Synthesis of Finite Time Stable Backstepping Control Systems

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Abstract—**A new nonlinear design technique for finite time stabilization of a class of nonlinear systems is developed using backstepping method. This method is able to apply strict feedback form systems. An example illustrates the theoretical results.**

Index Terms— **Finite time stabilization, nonlinear systems, backstepping, strict feedback form.**

I. INTRODUCTION

Finite time stability [1], [2] allows solving the finite time stabilization problem. This finite time stabilization was developed in [1]–[4], [9] for particular systems, for example the *n*-order integrator. Bhat et al provided an important contribution in [1] by proving that there is a necessary and sufficient condition for finite time stability involving the continuity of the settling-time function at the origin. Moulay et al developed in [2] a necessary and sufficient condition for finite time stability for continuous autonomous system. Then they developed a necessary and sufficient condition for finite time stabilization of class CL_k-affine systems involving a class CL^0-settling-time function for the closed-loop system. They extended Sontag formula [5] to design a feedback control for the finite time stabilization.

Backstepping control for continuous-time systems has recently received a great deal of attention in the nonlinear control literature [6]-[8]. The popularity of this control methodology can be explained in a large part due to the fact that it provides a framework for designing stabilizing non-linear controllers for a large class of nonlinear cascade systems. Specifically, this framework guarantees stability by providing a systematic procedure for finding a control Lyapunov function for the closed-loop system and choosing the control such that the time derivative of the control Lyapunov function along the trajectories of the closed-loop dynamic system is negative.

In this paper we develop finite time stabilization of strict feedback control systems with a method as [6]. The main result relies on Theorem 4.2 in [1].

The rest of paper is organized as follows: In Section 2, some notations and preliminary results on finite time stability is reviewed. Section3 presents finite time backstepping control for continuous-time systems. Simulation results are included in Section 4. Section 5 concludes the paper.

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II. PRELIMINARY RESULTS

In this section, we introduce notation and definitions, and present some key results needed for developing the main results of this paper. Let R denote the set of real numbers, R_+ denote the set of positive real numbers, $(.)^T$ and $|.|$ denote transpose and *1*-norm, respectively.
Consider the nonlinear dynamical system given by

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{1}$$

where $x(t) \in D \subseteq R^n$ is the state vector, D is an open set, $0 \in D$, $f(0) = 0$, and $f(.)$ is continuous on D.
The following result provides sufficient conditions involving a scalar Lyapunov function for finite-time stability of the nonlinear dynamical system (1).
Theorem 1 ([1]*):* Consider the nonlinear dynamical system (1). Assume there exists a continuously differentiable function $V : D \to R_+$, real numbers $c > 0$ and $\alpha \in (0,1)$, and an open neighborhood $\Omega \subseteq D$ of the origin such that $V(.)$ is positive definite and

$$\dot{V} \le -cV^{\alpha}, \qquad x \in \Omega$$

Then the zero solution $x(t) \equiv 0$ to (1) is finite-time stable. Moreover, if $N \subseteq D$ and $T(.)$ is the settling time function, then

$$T(x) \le \frac{1}{c(1-\alpha)}(V(x))^{1-\alpha}, \qquad x \in N$$

and $T(.)$ is continuous on N. If, in addition, $D = \Omega = R^n$ and $V(.)$ is radially unbounded, then the zero solution $x(t) \equiv 0$ to (1) is globally finite-time stable.

III. FINITE TIME BACKSTEPPING CONTROL

Let us consider the following system

$$\dot{\eta} = f(\eta) + g(\eta)\xi \tag{2}$$

$$\dot{\xi} = f_1(\xi) + g_1(\xi)u \tag{3}$$

where $[\eta^T, \xi]^T \in R^n$ is the state vector, $u \in R$ is the control input, $f : D \to R^n$ satisfies $f(0) = 0$, $g : D \to R^n$, and D is an open set with $0 \in D$. The goal is to design a control law to stabilize the origin($\eta = 0, \xi = 0$) for a finite time duration. If the following state feedback control law is chosen

$$u = \frac{1}{g_1(\xi)}(v - f_1(\xi)) \tag{4}$$

The following system is resulted

$$\dot{\eta} = f(\eta) + g(\eta)\xi \tag{5}$$

$$\dot{\xi} = v \tag{6}$$

Assume $\xi = \phi(\eta)$ exists such that the subsystem (2) is stabilized for a finite time duration. Also suppose that there exists a proper Lyapunov function $V : R^n \to R_+$ such that $\dot{V} \le -mV^\gamma$ on R^n where $m \ge 1$ and γ a rational number such that its denominator is an odd number. Same as [6] we have

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + g(\eta)z \tag{7}$$

$$\dot{z} = w \tag{8}$$

where $z = \xi - \phi$, and $w = v - \dot{\phi}$. Let $V_a(\eta, z) = V + |z|$ be a Lyapunov candidate for the above system. Then we have

$$\dot{V}_a = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta} g(\eta)z + \mathrm{sgn}(z)w \le$$
$$-mV^\gamma + \frac{\partial V}{\partial \eta} g(\eta)z + \mathrm{sgn}(z)w$$

Then with the choice of $w = -|z|\frac{\partial V}{\partial \eta} g(\eta) - mz^\gamma$ results in

$$\dot{V}_a \le -mV^\gamma - m|z|^\gamma \le -mV_a^\gamma \tag{9}$$

The inequality (9) shows the origin $(\eta = 0, z = 0)$ is globally finite time stable. Since $\phi(0) = 0$ the system (2)-(3) is globally finite time stable. By substituting w, z, $\dot{\phi}$, and (4) in u, the state feedback control law is determined

$$u = \frac{1}{g_1(\xi)}\left(\frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - |\xi - \phi|\frac{\partial V}{\partial \eta} g(\eta) - rz^\gamma - f_1(\xi)\right) \tag{8}$$

The main result is summarized in the following theorem.

Theorem 1: Consider the nonlinear dynamical system (2)-(3). Suppose that $\phi(\eta)$, $\phi(0) = 0$, be a state feedback finite time stabilizer for (2) and $V(\eta)$ be a Lyapunov function that $\dot{V} \le -mV^\gamma$ on R^n where $m \ge 1$ and γ a rational number such that its denominator is an odd number. Therefore, the control law (8) with the Lyapunov function $V_a(\eta, \xi) = V(\eta) + |\xi - \phi(\eta)|$ stabilizes for finite time duration. Moreover, the settling time function is

$$T(x) \le \frac{1}{m(1-\gamma)}(V_a(x))^{1-\gamma} \tag{9}$$

□

The same as [6] with repeat of backstepping method, the following strict feedback form can be stabilized in finite time

$$\begin{aligned}
\dot{x} &= f_0(x) + g_0(x)z_1 \\
\dot{z}_1 &= f_1(x, z_1) + g_1(x, z_1)z_2 \\
\dot{z}_2 &= f_2(x, z_1, z_2) + g_2(x, z_1, z_2)z_2 \\
&\vdots \\
\dot{z}_{k-1} &= f_{k-1}(x, z_1, \ldots, z_{k-1}) + g_{k-1}(x, z_1, \ldots, z_{k-1})z_{k-1} \\
\dot{z}_k &= f_k(x, z_1, \ldots, z_k) + g_k(x, z_1, \ldots, z_k)u
\end{aligned} \tag{11}$$

where $x \in R^n$, z_i, $1 \le i \le k$, is scalar value, and $f_i(0)=0$.

Multi-input case can also be determined based on *Theorem 1* and block backstepping method [10].

IV. EXAMPLE

To verify the theoretical derivations, we design state feedback control law for third-order integrator [9].
Consider the following dynamical system

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= u
\end{aligned} \tag{11}$$

In [9], a finite time stabilizer has been proposed for double integrators. Herein, we divide the above system in two subsystems. Therefore a feedback control for the finite time stabilization of first subsystem is [9]

$$\rho = x_3 = -\mathrm{sgn}(x_2)|x_2|^\lambda - \mathrm{sgn}(\phi_\lambda)|\phi_\lambda|^{\frac{\lambda}{2-\lambda}} \tag{12}$$

where $\phi_\lambda = x_1 - \frac{1}{2-\lambda}\mathrm{sgn}(x_2)|x_2|^{2-\lambda}$, and $\lambda = 3 - \frac{2}{\gamma}$.
Therefore, from *Theorem 1*, the finite time stabilizer feedback law is

$$u = w + \dot{\rho} = -|z|\frac{\partial V}{\partial x_2} - z^\gamma + \dot{\rho} \tag{13}$$

where $V = \frac{2-\lambda}{3-\lambda}|\phi_\lambda|^{\frac{3-\lambda}{2-\lambda}} + x_2\phi_\lambda - \frac{1}{3-\lambda}|x_2|^{3-\lambda}$ and $z = x_3 - \phi_\lambda$. Simulation runs for $\lambda = \frac{2}{3}$ and 20 seconds. Initial conditions of plant are set to $[1\ {-0.5}\ {-1}]^T$. Fig. 1 depicts state trajectories of the system. Fig.2 and Fig.3 show the control $w(t)$ and $z(t)$, respectively.

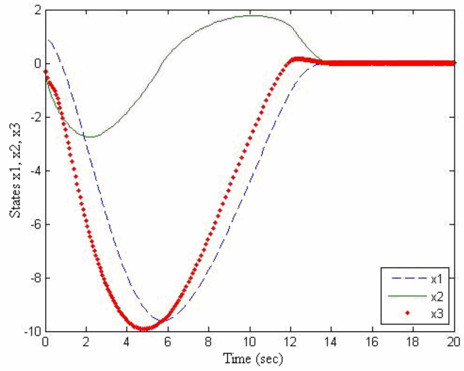

Fig. 1. Trajectories of the state x_1 (dashed line), the state x_2 (solid line), and the state x_3 (dotted line)

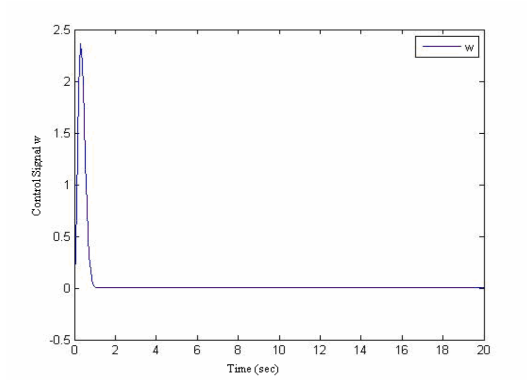

Fig. 2. Trajectory of the control signal w

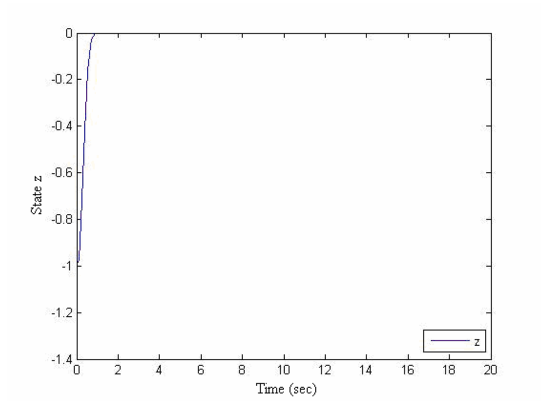

Fig. 3. Trajectory of the state z

V. Conclusion

In this paper, we have extended the backstepping control for finite time stabilization of continuous-time systems. The main result is a variable structure control which can be applied to a large class of nonlinear systems. The simulation results show that the proposed method is effective.

References

[1] S. P. Baht, D. S. Bernstein, "Finite time stability of continuous autonomous systems, SIAM J. Control Optim. 38 (3), pp. 751-766, 2000.

[2] E. Moulay, W. Perruquetti, "Finite time stability and stabilization of a class of continuous systems," *J. Math. Anal. Appl*. 323, pp. 1430–1443, 2006.

[3] Y. Hong, Finite-time stabilization and stabilizability of a class of controllable systems, *Systems Control Lett*. 46, pp. 231–236, 2002.

[4] Y. Hong, Y. Xu, J. Huang, Finite-time control for robot manipulators, *Systems Control Lett*. 46, pp. 243–253, 2002.

[5] E. D. Sontag, "A universal construction of Artstein's theorem on nonlinear stabilization," *Syst. Control Lett.*, vol. 13, pp. 117–123, 1989.

[6] P.V. Kokotovic, "The joy of feedback: nonlinear and adaptive," *IEEE Control Systems Magazine*, vol. 12, 1992, pp. 7–17.

[7] M. Krstic, I. Kanellakopoulos and P. V. Kokotovi6, "'Adaptive nonlinear control without overparameterization," *Systems Control Lett.*, Vol. 19, pp. 177 185, 1992.

[8] M. Krstic, J. M. Protz, J. D. Paduano and P. V. Kokotovic, "Backstepping designs for jet engine stall and surge control," In *Proc. 1EEE Conf. Dec. Contr.*, New Orleans, LA, 1995, pp. 3049-3055.

[9] S.P. Bhat, D. Bernstein, Continuous, bounded, finite-time stabilization of the translational and rotational double integrators, *IEEE Trans. Automatic Control*, Vol. 43, No. 5, pp. 678-682, May 1998.

[10] H.K. Khalil, *Nonlinear Systems*. 2nd ed. Prentice Hall. Englewood Cliffs, NJ. 1996, ch. 13.