_{ff} = \int_V \rho \bar{\mathbf{N}}^T \bar{\mathbf{N}} dV \quad (14)$$

From the definitions of  $\mathbf{N}$  and  $\bar{\mathbf{N}}$  follows that

$$\bar{\mathbf{N}} = \mathbf{A}^T \mathbf{N} \mathbf{A}^n \quad (15)$$

Substituting  $\bar{\mathbf{N}}$  in (14), we get:

$$\bar{\mathbf{S}}_{ff} = \int_V \rho \mathbf{A}^n \mathbf{A}^T \mathbf{N}^T \mathbf{A} \mathbf{A}^T \mathbf{N} \mathbf{A}^n dV = \mathbf{A}^n \mathbf{M}_A \mathbf{A}^n \quad (16)$$

Therefore

$$\mathbf{M}_A = \mathbf{A}^n \bar{\mathbf{S}}_{ff} \mathbf{A}^n \quad (17)$$

This equation can be used for the calculation of  $\bar{\mathbf{S}}_{ff}$  from of the mass matrix  $\mathbf{M}_A|_{t=0}$  and the rotation matrix  $\mathbf{A}|_{t=0}$ , obtained from ANSYS. In future we will assume that the axes  $\bar{\mathbf{e}}^1, \bar{\mathbf{e}}^2, \bar{\mathbf{e}}^3$  are initially parallel to the axes  $\mathbf{e}^1, \mathbf{e}^2, \mathbf{e}^3$ , i.e.  $\mathbf{A}|_{t=0} = \mathbf{E}$  where  $\mathbf{E}$  is the identity (3,3) matrix. Hence

$$\bar{\mathbf{S}}_{ff} = \mathbf{M}_A|_{t=0} \quad (18)$$

It can be also shown that from the absence of deformations in the free-fall case follows that

$$\left( \sum_{j=1}^n \bar{\mathbf{S}}_{ff}^{ij} \right) = k^i \mathbf{E} \quad i = 1..n \quad (19)$$

where  $k^i$  is a coefficient, corresponding to the  $i$ -th node.

The symmetry of  $\bar{\mathbf{S}}_{ff}$  implies that

$$\left( \sum_{i=1}^n \bar{\mathbf{S}}_{ff}^{ij} \right) = k^j \mathbf{E} \quad j = 1..n \quad (20)$$

### IV. MASS AND STIFFNESS MATRICES

#### A. Mass matrix $\mathbf{M}$

The kinetic energy  $T$  of the body in the IFF can be written as [4]:

$$T = \frac{1}{2} \dot{\mathbf{u}}^T \mathbf{M}_A \dot{\mathbf{u}} = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}_A \dot{\mathbf{q}} \quad (21)$$

Our goal is to find from the initial value of  $\mathbf{M}_A|_{t=0} = \bar{\mathbf{S}}_{ff}$  the formula for the calculation of the mass matrix  $\mathbf{M}$  in FFRE, used for the calculation of kinetic energy

$$T = \frac{1}{2} \dot{\mathbf{p}}^T \mathbf{M} \dot{\mathbf{p}} \quad (22)$$

where  $\mathbf{p} = (\mathbf{x}_0^T \quad \boldsymbol{\theta}_0^T \quad \bar{\mathbf{q}}_f^T)^T$ ,  $\mathbf{q}_f = (\mathbf{q}_f^1^T \quad \dots \quad \mathbf{q}_f^n^T)^T$   
Differentiating (9), we get:

$$\dot{\mathbf{q}}^i = \dot{\mathbf{x}}_0 - \mathbf{A}^n \tilde{\mathbf{r}}^i \bar{\boldsymbol{\omega}} + \mathbf{A}^n \dot{\mathbf{q}}_f^i \quad i = 1..n \quad (23)$$

where  $\bar{\boldsymbol{\omega}}$  is the angular velocity vector defined in  $\{\bar{\mathbf{O}} \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ . The vector  $\bar{\boldsymbol{\omega}}$  can be expressed using the time derivation  $\dot{\boldsymbol{\theta}}_0$  of the body rotational coordinates:

$$\bar{\boldsymbol{\omega}} = \bar{\mathbf{T}}(\boldsymbol{\theta}_0) \dot{\boldsymbol{\theta}}_0 \quad (24)$$

Substituting (24) in (23), we get

$$\dot{\mathbf{q}}^i = \dot{\mathbf{x}}_0 - \mathbf{A}^n \tilde{\mathbf{r}}^i \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 + \mathbf{A}^n \dot{\mathbf{q}}_f^i \quad i = 1..n \quad (25)$$

It will be more convenient to rewrite this system of equations as a matrix equation:

$$\dot{\mathbf{q}} = \dot{\mathbf{x}}_0^n - \mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 + \mathbf{A}^n \dot{\mathbf{q}}_f \quad (26)$$

where  $\tilde{\mathbf{r}} = (\tilde{\mathbf{r}}^1^T \dots \tilde{\mathbf{r}}^n^T)^T$ .

Substituting (26), (17) in (21), we get:

$$\mathbf{T} = (\dot{\mathbf{x}}_0^n - \mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 + \mathbf{A}^n \dot{\mathbf{q}}_f)^T \cdot \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \mathbf{A}^n \cdot (\dot{\mathbf{x}}_0^n - \mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 + \mathbf{A}^n \dot{\mathbf{q}}_f) \quad (27)$$

Or, in another form:

$$\mathbf{T} = \mathbf{T}_{xx} + 2\mathbf{T}_{x\theta} + 2\mathbf{T}_{xf} + \mathbf{T}_{\theta\theta} + 2\mathbf{T}_{\theta f} + \mathbf{T}_{ff} \quad (28)$$

where

$$\begin{aligned} \mathbf{T}_{xx} &= \frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \mathbf{A}^n \dot{\mathbf{x}}_0^n \\ \mathbf{T}_{x\theta} &= -\frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 \\ \mathbf{T}_{xf} &= \frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \dot{\mathbf{q}}_f \\ \mathbf{T}_{\theta\theta} &= \frac{1}{2} (\mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0)^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 \\ \mathbf{T}_{\theta f} &= -\frac{1}{2} (\mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0)^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \dot{\mathbf{q}}_f \\ \mathbf{T}_{ff} &= \frac{1}{2} \dot{\mathbf{q}}_f^T \bar{\mathbf{S}}_{f,f} \dot{\mathbf{q}}_f \end{aligned}$$

Let us consider now the component of the sum in (27) more precisely.

The part  $\mathbf{T}_{xx}$  is equal to the kinetic energy of the translation of the non-deformed body:

$$\begin{aligned} \mathbf{T}_{xx} &= \frac{1}{2} \dot{\mathbf{x}}_0^T \mathbf{A} (\sum_{i,j} \bar{\mathbf{S}}_{ff}^{ij}) \mathbf{A}^T \dot{\mathbf{x}}_0 = \\ &= \frac{1}{2} \dot{\mathbf{x}}_0^T \begin{pmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \dot{\mathbf{x}}_0 = \frac{1}{2} \dot{\mathbf{x}}_0^T \bar{\mathbf{S}}_{xx} \dot{\mathbf{x}}_0 \end{aligned} \quad (29)$$

where  $m$  is the body mass. Consider now the component  $\mathbf{T}_{x\theta}$ :

$$\begin{aligned} \mathbf{T}_{x\theta} &= -\frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A}^n \bar{\mathbf{S}}_{ff} \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 = \\ &= -\frac{1}{2} \dot{\mathbf{x}}_0^T \mathbf{A} \left( \sum_{i,j=1}^n \bar{\mathbf{S}}_{ff}^{ij} \tilde{\mathbf{r}}^j \right) \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 \end{aligned} \quad (30)$$

Using (20), we obtain:

$$\sum_{i,j=1}^n \bar{\mathbf{S}}_{ff}^{ij} \tilde{\mathbf{r}}^j = \sum_{i=1}^n \left( \sum_{j=1}^n \bar{\mathbf{S}}_{ff}^{ij} \right) \tilde{\mathbf{r}}^j = \bar{\mathbf{S}}_{x\theta} \quad (31)$$

where

$$\bar{\mathbf{S}}_{x\theta} = \sum_{j=1}^n k^j \tilde{\mathbf{r}}^j \quad (32)$$

Hence, we can rewrite (30) as

$$\mathbf{T}_{x\theta} = -\frac{1}{2} \dot{\mathbf{x}}_0^T \mathbf{A} \bar{\mathbf{S}}_{x\theta} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 \quad (33)$$

The formula for  $\mathbf{T}_{xf}$  can be rewritten as

$$\mathbf{T}_{xf} = \frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \dot{\mathbf{q}}_f = \frac{1}{2} \dot{\mathbf{x}}_0^n^T \mathbf{A} \bar{\mathbf{S}}_{xf} \dot{\mathbf{q}}_f \quad (34)$$

where  $\bar{\mathbf{S}}_{xf}$  is a constant  $3 \times 3n$  matrix

$$\bar{\mathbf{S}}_{xf} = \left( \sum_{i=1}^n \bar{\mathbf{S}}_{ff}^{i1} \dots \sum_{i=1}^n \bar{\mathbf{S}}_{ff}^{in} \right) = (k^1 \mathbf{E} \dots k^n \mathbf{E}) \quad (35)$$

From the definition of  $\mathbf{T}_{\theta f}$  follows that:

$$\mathbf{T}_{\theta f} = -\frac{1}{2} \dot{\boldsymbol{\theta}}_0^T \bar{\mathbf{T}}^T \tilde{\mathbf{r}}^T \bar{\mathbf{S}}_{ff} \dot{\mathbf{q}}_f = -\frac{1}{2} \dot{\boldsymbol{\theta}}_0^T \bar{\mathbf{T}}^T \mathbf{S}_{\theta f} \dot{\mathbf{q}}_f \quad (36)$$

where

$$\mathbf{S}_{\theta f} = \tilde{\mathbf{r}}^T \bar{\mathbf{S}}_{ff} = \left( \sum_{i=1}^n \tilde{\mathbf{r}}^i^T \bar{\mathbf{S}}_{ff}^{i1} \dots \sum_{i=1}^n \tilde{\mathbf{r}}^i^T \bar{\mathbf{S}}_{ff}^{in} \right) \quad (37)$$

We also simplify equations for  $\mathbf{T}_{\theta\theta}$ :

$$\begin{aligned} \mathbf{T}_{\theta\theta} &= \frac{1}{2} (\mathbf{A}^n \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0)^T \mathbf{A}^n \bar{\mathbf{S}}_{f,f} \tilde{\mathbf{r}} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 = \\ &= \frac{1}{2} \dot{\boldsymbol{\theta}}_0^T \bar{\mathbf{T}}^T \mathbf{S}_{\theta\theta} \bar{\mathbf{T}} \dot{\boldsymbol{\theta}}_0 \end{aligned} \quad (38)$$

where

$$\mathbf{S}_{\theta\theta} = \tilde{\mathbf{r}}^T \bar{\mathbf{S}}_{f,f} \tilde{\mathbf{r}} = \mathbf{S}_{\theta f} \tilde{\mathbf{r}} \quad (39)$$

Substituting  $\mathbf{T}_{xx}$ ,  $\mathbf{T}_{x\theta}$ ,  $\mathbf{T}_{xf}$ ,  $\mathbf{T}_{\theta\theta}$ ,  $\mathbf{T}_{\theta f}$ ,  $\mathbf{T}_{ff}$  in (28), we obtain the mass matrix  $\mathbf{M}$  in the FFRF:

$$\mathbf{M} = \begin{pmatrix} \bar{\mathbf{S}}_{xx} & -\mathbf{A} \bar{\mathbf{S}}_{x\theta} \bar{\mathbf{T}} & \mathbf{A} \bar{\mathbf{S}}_{xf} \\ -(\mathbf{A} \bar{\mathbf{S}}_{x\theta} \bar{\mathbf{T}})^T & \bar{\mathbf{T}}^T \mathbf{S}_{\theta\theta} \bar{\mathbf{T}} & -\bar{\mathbf{T}}^T \mathbf{S}_{\theta f} \\ (\mathbf{A} \bar{\mathbf{S}}_{xf})^T & -\mathbf{S}_{\theta f}^T \bar{\mathbf{T}} & \bar{\mathbf{S}}_{ff} \end{pmatrix} \quad (40)$$

## B. Stiffness matrix $\bar{\mathbf{K}}$ .

The potential energy of the elastic forces can be written as

$$\Pi_f = \frac{1}{2} \bar{\mathbf{q}}_f^T \bar{\mathbf{K}}_{ff} \bar{\mathbf{q}}_f \quad (41)$$

where  $\bar{\mathbf{K}}_{ff}$  is a constant stiffness matrix, equal in the beginning to stiffness matrix  $\mathbf{K}_A$  generated by ANSYS  $\bar{\mathbf{K}}_{ff} = \mathbf{K}_A|_{t=0}$ .

Then the elastic forces  $\mathbf{f}_k$  can be written as:

$$\mathbf{f}_k = - \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{K}}_{ff} \end{pmatrix} \begin{pmatrix} \mathbf{x}_0 \\ \boldsymbol{\theta}_0 \\ \bar{\mathbf{q}}_f \end{pmatrix} = -\bar{\mathbf{K}}\mathbf{p} \quad (42)$$

## V. EQUATIONS OF MOTION IN THE FFRF

Lagrange's equation for a flexible body can be written in the following form [3]

$$\begin{pmatrix} \mathbf{M}_{xx} & \mathbf{M}_{x\theta} & \mathbf{M}_{xf} \\ & \mathbf{M}_{\theta\theta} & \mathbf{M}_{\theta f} \\ \text{symmetric} & & \mathbf{M}_{ff} \end{pmatrix} \ddot{\mathbf{p}} + \mathbf{G}^T \boldsymbol{\lambda} = \mathbf{f}_e + \mathbf{f}_v - \bar{\mathbf{K}}\mathbf{p} \quad (43)$$

where  $\mathbf{G}$  is the constraint Jacobian matrix,  $\mathbf{f}_e$  is the vector of externally applied forces and  $\boldsymbol{\lambda}$  is the vector of Lagrange multipliers,  $\mathbf{M}_{xx}, \mathbf{M}_{x\theta}, \mathbf{M}_{xf}, \mathbf{M}_{\theta\theta}, \mathbf{M}_{\theta f}, \mathbf{M}_{ff}$  are parts of  $\mathbf{M}$ , defined by (40) and  $\mathbf{f}_v$  is the quadratic velocity vector

$$\mathbf{f}_v = \begin{pmatrix} \mathbf{f}_{vr} \\ \mathbf{f}_{v\theta} \\ \mathbf{f}_{vf} \end{pmatrix} = -\dot{\mathbf{M}}\dot{\mathbf{p}} + \left[ \frac{\partial}{\partial \mathbf{p}} \left( \frac{1}{2} \dot{\mathbf{p}}^T \mathbf{M} \dot{\mathbf{p}} \right) \right]^T \quad (44)$$

Using the formula [3]

$$\boldsymbol{\alpha}_0 = \mathbf{A}^T \ddot{\boldsymbol{\theta}}_0 \quad (45)$$

we can transform (43) to the equations of motion in the following form

$$\begin{pmatrix} \mathbf{M}_{xx} & \mathbf{M}_{x\alpha} & \mathbf{M}_{xf} \\ & \mathbf{M}_{\alpha\alpha} & \mathbf{M}_{\alpha f} \\ \text{symmetric} & & \mathbf{M}_{ff} \end{pmatrix} \begin{pmatrix} \ddot{\mathbf{x}}_0 \\ \boldsymbol{\alpha}_0 \\ \ddot{\bar{\mathbf{q}}}_f \end{pmatrix} + \begin{pmatrix} \mathbf{G}_x^T \\ \mathbf{G}_\alpha^T \\ \mathbf{G}_f^T \end{pmatrix} \boldsymbol{\lambda} = \begin{pmatrix} \mathbf{f}_{ex} \\ \mathbf{f}_{e\alpha} \\ \mathbf{f}_{ef} - \bar{\mathbf{K}}_{ff} \bar{\mathbf{q}}_f \end{pmatrix} + \begin{pmatrix} \mathbf{f}_{vx} \\ \mathbf{f}_{v\alpha} \\ \mathbf{f}_{vf} \end{pmatrix} \quad (46)$$

where  $\mathbf{M}_{x\alpha} = -\mathbf{A} \tilde{\mathbf{s}}_{x\theta} \mathbf{A}^T$ ,  $\mathbf{M}_{\alpha\alpha} = \mathbf{A} \mathbf{S}_{\theta\theta} \mathbf{A}^T$ ,  $\mathbf{M}_{\alpha f} = -\mathbf{A} \mathbf{S}_{\theta f}$  and

$$\begin{aligned} \bar{\mathbf{T}}^T \mathbf{A}^T \mathbf{f}_{e\alpha} &= \mathbf{f}_{e\theta} \\ \bar{\mathbf{T}}^T \mathbf{A}^T \mathbf{f}_{v\alpha} &= \mathbf{f}_{v\theta} \end{aligned} \quad (47)$$

## VI. FORCES AND QUADRATIC VELOCITY VECTOR

### A. Gravitational forces

Using (35) and (10), we can formulate the potential energy of the gravitational forces in the inertial frame formulation in the following form:

$$\begin{aligned} \Pi &= -\mathbf{g}^T \sum_i k^i \mathbf{q}^i = -\mathbf{g}^T \bar{\mathbf{S}}_{xf} \mathbf{q} \\ &= \mathbf{g}^T (\bar{\mathbf{S}}_{xf} \mathbf{x}_0^n + \bar{\mathbf{S}}_{xf} \mathbf{A}^n \bar{\mathbf{r}}) \end{aligned} \quad (48)$$

where  $\mathbf{g} = g\mathbf{e}$ ,  $g$  is the value of the free fall acceleration and  $\mathbf{e}$  is the gravitational direction. The gravitational forces  $\mathbf{f}_g = (\mathbf{f}_{gx}^T \ \mathbf{f}_{g\alpha}^T \ \mathbf{f}_{gf}^T)^T$  in floating frame formulation can be calculated from

$$\begin{pmatrix} \mathbf{f}_{gx} \\ \mathbf{f}_{g\theta} \\ \mathbf{f}_{gf} \end{pmatrix} = - \begin{pmatrix} \frac{\partial \Pi}{\partial \mathbf{x}_0} & \frac{\partial \Pi}{\partial \boldsymbol{\theta}_0} & \frac{\partial \Pi}{\partial \bar{\mathbf{q}}_f} \end{pmatrix}^T \quad (49)$$

where  $\mathbf{f}_{g\alpha}^T \mathbf{A}^T = \mathbf{f}_{g\theta}^T$ .

Substituting the value of  $\Pi$  from (48) and simplifying, we obtain the following formula for the calculation of gravitational forces:

$$\mathbf{f}_g = \begin{pmatrix} m\mathbf{g} \\ \mathbf{A} \tilde{\mathbf{s}}_{x\theta} \mathbf{A}^T \mathbf{g} \\ \bar{\mathbf{S}}_{xf}^T \mathbf{A}^T \mathbf{g} \end{pmatrix} = \begin{pmatrix} m\mathbf{g} \\ -\mathbf{M}_{x\alpha} \mathbf{g} \\ \mathbf{M}_{xf}^T \mathbf{g} \end{pmatrix} \quad (50)$$

### B. External force $\mathbf{f}_p$

Let us consider an vector of external force  $\mathbf{F}_p$ , acting on  $i$ -th node on the deformable body. The virtual work of the force is defined as

$$\delta W_i = \mathbf{F}_p^T \delta \mathbf{q}^i \quad (51)$$

From (9) follows that

$$\delta \mathbf{q}^i = \delta \mathbf{x}_0 + \frac{\partial (\mathbf{A} \mathbf{r}^i)}{\partial \boldsymbol{\theta}_0} \delta \boldsymbol{\theta}_0 + \mathbf{A} \delta \bar{\mathbf{q}}_f \quad (52)$$

It can be proved that

$$\frac{\partial (\mathbf{A} \mathbf{r}^i)}{\partial \boldsymbol{\theta}_0} = -\mathbf{A} \tilde{\mathbf{r}}^i \bar{\mathbf{T}} \quad (53)$$

Therefore, the vector  $\mathbf{f}_p$  of generalized forces, correspondent to the force  $\mathbf{F}_p(\mathbf{q}, t)$ , can be written as

$$\mathbf{f}_p(\mathbf{F}_p) = (\mathbf{f}_{px}^T \ \mathbf{f}_{p\alpha}^T \ \mathbf{f}_{pf}^T)^T \quad (54)$$

where

$$\begin{aligned} \mathbf{f}_{px} &= \mathbf{F}_p \\ \mathbf{f}_{p\alpha} &= \mathbf{A} \tilde{\mathbf{r}}^i \mathbf{A}^T \mathbf{F}_p \\ \mathbf{f}_{pf} &= \begin{pmatrix} \mathbf{0} & \dots & \mathbf{0} & (\mathbf{f}_{pf}^i)^T & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}^T \end{aligned} \quad (55)$$

$$\mathbf{f}_{pf}^i = \mathbf{A}^T \mathbf{F}_p$$

### C. Quadratic velocity vector $\mathbf{f}_v$

From (44) follows that

$$\mathbf{f}_v = \begin{pmatrix} \mathbf{f}_{vx} \\ \mathbf{f}_{v\theta} \\ \mathbf{f}_{vf} \end{pmatrix} = - \begin{pmatrix} \mathbf{0} & \dot{\mathbf{M}}_{x\theta} & \dot{\mathbf{A}}\bar{\mathbf{S}}_{xf} \\ \dot{\mathbf{M}}_{\theta x}^T & \dot{\mathbf{M}}_{\theta\theta} & \dot{\mathbf{M}}_{\theta f} \\ (\dot{\mathbf{A}}\bar{\mathbf{S}}_{xf})^T & \dot{\mathbf{M}}_{\theta f}^T & 0 \end{pmatrix} \begin{pmatrix} \dot{\mathbf{x}}_0 \\ \dot{\theta}_0 \\ \dot{\bar{\mathbf{q}}}_f \end{pmatrix} + \left[ \frac{\partial}{\partial \mathbf{p}} \left( \frac{1}{2} \mathbf{p}^T \mathbf{M} \mathbf{p} \right) \right]^T \quad (56)$$

If we calculate  $\mathbf{f}_v$  then, using (47), we can easily calculate the desired vector  $(\mathbf{f}_{vx}^T \ \mathbf{f}_{v\alpha}^T \ \mathbf{f}_{vf}^T)^T$ .

Formulas for calculation of  $\mathbf{f}_{vx}$  and  $\mathbf{f}_{v\theta}$  are written in [3]:

$$\mathbf{f}_{vx} = -\mathbf{A}[(\bar{\omega})^2 \mathbf{s}_{x\theta} + 2\bar{\omega}\bar{\omega}_{xf} \dot{\bar{\mathbf{q}}}_f] \quad (57)$$

$$\mathbf{f}_{v\theta} = -\bar{\mathbf{T}}^T(2\mathbf{s}_{\theta\theta}\bar{\omega} + 2\mathbf{s}_{\theta f}\dot{\bar{\mathbf{q}}}_f) - \bar{\mathbf{T}}^T \dot{\mathbf{s}}_{\theta\theta} \bar{\omega} \quad (58)$$

Using (47) and simplifying, we get

$$\mathbf{f}_{v\alpha} = -\mathbf{A}(\bar{\omega}[\mathbf{s}_{\theta\theta}\bar{\omega} + \mathbf{s}_{\theta f}\dot{\bar{\mathbf{q}}}_f] + \dot{\mathbf{s}}_{\theta\theta}\bar{\omega}) \quad (59)$$

where  $\dot{\mathbf{s}}_{\theta\theta}$  we can calculate using the formula:

$$\begin{aligned} \dot{\mathbf{s}}_{\theta\theta} &= \frac{d}{dt}(\bar{\mathbf{r}}^T \bar{\mathbf{S}}_{f,f} \bar{\mathbf{r}}) = \bar{\mathbf{r}}^T \bar{\mathbf{S}}_{f,f} \frac{d\bar{\mathbf{r}}}{dt} + \left( \bar{\mathbf{r}}^T \bar{\mathbf{S}}_{f,f} \frac{d\bar{\mathbf{r}}}{dt} \right)^T \\ &= \bar{\mathbf{r}}^T \bar{\mathbf{S}}_{f,f} (\dot{\bar{\mathbf{q}}}_f) + (\dot{\bar{\mathbf{q}}}_f)^T \bar{\mathbf{S}}_{f,f} \bar{\mathbf{r}} \end{aligned} \quad (60)$$

The equations for  $\mathbf{f}_{vx}$  and  $\mathbf{f}_{v\alpha}$  can be proved by substituting in (56) correspondent equations for parts of  $\mathbf{M}$ .

The formula for the calculation of  $\mathbf{f}_{vf}$  can be obtained from (56):

$$\mathbf{f}_{vf} = -(\dot{\mathbf{A}}\bar{\mathbf{S}}_{xf})^T \dot{\mathbf{x}}_0 - \dot{\mathbf{M}}_{\theta f}^T \dot{\theta}_0 + \left[ \frac{\partial}{\partial \bar{\mathbf{q}}_f} \left( \frac{1}{2} \mathbf{p}^T \mathbf{M} \mathbf{p} \right) \right]^T \quad (61)$$

It can be shown that the substitution of  $\mathbf{M}$ ,  $\dot{\mathbf{A}}$ , etc. in (61) leads to the following formula for  $\mathbf{f}_{vf}$ :

$$\mathbf{f}_{vf} = 2\bar{\mathbf{S}}_{f,f}(\dot{\bar{\mathbf{q}}}_f)\bar{\omega} + \mathbf{J}_{ww}\bar{\mathbf{r}} \quad (62)$$

where  $\mathbf{J}_{ww}$  is a symmetric matrix, consisting of  $n$  blocks size  $3 \times 3$ , which can be calculated using the formula:

$$\mathbf{J}_{ww}^{ij} = \bar{\omega}^T \bar{\mathbf{S}}_{f,f}^{ij} \bar{\omega} \quad i, j = 1..n \quad (63)$$

## VII. DEFINITION OF BODY REFERENCE FRAME

The equations of motion (46) are valid for each body frame  $\{\bar{O} \ \bar{\mathbf{e}}^1 \bar{\mathbf{e}}^2 \bar{\mathbf{e}}^3\}$ , initially coinciding with the global inertia frame  $\{O \ \mathbf{e}^1 \mathbf{e}^2 \mathbf{e}^3\}$ . A unique representation of the body motion requires six additional equations, defining the body reference frame. We propose to use the definition of the body

frame based on three points of the body, proposed by [5].

Let in the beginning the absolute coordinates of the first node one are equal to null, the second node lies on the  $x$ -axis and the third node lies on the  $x$ - $y$  plane:

$$\begin{aligned} \mathbf{q}^1|_{t=0} &= \mathbf{0} \\ \mathbf{q}^2|_{t=0} &= (q_x^2|_{t=0} \ 0 \ 0)^T \\ \mathbf{q}^3|_{t=0} &= (q_x^3|_{t=0} \ q_y^3|_{t=0} \ 0)^T \end{aligned} \quad (64)$$

Then we can set the local frame, using the following conditions:

1. the origin  $\bar{O}$  is attached to the first node:

$$\bar{\mathbf{r}}^1 = \mathbf{0} \quad (65)$$

2.  $\bar{\mathbf{e}}^1$  is on the line between  $\bar{O}$  and the second node:

$$\bar{\mathbf{r}}^{2,y} = 0 \quad \bar{\mathbf{r}}^{2,z} = 0 \quad (66)$$

3.  $\bar{\mathbf{e}}^3$  is perpendicular to the line between  $\bar{O}$  and the third node:

$$\bar{\mathbf{r}}^{3,z} = 0 \quad (67)$$

This implies six limitations on the vector of nodal deformations  $\bar{\mathbf{q}}_f$ :

$$\bar{\mathbf{q}}_f^1 = \mathbf{0}, \bar{\mathbf{q}}_f^{2,y} = 0, \bar{\mathbf{q}}_f^{2,z} = 0, \bar{\mathbf{q}}_f^{3,z} = 0 \quad (68)$$

Therefore, we can easily exclude these six positions from  $\bar{\mathbf{q}}_f$  and six correspondent rows and columns from matrices and vectors in (46). The matrix  $\mathbf{M}$  after this procedure becomes invertible. Hence, the values of accelerations can be uniquely calculated from the modified (46).

## VIII. MODAL TRANSFORMATION

A major advantage of using the FFRF is that the finite element nodal coordinates can be easily reduced using modal analysis techniques, based on using a reduced set of eigenvectors  $\mathbf{a}$  (also called *mode shapes*) of the free vibration equations of motion [3]:

$$\bar{\mathbf{K}}_{ff} \mathbf{a} = \omega^2 \mathbf{M}_{ff} \mathbf{a} \quad (69)$$

The effect of elimination of high-frequency mode shape on the computational speed is twofold. Firstly, the number of numerical operations on each time step decreases because the size of matrices in the equations of motions is much less than in non-reduced case. Secondly, a larger integration time step can be used.

## IX. IMPLEMENTATION

We have implemented the method and developed in Maple a module VSDFlex for the simulation of dynamics of flexible multibodies. The module is integrated with ANSYS, where we get matrices of the initial values of  $\mathbf{q}|_{t=0}$ ,  $\mathbf{M}_A|_{t=0}$ ,  $\mathbf{K}_A|_{t=0}$ , describing the motion of bodies in the absence of constraints. The types and parameters of constraints should be defined in VSDFlex.

The simulation in VSDFlex consists on three standard steps:

1. **Preprocessing Step:** Starting from the values of  $\mathbf{q}|_{t=0}$ ,  $\mathbf{M}_A|_{t=0}$ ,  $\mathbf{K}_A|_{t=0}$ , VSDFlex calculates integrals  $\bar{\mathbf{S}}_{xf}$ ,  $\bar{\mathbf{S}}_{ff}$ ,  $\bar{\mathbf{S}}_{xx}$  and a modal transformation matrix  $\mathbf{B}$ .
2. **Dynamic analysis:** VSDFlex calculates the dynamics of the simulated system, using the modal transformation.
3. **Postprocessing Step:** Starting from trajectory of modal coordinates, obtained during the dynamic analysis, the trajectories positions, velocities and accelerations of nodes, applied forces as well as stresses and strains can be evaluated.

#### X. FLEXIBLE PENDULUM EXAMPLE

The example we considered is a flexible pendulum, articulated to the foundation through a revolute joint. The model of the pendulum, shown in Fig. 2, was developed in ANSYS as a rod, consisting of 16 SOLID45 elements.

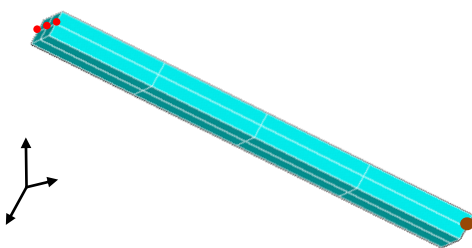


Fig. 2 Model of flexible pendulum

The revolute joint is modeled in VSDFlex as a fixation of three nodes, colored in Fig. 2 by red, in the inertial system of coordinates. The physical properties, expressed in SI are: length 0.1, wide 0.01, thickness 0.015, elasticity modulus  $10^7$ , Poisson ratio 0.49, mass 0.26, moment of inertia  $3.3 \cdot 10^{-3}$ . The simulation was performed using the modal transformation, including the nine mode shapes, correspondent to the first nine eigenfrequencies (from 5 to 124 Hz).

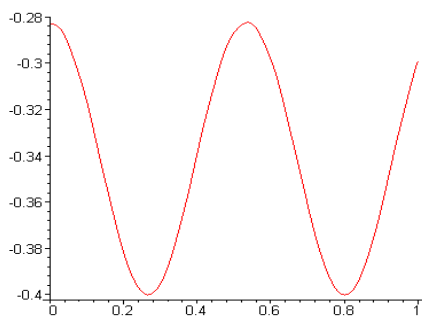


Fig. 3 Z-coordinate of the end node

Fig. 3 shows the trajectory of the  $z$ -coordinate of the node at the end of the rod (marked in Fig. 2 by brown). Due to model order reduction a simulation step size of  $5 \cdot 10^{-5}$  ms guarantees stable results. Obviously, the high elasticity modulus implies little displacements on the coordinate level.

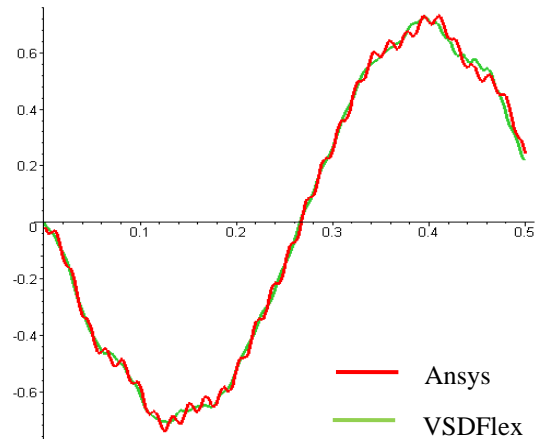


Fig. 4  $z$ -velocities of the end node

In order to prove the VSDFlex simulation we simulated the dynamics of the model in ANSYS. In Fig. 4 are shown the velocity in  $z$ -direction of the same node, obtained in VSDFlex and in ANSYS. The difference between ANSYS and VSDFlex solutions appears because of the accounting of non-linear large-deflection effects in ANSYS. The divergence on the coordinate and on the velocity levels is limited by  $4 \cdot 10^{-4}$  and  $4 \cdot 10^{-2}$  correspondently.

#### XI. CONCLUSION

The article shows a method of transformation of equations of motion in the inertial frame formulation (IFF), to the equations in the floating frame of reference formulation (FFRF). The method was implemented in VSDFlex module, performing the simulation of flexible multibody systems. Starting from the initial values of mass matrix  $\mathbf{M}_A|_{t=0}$  and stiffness matrix  $\mathbf{K}_A|_{t=0}$ , generated by ANSYS in the IFF, the module generates the FFRF equations of motion. The use of the modal transformation during the simulation greatly reduces the computation time. Results of the simulation of flexible pendulum example show the method's stability.

It seems needful to integrate VSDFlex with the object-oriented simulation software VSD in order to improve the software usability and areas of implementation.

#### REFERENCES

- [1] T. Wasfy and A. Noor. "Computational strategies for flexible multibody systems". *Applied Mechanics Review*, 56:553–613, 2003.
- [2] Shabana, A.A., "Flexible multibody dynamics: review of past and recent developments", *Journal of Multibody System Dynamics*, vol. 1, no. 2, pp. 189–222, 1997
- [3] A.A. Shabana. *Dynamics of Multibody Systems*. Cambridge University Press, 2. nd. edition, 1998.
- [4] ANSYS® Academic Teaching , Release 11.0, Help System
- [5] R. Schwertassek, O. Wallrapp and A.A. Shabana, "Flexible multibody simulation and choice of shape functions", *Nonlinear Dynamics*, 20, 361–380, (1999).