Family of Mapped Elastodynamic Infinite Elements, Applicable to FEM

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Abstract— **Many engineering problems can be regarded as unbounded. In structural mechanics Soil-Structure Interaction problems can be modeled more efficiently using infinite elements on the some artificial boundary of the studied system. Generally, the domain of the problem is divided into two parts: *near* and *far* field. The solution is based on finite elements for the near field and infinite elements for the far field.**

This paper is devoted to the application of *mapped elastodynamic infinite elements with union shape functions* to the Finite element models and demonstration of the some advantages of such an approach. Very often methods based on the infinite element are called infinite element methods. However, the infinite element computational models of unbounded problems usually contain finite and infinite elements in conjunction. By reason of that, in the paper, this technique is called *Infinite elements to the Finite element method.*

In short, an extension of dynamic infinite elements for transient dynamic analysis is also presented in the paper. The realistic simulation of the transient Soil-Structure Interaction in a time domain is compared with a formulation in frequency domain.

***Index Terms*— Finite element method, Infinite elements, transient dynamic Soil-Structure Interaction**.

I. Introduction

The realistic models of some structural systems impose using of unbounded domain. Methods for dealing with unbounded domains in conjunction with finite elements can be classified into two types. In the first type models some local boundary conditions such dampers, absorbers are used or truncation of the domain is applied. These techniques are called *local boundary approach*. The second type, called *global solution approach*, uses boundary integrals linked to the finite element mesh. Infinite elements keep the advantages of the above methods such a retaining the bandness, being effectively by refinement and easy for implementation into the Finite element method.

Infinite elements can be systematized [5] into five classes:

- *Classical;*
- *Decay;*
- *Mapped;*
- *Elastodynamic;*
- *Wave envelope.*

Mapped infinite elements use appropriate transformation functions [6] to map the infinite element domain into a finite in one, two or three direction. Such a formulation makes mapped infinite elements directly applicable in the Finite element method. In realistic simulations of dynamic Soil-Structure Interaction dynamic mapped infinite element can be used. Such elements are comparatively cheep, accurate and flexible for many cases.

Classical, decay and mapped infinite elements cannot be directly used to truly transient problems. Practically cannot be covered all or even most patterns of displacements into the element domain. Only some recently developed elastodynamic [7] and some wave enveloped infinite element are applicable in the frequency and time domain, in common. In the low frequency domain case the realistic of the finite element model depends on the accurate stiffness formulation. However in the high frequency domain case also the wave propagation properties of the elements take an important role. It can be generalized that appropriate coupling the both stiffness and wave propagation properties is a key to adequate infinite element formulation to the high frequency domain case.

This paper proposes an extension of the multi-wave infinite element proposed by the author in work [8]. The basic idea is to be taking into account the damped properties of the system by constructing a complex stiffness matrix. Such a matrix is base on the complex modulus.

II. Formulation of Elastodynamic Infinite Element with Union Shape Functions

A. Displacement Field

The displacement field in elastodynamic infinite element can be described in the standard form of the shape functions based on finite number of wave propagation functions [2] as

$$u(x,z,\omega)=\sum_{i=1}^{n}\sum_{q=1}^{m}N_{iq}(x,z,\omega)p_{iq}(\omega) \qquad (1)$$

where $N_{iq}(x,z,\omega)$ are the standard shape displacement functions, $p_{iq}(\omega)$ are generalized coordinates, associated with corresponding $N_{iq}(x,z,\omega)$, n is the number of nodes and m is the number of wave functions included in the formulation of the infinite element. For horizontal wave propagation basic or standard shape functions for the *HIE* (horizontal infinite element) type of infinite element can be expressed as:

$$N_{iq}(x,z,\omega)=T(x,z,\xi,\eta)L_i(\eta)W_q(\xi,\omega) \qquad (2)$$

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where $W_q(\xi,\omega)$ are wave functions, corresponding to the horizontal wave propagation (wave propagation direction is parallel to local coordinate ξ). Here $T(x,z,\xi,\eta)$ assures the geometrical transformation of local (ξ,η) to global (x,z) coordinates.

B. Wave Functions

Taking into account only the real parts of the wave functions the equations of the wave propagation can be written as

$$\mathrm{Re}W_q(\xi,\omega)=\cos\left(\frac{i\omega}{c_s}\xi\right)e^{-\alpha\xi} \quad \text{or}$$
$$\mathrm{Re}W_q(\xi,\omega)=\cos\left(\frac{i\omega}{c_p}\xi\right)e^{-\alpha\xi}, \qquad (3)$$

where c_s and c_p are *S-waves* and *P-waves* velocities respectively.
Suppose now that the number of waves, *m*, and the frequencies ω_q, *q*=1, 2,…,*m* are known then

$$\mathrm{Re}W(\xi)=\sum_{q=1}^{m}A_q\cos\left(\frac{i\omega_q}{c_s}\xi\right)e^{-\alpha\xi}$$
$$\mathrm{Re}W(\xi)=\sum_{q=1}^{m}A_q\cos\left(\frac{i\omega_q}{c_p}\xi\right)e^{-\alpha\xi}. \qquad (4)$$

C. Construction of Union Shape Functions

Coefficients A_q can be treated as weight coefficients and for construction of the *united shape functions* an additional condition for these coefficients is applied [8], written here in the form $\sum_{q=1}^{m}A_q=1$. Such a requirement assures value unit of the united shape function on the node, corresponding to the same function.

United shape function in global coordinate $N_i(x,z)$ now can be written as

$$N_i(x,z)=\sum_{q=1}^{m}N_{iq}(x,z,\omega)=T(x,z,\xi,\eta)L_i(\eta)\mathrm{Re}W(\xi) \qquad (5)$$

Finally, eq. (1) can be given as

$$u(x,z,\omega)=\sum_{i=1}^{n}N_i(x,z,)p_i(\omega) \qquad (6)$$

or in the matrix form

$$u(x,z,\omega)=\mathbf{N}(x,z)\mathbf{p}. \qquad (7)$$

Equation (5) gives so-called ***united shape function*** of node *i* for the proposed infinite element, based on the finite number of wave propagation functions. ***United basis***, used in eq. (6) or eq. (7), contains *n* united shape functions. In comparison, the standard basis, used in eq. (1), contains *n.m* terms.
With similar techniques also vertical (*VIE*) and corner (*CIE*) elastodynamic infinite elements with united shape functions can be formulated.

III. Stiffness and Mass element matrices

The stiffness and mass matrices of the proposed infinite element can be written as

$$[k_e]=\int_{\Omega_e}[\overline{B}]^T[D][\overline{B}]d\Omega_e$$

and

$$[m_e]=\left(\int_{\Omega_e}[\overline{N}]^T[\overline{N}]d\Omega_e\right)I, \qquad (8)$$

where matrix $[\overline{N}]$ contains the united shape functions. The vectors $\{\overline{B}_i\}$ in the matrix $[\overline{B}]$ are written as

$$\{\overline{B}_i\}=\begin{Bmatrix}\dfrac{\partial\overline{N}_i}{\partial x}\\ \dfrac{\partial\overline{N}_i}{\partial y}\end{Bmatrix} \quad \text{or} \quad \{\overline{B}_i\}=[J]^{-1}\begin{Bmatrix}\dfrac{\partial\overline{N}_i}{\partial\eta}\\ \dfrac{\partial\overline{N}_i}{\partial\xi}\end{Bmatrix}. \qquad (9)$$

IV. Global System Equations

Global system equations, given as FEM equilibrium equations, in matrix style, are written here in frequency domain as

$$\begin{bmatrix}\mathbf{S}_{ss}(\omega,t) & \mathbf{S}_{sb}(\omega,t)\\ \mathbf{S}_{bs}(\omega,t) & \mathbf{S}_{bb}(\omega,t)+\mathbf{S}_{bb}^{g}(\omega,t)\end{bmatrix}\begin{Bmatrix}\mathbf{U}_s(\omega,t)\\ \mathbf{U}_b(\omega,t)\end{Bmatrix}=\begin{Bmatrix}\mathbf{F}_s(\omega,t)\\ \mathbf{F}_b(\omega,t)\end{Bmatrix} \qquad (10)$$

where $\mathbf{U}(\omega,t)$, $\mathbf{F}(\omega,t)$ and $\mathbf{S}(\omega,t)$ respectively are: displacement vector, force vector and dynamic stiffness matrix in frequency domain. Subscripts *b* and *s* stands for the nodes along the artificial boundary between the near and the far field soil region and for those of the structure and near field soil region respectively. In time domain eq.10 is written as

$$\begin{bmatrix}\mathbf{M}_{ss} & \mathbf{M}_{sb}\\ \mathbf{M}_{bs} & \mathbf{M}_{ss}\end{bmatrix}\begin{Bmatrix}\ddot{\mathbf{u}}_s(t)\\ \ddot{\mathbf{u}}_b(t)\end{Bmatrix}+\begin{bmatrix}\mathbf{K}_{ss} & \mathbf{K}_{sb}\\ \mathbf{K}_{bs} & \mathbf{K}_{bb}+\mathbf{S}_0^g\end{bmatrix}\begin{Bmatrix}\mathbf{u}_s(t)\\ \mathbf{u}_b(t)\end{Bmatrix}=$$
$$=\begin{Bmatrix}\mathbf{f}_s(t)\\ \mathbf{f}_b(t)-\int_0^t\left\{S_2^g+(t-\tau)S_3^g\exp(-a(t-\tau))\mathbf{u}_b(\tau)d\tau\right\}\end{Bmatrix} \qquad (11)$$

where $\mathbf{u}(t)$ and $\mathbf{f}(t)$ are respectively displacement and force vectors, and $\mathbf{S}_j^g$ represent the mechanical characteristics of the far field soil region.

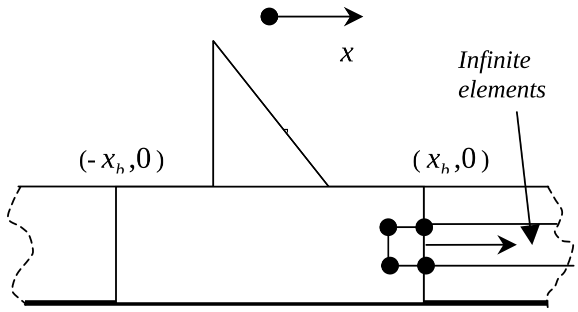

Fig.1 Computational model idea

Fig.1 demonstrates the computational model idea.

V. Mapping Functions

Using the procedure, demonstrated above a family of elastodynamic infinite elements with united shape functions can be obtained. Such elements can be: *four node elastodynamic infinite element* and *five node elastodynamic infinite element* or *six node elastodynamic infinite element* and *seven node elastodynamic infinite element.* The element domains of those elements, before mapping, are shown on fig.2.

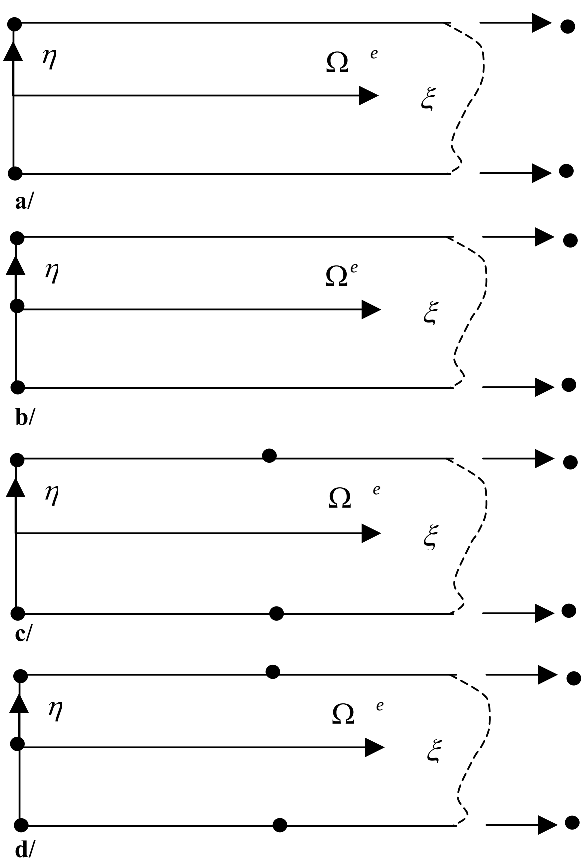

Fig.2 Infinite element domains

a/ four node IE

b/ five node IE (one mid node on line $\xi = 0$)

c/ six node IE (with two mid nodes on lines $\eta = \pm 1$)

d/ seven node IE (one mid node on line $\xi = 0$) and two mid nodes on lines $\eta = \pm 1$)

Local domain to be mapped into global, mapping functions must be used. In general form, the mapping of local to global coordinates along the infinite element domain can be written as

$$\mathbf{x} = \mathbf{T}\boldsymbol{\xi}, \tag{12}$$

where $\mathbf{x}$ and ξ are vectors which contain components of the two coordinate systems. Such vectors can be one-, two- or three-dimensional. The matrix $\mathbf{T}$ is a matrix which accomplishes unique transformation of a point from the local to the global domain. If reverse transformation exists then

$$\{\xi\} = [T]^{-1}\{x\}. \tag{13}$$

An alternative geometrical transformation can be written in the form

$$x = \sum_{i=1}^{n} M_i x_i \,,\ y = \sum_{i=1}^{m} M_i y_i \ \text{ and } \ z = \sum_{i=1}^{q} M_i z_i \tag{14}$$

where M_i are mapping functions, and *n*, *m* and *q* are the number of node points used respectively in every one direction. The two forms give identical transformation in many cases. The mapping in much number of infinite element formulations is based on the special form given by (14), e.q [9].

First, one dimensional mapping functions are given and applied in the work of Zienkiewicz [11]. The geometrical transformation is realized by

$$x = M_0(\xi)x_0 + M_k(\xi)x_k \,, \tag{15}$$

where

$$M_0(\xi) = -\frac{\xi}{1-\xi} \tag{16}$$

and

$$M_k(\xi) = 1 - M_0(\xi) = 1 + \frac{\xi}{1-\xi} \,. \tag{17}$$

The mapping described by eq. (15), eq. (16) and eq. (17) is based on *pole point* and *mid point*.

In the work [12] Zhao and Valliappan applied a transformation in the case corresponding *n=m=q=8.*

Three types of mapped infinite elements can be constructed by proposed in this paper scheme. The first and the second type have element domain which extends to infinity in only one direction, respectively. In the third, the domain extends to infinity in two directions. Here, some details, concerning the transformation of the first type element are presented.

The geometrical transformation along the infinite directions is realized through four nodes by the relation

$$x = M_1(\xi)x_J + M_2(\xi)x_K + M_3(\xi)x_L + M_4(\xi)x_M \quad (18)$$

Two variants of mapping functions can be used:

Variant I

$$M_1(\xi) = -\frac{2\xi}{1-\xi},$$
$$M_2(\xi) = (1+\xi)\left(\frac{1}{3}-\xi\right)\left(\frac{1}{3}+\xi\right),$$
$$M_3(\xi) = \frac{3}{4}(1+\xi),$$
$$M_4(\xi) = (1+\xi)\left(\frac{1}{3}-\xi\right)\left(\frac{1}{3}+\xi\right) \quad (19)$$

and

Variant II

$$M_1(\xi) = -\frac{2\xi}{1-\xi} + \frac{9}{8}(1+\xi)\left(\frac{1}{3}+\xi\right),$$
$$M_2(\xi) = (1+\xi)\left(\frac{1}{3}-\xi\right)\left(\frac{1}{3}+\xi\right),$$
$$M_3(\xi) = \frac{3}{4}(1+\xi),$$
$$M_4(\xi) = (1+\xi)\left(\frac{1}{3}-\xi\right)\left(\frac{1}{3}+\xi\right) \quad (20)$$

The proposed functions give a unique transformation of the coordinates. The local coordinates are transformed to a global correctly except only for **Variant I**, node L with $\xi = 1/3$ (local) and $x = x_l$ (global) coordinates. In spite of this, the mapping based on eq. (19) or eq. (20) is applicable in the formulations of infinite elements because the global positions of the mid nodes in more cases are not of great importance.

By using similar techniques, the transformations related to the second and to the third type can also be obtained. For the element, which domain extends to infinity in only one direction (the first and the second type) Lagrange polynomial interpolation gives the distribution of the basic unknown on the other directions. More detailed discussion, concerning the domain transformation is given in work [8].

VI. Conclusion

The paper describes the idea of so-called mapped elastodynamic infinite elements with union shape functions. The mapping of local to global coordinates can be realized by different number of points, called mapping support points. Such points can also be used as additional nodes. Then the union basis increases with the same number of union shape functions as the number of the additional used nodes.

The proposed **mapped elastodynamic infinite elements with union shape functions** can be directly used in the computational model created by the Finite element method.

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