Analytical Solution of Temperature Field in a Spherical Powder Particle Subjected to Gaussian Heat Flux

Gholamali Atefi, Shokran Khadem alsharieh, Jr.

***Abstract*—The analytical two dimensional temperature field for a spherical metal powder particle subjected to Gaussian heat flux is derived. The particle is considered to be spherical, homogeneous and isotropic with time-independent thermal properties. The heat transfer equation is solved, the temperature distribution in the spherical particle is derived, the 3-D temperature charts are drawn, the results are compared with available resources and good agreements have been observed. As several conduction heat transfer problems can be modeled by a sphere subjected to Gaussian heat flux, the results are used to approximate the problems and the time consuming complex numerical calculations avoided. Good example for this problem is Rapid prototyping with low frequency Selective Electrical Discharge Sintering (SEDS). In SEDS, electrical plasma arcs provide the thermal energy for initial binding. Binding is a heat transfer process from the energy source to the raw material which cause the separated particles to unify. Heat fluxes entering the powder particle from concentrated energy sources such as electrical plasma arcs or lasers have Gaussian distribution. Metal powder which is used as raw material in SEDS process is considered as spherical particle.**

***Index Terms*— Analytical solution, Gaussian boundary condition, SEDS, Temperature Field, Heat Transfer.**

I. Introduction

Several conduction heat transfer problems can be modeled by a sphere subjected to Gaussian heat flux. An example which could be modeled as it said, is Rapid prototyping with low frequency Selective Electrical Discharge Sintering (SEDS), which is a new method for manufacturing different parts and molds with complicated geometry. It gives the possibility to make complex parts and molds in a faster time and a considerably lower cost. [1]. In SEDS, an electrical plasma arcs provide the thermal energy for initial binding of the powder particles. Binding is a heat transfer process from the energy source to the raw material which cause the separated particles to unify. Heat fluxes entering the powder from concentrated energy source such as plasma arcs or lasers have been usually modeled by constant [2], [3] or Gaussian distribution [4], [6]. Controlling the temperature in initial binding stage would result in better binding and in turn higher part quality. Thus calculating the temperature field in this matter is of great importance [1].

In this survey it is assumed that the metal powder which is used as raw material in SEDS process is spherical, homogenous and isotropic with time-independent thermal properties. Heat transfer process for each powder particle is analyzed separately, conduction and convection have been taken into account and radiation is neglected, the heat flux is considered to be from concentrated source with Gaussian distribution; solving the conduction heat transfer equation analytically in the spherical coordinates, the two dimensional temperature field for a spherical particle subjected to Gaussian heat flux is derived. The results in this method are exact and they can be used to approximate different problems with Gaussian boundary condition; in addition time-consuming and non-exact complex numerical calculations are avoided.

II. Modeling

The problem geometry is simulated as shown in Fig. 1. As mentioned before, the heat flux has Gaussian distribution. b is the plasma arc radius and q_0 is the maximum heat flux at the center of the arc.

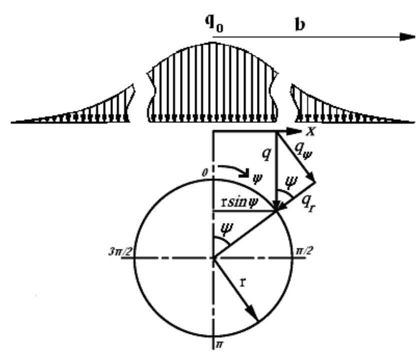

Fig. 1, Simulation of powder particle

The entering heat flux can be written as followed, Q is the total amount of the heat from the plasma arc. [1]

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$$q_{in} = q_{\circ} \exp[-(\frac{x}{b})^2]$$
$$q_0 = q_{\max\ @x=0} = \frac{Q}{\pi b^2} \tag{1}$$

From the total amount of the heat flux only the radial vector, is absorbed by the metal powder, thus the amount of heat entering the sphere upper half is equal to:

$$q = \alpha_p q_0 \exp[-(\frac{R_0 \sin\psi}{b})^2]\cos\psi \tag{2}$$

α_p is the absorbent factor of metal powder.

III. Governing Equations

The spherical coordinate is located at the center of the spherical particle. The heat conduction equation in spherical coordinates for an isotropic material that has temperature and time-independent properties, with the absence of heat source under asymmetric condition, is [7]:

$$\frac{\partial^2 T}{\partial r^2} + \frac{2}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}(\cot\psi \frac{\partial T}{\partial \psi} + \frac{\partial^2 T}{\partial \psi^2}) = a^2 \frac{\partial T}{\partial t}$$
$$a^2 = \frac{1}{\alpha} \quad , \quad \alpha = \frac{k}{\rho C_p} \tag{3}$$

The initial sphere temperature is constant equal to T_0, the ambient temperature is T∞, and therefore there is convection between the sphere and the ambient. Only the sphere upper half is subjected to heat flux and the temperature at the center of the sphere is limited, thus the boundary and initial conditions are:

$$h[T(r_o,\psi,t) - T_\infty] + k\frac{\partial T}{\partial r}\Big|_{r_o,\psi,t} = g(\psi) \tag{4}$$

$$g(\psi) = \begin{cases} \alpha_p q_0 \exp[-(\frac{R_0 \sin\psi}{b})^2]\cos\psi & \frac{3\pi}{2} < \psi < \frac{\pi}{2} \\ 0 & \frac{\pi}{2} < \psi < \frac{3\pi}{2} \end{cases} \tag{5}$$

$$\frac{\partial T}{\partial r}\Big|_{0,\psi,t} = 0 \tag{6}$$

$$T(r,\psi,0) = T_0 \tag{7}$$

Assuming $\theta = T - T_0$ for having homogenous initial condition, we will have the following boundary and initial condition:

$$h\theta(r_o,\psi,t) + k\frac{\partial \theta}{\partial r}\Big|_{r_o,\psi,t} = f(\psi) \tag{8}$$

$$f(\psi) = g(\psi) + h\theta_\infty \quad ,\theta = T_\infty - T_0 \tag{9}$$

$$\theta(r,\psi,0) = 0 \tag{10}$$

$$\frac{\partial \theta}{\partial r}\Big|_{0,\psi,t} = 0 \tag{11}$$

IV. Analytical Solution

The problem cannot be solved directly because of the non-homogeneous term $f(\psi)$ [8]. Using superposition principle; the solution of the problem is the summation of a steady-state solution $\theta_0(r,\psi)$ and a transient solution $\theta_1(r,\psi,t)$.

$$\theta(r,\psi,t) = \theta_0(r,\psi) + \theta_1(r,\psi,t) \tag{12}$$

The differential heat conduction equation in the steady-state case is:

$$\frac{\partial^2 \theta_0}{\partial r^2} + \frac{2}{r}\frac{\partial \theta_0}{\partial r} + \frac{1}{r^2}(cot\psi \frac{\partial \theta_0}{\partial \psi} + \frac{\partial^2 \theta_0}{\partial \psi^2}) = 0 \tag{13}$$

Boundary conditions are the same as (9), (10), and (11).

Transient conduction differential equation is:

$$\frac{\partial^2 \theta_1}{\partial r^2} + \frac{2}{r}\frac{\partial \theta_1}{\partial r} + \frac{1}{r^2}(cot\psi \frac{\partial \theta_1}{\partial \psi} + \frac{\partial^2 \theta_1}{\partial \psi^2}) = a^2\frac{\partial \theta_1}{\partial t} \tag{14}$$

The following condition should be satisfied in the transient solution:

$$h\theta_1(r_o,\psi,t) + k\frac{\partial \theta_1}{\partial r}\Big|_{r_o,\psi,t} = 0 \tag{15}$$

$$\frac{\partial \theta_1}{\partial r}\Big|_{0,\psi,t} = 0 \tag{16}$$

$$\theta_1(r,\psi,0) = -\theta_0(r,\psi) \tag{17}$$

A. Steady-State Solution

Variables' separation method is used to solve (13),

$$\theta_0(r,\psi) = R(r)\Psi(\psi) \tag{18}$$

Choosing the constant *n(n+1)*; two differential equations are obtained, an Euler type (19) and a Legendre type (20).

$$r^2\frac{d^2 R_n}{dr^2} + 2r\frac{dR_n}{dr} - n(n+1)R_n(r) = 0 \tag{19}$$

$$\frac{d^2\Psi_n}{d\psi^2} + cot\psi\frac{d\Psi_n}{d\psi} + n(n+1)\Psi_n = 0 \qquad n = 0,1,2,... \tag{20}$$

Equation (19), is an Euler type differential equation with the following solution:

$$R_n(r) = M_{1n}r^n + \frac{N_{1n}}{r^{n+1}} \tag{21}$$

Because of the constant temperature in the center of the sphere (11), we have $N_{1n} = 0$.

Equation (20), is a Legendre equation which could be solved by defining $\zeta = \cos(\psi)$

$$\Psi_n(\zeta) = M_{2n}P_n(\zeta) + N_{2n}Q_n(\zeta) \tag{22}$$

$P_n(\zeta), Q_n(\zeta)$ are the Legendre functions. $Q_n(\zeta)$ functions are not defined in $|\zeta|<1$ so we will have $N_{2n} = 0$, and therefore from (18), we will have:

$$\theta_0(r,\psi) = \sum_{n=0}^{\infty} M_n r^n P_n(\cos\psi) \tag{23}$$

the constants M_n could be found by applying (9), and the steady-state solution is:

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$$\theta_0(r,\varsigma)=\sum_{n=0}^{\infty}\eta_n(r)p_n(\varsigma)\quad,n=0,1,2,3,... \tag{24}$$

$$\eta_n(r)=\frac{C_n r^n}{hr_o^n+nr_o^{n-1}k} \tag{25}$$

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}f(\zeta)P_n(\zeta)d\zeta \tag{26}$$

B. Transient Solution

Using the variables' separation method for solving (14), assuming $\theta_1(r,\psi,t)=R(r)\Psi(\psi)\tau(t)$ and using the constant *n(n+1)* and ω^2 the following equations will be obtained:

$$\frac{d\tau}{dt}+(\frac{\omega}{a})^2\tau(t)=0\quad\Rightarrow\quad\tau(t)=\overline{A}e^{-(\frac{w}{a})^2t} \tag{27}$$

$$\frac{d^2R_n}{dr^2}+\frac{2}{r}\frac{dR_n}{dr}+(\omega^2-\frac{n(n+1)}{r^2})R_n=0\Rightarrow$$
$$R_n(r)=\frac{1}{\sqrt{r}}\left(C_{1n}J_{n+\frac{1}{2}}(\omega r)+C_{2n}J_{-(n+\frac{1}{2})}(\omega r)\right) \tag{28}$$

$C_{2n}=0$ is obtained from (16),

$$\frac{d^2\Psi_n}{d\psi^2}+\cot g\psi\frac{d\Psi_n}{d\psi}+n(n+1)\Psi_n=0\Rightarrow$$
$$\Psi_n(\zeta)=\overline{A_n}P_n(\zeta)+\overline{B_n}Q_n(\zeta) \tag{29}$$

Again $Q_n(\zeta)$ functions are not defined in |ζ|<1 so we will have $\overline{B_n}=0$,

$$\theta_1(r,\psi,t)=e^{-(\frac{\omega}{a})^2t}\sum_{n=0}^{\infty}\left(A_n\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(\omega r)\right)P_n(\zeta) \tag{30}$$

Boundary equations (15), should be satisfied, therefore eigen values ω_{kn} could be calculated. The final solution of the transient problem can be expressed as:

$$\theta_1(r,\psi,t)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}A_n\Phi(\omega_{kn}r)P_n(\zeta)e^{-(\frac{\omega}{a})^2t}$$
$$\Phi(\omega_{kn}r)=\frac{1}{\sqrt{r}}J_{n+\frac{1}{2}}(\omega_{kn}r) \tag{31}$$

Satisfying (17), A_n can be calculated and (31) will be expressed as:

$$\theta_1(r,\psi,t)=-\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{\gamma}{\delta_{kn}}\Phi_n(r\omega_{kn})P_n(\varsigma)e^{-(\frac{\omega_{kn}}{a})^2t} \tag{32}$$

$$\gamma=\int_0^{r_o}r^2\eta_n(r)\Phi_n(r\omega_{kn})dr \tag{33}$$

$$\delta_{kn}=\frac{r}{2\omega_{kn}^2}\left\{r^2\Phi_n{}^2\omega_{kn}^2-n(n+1)\Phi_n{}^2+r\omega_{kn}\Phi_n\Phi_n{}'+r^2\omega_{kn}^2\Phi_n{}'^2\right\}_0^{r_o} \tag{34}$$
$$\Phi_n=\Phi_n(r\omega_{kn})$$

C. Final Temperature Field

The total temperature field will be the summation of steady-state (24) and the transient solution,(32) :

$$\theta(r,\psi,t)=\theta_0(r,\psi)+\theta_1(r,\psi,t) \tag{35}$$

$$\theta(r,\psi,t)=\sum_{n=0}^{\infty}\left(\eta_n(r)-\sum_{k=0}^{\infty}\frac{\gamma}{\delta_{kn}}e^{-(\frac{\omega_{kn}}{a})^2t}\Phi_n(r\omega_{kn})\right)P_n(\varsigma) \tag{36}$$

Finding (36) , the temperature field according to the radial position, polar angle and time is derived. Temperature at any time can be calculated anywhere in the sphere.

V. Results

From (36) temperature at any time can be calculated at any point in the sphere. The sphere diameter is considered to be 1 "mm". In Fig. 2, the three dimensional temperature filed is drawn according to the polar angle ψ and the sphere radius.

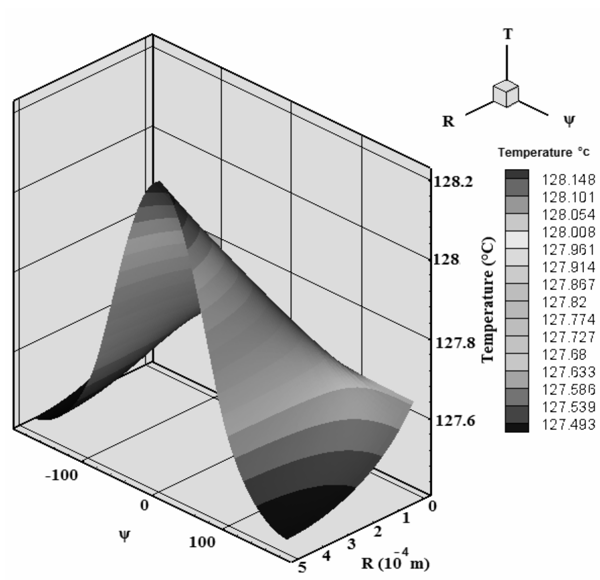

Fig.2(a), 3-D Temperature distribution at t=0.5 s

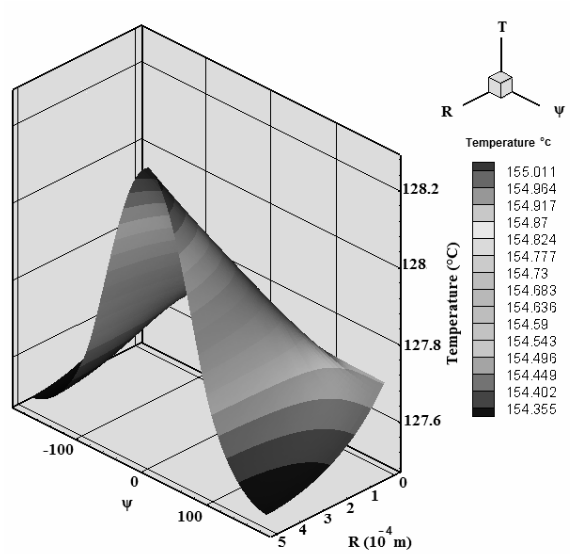

Fig.2(b), 3-D Temperature distribution at t=1 s

In Fig. 3, the temperature constant contours are drawn according to the polar angle ψ and the sphere radius R, at t=0.5 s and t=1 s. It is observed that the temperature at the upper part of the sphere (R=r_0 and ψ=0) which is subjected to heat flux is maximum. The temperature will be reduced by increasing the polar angle ψ. The temperature filed will also be reduced by decreasing the radius r. Fig. 4, shows the temperature charts at different radius. Results are compared to Ref. [1] & [9] and good agreements have been observed.

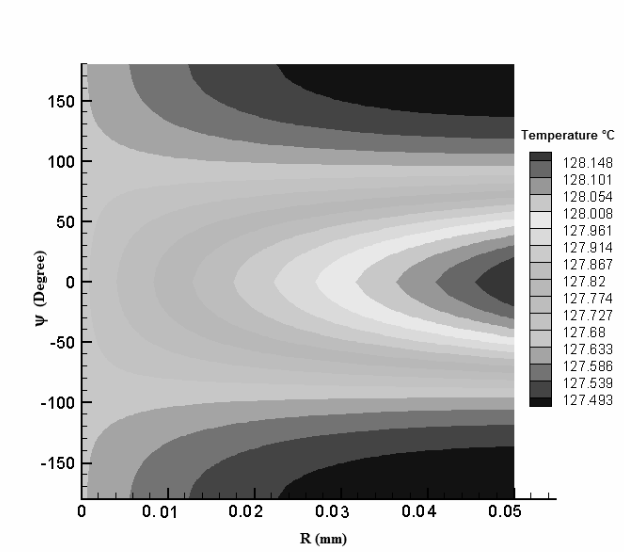

Fig. 3(a), Temperature contour in the sphere at t=0.5

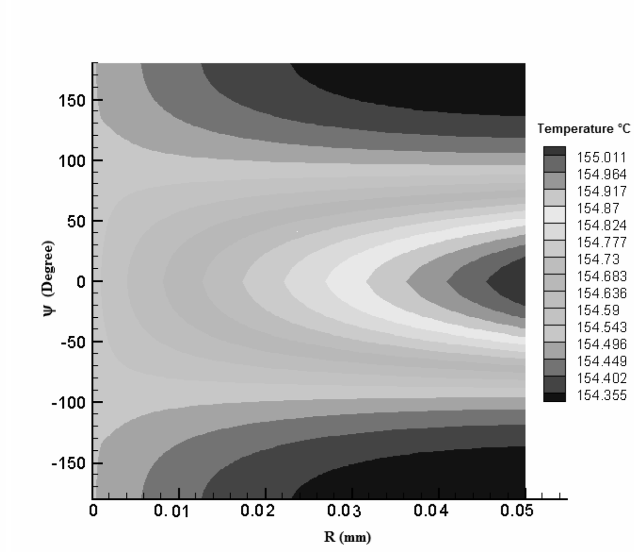

Fig. 3(b), Temperature contour in the sphere at t=1 s

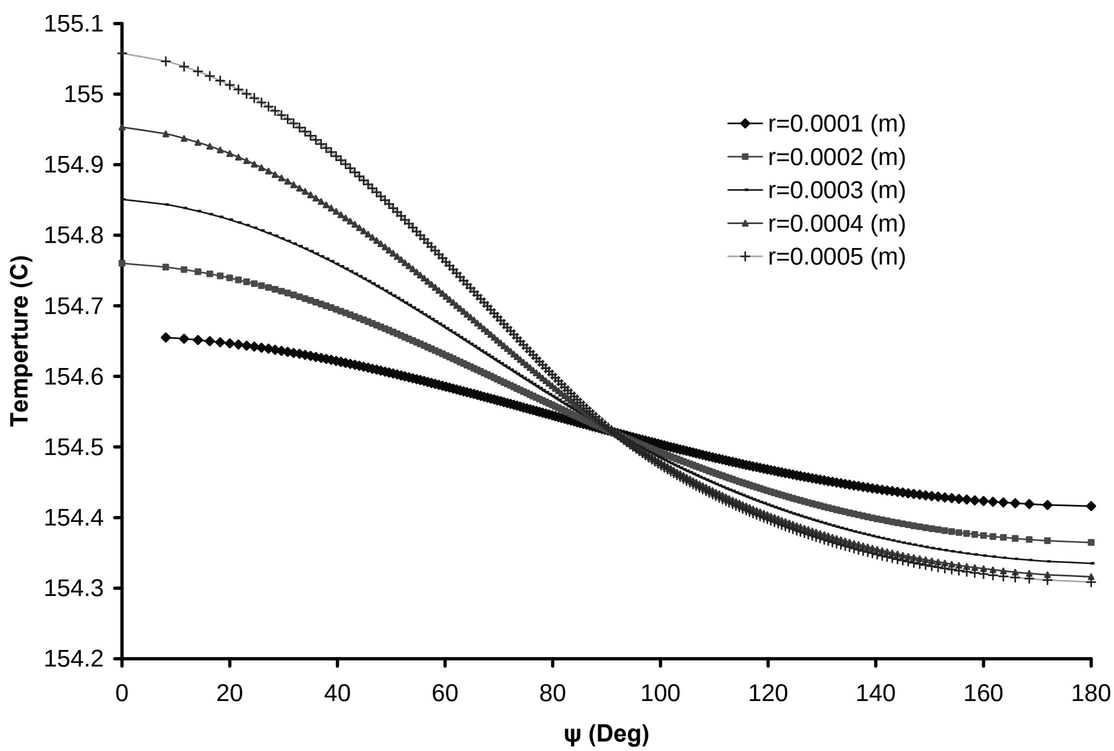

Fig. 4, Temperature according to polar angle at different sphere radius, t=1 s

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I. NOMENCLATURE

a^2	Inverse of thermal diffusivity (1/α)	s /m²
b	Plasma arc radius	m
C_p	Specific heat capacity	J/kg.K
h	Convection heat transfer coefficient	W/m².K
J	Bessel J function	-
k	Thermal conductivity	W/m.K
P	Legendre function	-
Q	Total thermal energy	W
q_0	Heat Flux	W/ m²
r,ψ,φ	Spherical coordinate	-
R_0, r_0	Sphere radius	m
T	Temperature	C
T_∞	Ambient temperature	C
T_0	Initial temperature	C
t	Time	s
x	Radial distance from arc center	m
α	Powder thermal diffusivity	m²/s
α_p	Absorptive coefficient of powder	-
θ	Temperature	C
ρ	Density	kg/m³
ω	Eigenvalue	-

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