

# Model Reduction of a Perturbative Representation of a Weakly Nonlinear System

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**Abstract** – The paper explores the model reduction of a perturbative representation of a weakly nonlinear system. It examines the behaviour of the perturbative model under several transformations with a view to determining a suitable projection matrix for use in Krylov model reduction routines. The paper proceeds to illustrate the determination of such a projection matrix. Finally, suitable examples illustrate the efficacy of the proposed approach.

**Index terms:** Circuits and Networks, Model reduction., Krylov techniques, Nonlinear circuits

## 1. INTRODUCTION

As the level of detail and intricacy of dynamical system models continues to grow, the essential behaviour of such systems can be lost in a cloud of complexity. Furthermore, numerical simulation can consume extensive memory resources and take inordinate amounts of time. Consequently, to cope with the growing complexity and dimensionality of systems, model reduction has become a vital aspect of modern system simulation. Model reduction techniques for linear systems are well studied (e.g. [1-2] and references therein). However, the study of nonlinear systems is much more complicated and the development of model reduction methods for large-scale nonlinear systems represents a formidable challenge.

In general, model reduction techniques fall into two categories – moment-matching and Krylov techniques and balancing techniques. The present contribution is concerned with the former. The advantage of Krylov-based methods is that matrix-vector multiplications are all that are involved in the formation of the projection matrices that are used to project the system onto a reduced system. Also sparse linear solvers and iterative approaches can be used to improve the computational efficiency [6].

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The present contribution proposes the use of a perturbative representation as an alternative to the bilinear representation as an approximation of the nonlinear functions [4]. For weakly nonlinear systems, either approximation is satisfactory. However, it will be seen that the perturbative method has several advantages over the commonly employed bilinear representation. The use of the perturbative representation in reduction techniques based on Krylov methods has been addressed in [4]. In this contribution, an improved reduction method is proposed. Illustrative examples are chosen, which are typical of examples employed for comparing model reduction approaches [4]. The errors obtained from the different reduction strategies will be compared.

## II. PERTURBATIVE APPROXIMATION OF NONLINEAR SYSTEMS

Let the nonlinear dynamical system under consideration be of the form:

$$\begin{aligned}\dot{x}(t) &= f(x(t)) + Bu(t) \\ y(t) &= Cx(t)\end{aligned}\quad (1)$$

where  $f : \mathfrak{R}^n \rightarrow \mathfrak{R}^n$  is a non-linear function,  $x \in \mathfrak{R}^n$  are the state-space variables. The initial condition is  $x(0) = x_0$  and  $u(t), y(t) \in \mathfrak{R}^1$ .  $B, C \in \mathfrak{R}^n$  are constant vectors ( $C$  is a vector-row and  $B$  is a vector-column). It is assumed that  $x = 0$  is a stable equilibrium point of the system (1) and  $x_0 = 0$ . Under this assumption,  $f(x)$  can be expanded in a generalised Taylor's series about  $x = 0$

$$f(x) = A_1 x^{(1)} + A_2 x^{(2)} + A_3 x^{(3)} + \dots \quad (2)$$

where  $x^{(1)} = x$ ,  $x^{(2)} = x \otimes x$ ,  $x^{(3)} = x \otimes x \otimes x$ , etc. and  $\otimes$  denotes the Kronecker product. Since  $x = 0$  is a stable equilibrium point,  $A_1$  is a stable matrix, i.e. all of its eigenvalues have negative real parts. It is also assumed that each term in the Taylor's expansion is small compared to the previous one.

Now consider the case where a variational parameter  $\alpha$  is introduced, i.e.  $\dot{x}(t) = f(x(t)) + Bau(t)$  and let the response of the system  $x(t)$  be *perturbatively* expanded in a power series in  $\alpha$  [4]:

$$x(t) = \alpha x_1(t) + \alpha^2 x_2(t) + \alpha^3 x_3(t) + \dots \quad (3)$$

On comparing terms in the variational parameter  $\alpha$ , the following set of  $n$ -dimensional differential equations can be derived:

$$\begin{aligned} \dot{x}_1 &= A_1 x_1 + Bu \\ \dot{x}_2 &= A_1 x_2 + A_2 (x_1 \otimes x_1) \\ \dot{x}_3 &= A_1 x_3 + A_2 (x_1 \otimes x_2 + x_2 \otimes x_1) + A_3 (x_1 \otimes x_1 \otimes x_1) \\ &\vdots \end{aligned} \quad (4)$$

Each  $n$ -dimensional equation describes the time evolution of an  $x_i$  where  $x_i$  represents the  $i^{th}$  order perturbative term in the expansion (3). Defining a vector  $\underline{x}(t)$ :

$$\underline{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \end{bmatrix} \quad (5)$$

the system in (4) acquires the form:

$$\begin{aligned} \dot{\underline{x}} &= \underline{A}\underline{x} + \underline{B}u \\ y &= \underline{C}\underline{x} \end{aligned} \quad (6)$$

where

$$\underline{A} = \begin{bmatrix} A_1 & & & \\ & A_1 & & \\ & & A_1 & \\ & & & \ddots \end{bmatrix}, \underline{B} = \begin{bmatrix} B & 0 & 0 & 0 \dots \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_2 & A_3 \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}, \underline{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ \vdots \end{bmatrix} \quad (7)$$

$$\underline{C} = [C \ C \ C \ \dots]$$

$$u_1 = u(t), \quad u_2 = x_1 \otimes x_1, \quad u_3 = x_1 \otimes x_2 + x_2 \otimes x_1$$

$$u_4 = x_1 \otimes x_1 \otimes x_1 \dots$$

The source  $u_2$  for the second equation in (4) depends only on the state vector  $x_1$  determined from the first equation and so on. Note that since  $A_1$  is a stable matrix,  $\underline{A}$  is also automatically stable. Now,  $u_2, u_3$ , etc. are not independent inputs like  $u_i$  and therefore, linear system theory cannot be applied directly to the representation in (6). However, subsequent sections will show how the representation may be adapted so that linear system theory and consequently, linear model reduction may be applied to the representation in (6).

The well-known bilinear representation (Carleman bilinearization) [4, 5] is an alternative approach to approximation of (1) for weakly nonlinear systems:

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{N}\hat{x}(t)u(t) + \hat{B}u(t) \\ y(t) &= \hat{C}\hat{x}(t) \end{aligned} \quad (8)$$

$$\text{where } \hat{x}(t) = \begin{bmatrix} x^{(1)} \\ x^{(2)} \\ \vdots \end{bmatrix}$$

$\hat{A}$  and  $\hat{N}$  are square matrices of dimension  $n + n^2 + \dots + n^K$ ,  $\hat{B}$ ,  $\hat{C}$  are vectors with  $n + n^2 + \dots + n^K$  components if  $K$  terms in the Taylor's

series expansion are taken into account. The matrices are defined in [4,5].

However, the perturbative representation has several advantages over the bilinear representation – namely -

1. The system (6) has a simple linear form unlike (8).
2. The size of the system (6) with  $K$  perturbative terms is  $nK$ . The size of (8) with  $K$  terms in the series expansion is  $n + n^2 + \dots + n^K$ .
3. There is no need to restrict the input to the perturbative system to guarantee stability. However, a restriction exists on the input to guarantee stability of the bilinear system [5]. A sufficient condition for stability on the interval  $[0, T]$  is  $|u(t)| \leq K_c$  for all  $t \in [0, T]$  where  $\hat{A} + \lambda \hat{N}$  is stable for all  $\lambda \in [-K_c, K_c]$ .

### III. KRYLOV-BASED MODEL REDUCTION

The goal of any model reduction technique is to replace the  $n$ -dimensional system (1) with a system of much smaller dimension  $k \ll n$ , such that the behaviour of the reduced order system satisfactorily represents the behaviour of the full system. In projection based reduction schemes, a projection matrix,  $V$ , is selected such that its columns span the required subspace [4]. The reduced system is then formed from approximating the state vector  $x$  with  $\hat{x}$  where  $\hat{x} = Vx$ .

Consider a linear state-space representation:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned} \quad (9)$$

The state-space equations for the reduced system are then given as:

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}u(t) \\ \hat{y}(t) &= \hat{C}\hat{x}(t) \end{aligned} \quad (10)$$

where

$$\hat{A} = V^t A V \quad \hat{B} = V^t B \quad \text{and} \quad \hat{C} = C V$$

In Krylov-based methods, the projection matrix is chosen to span the columns of the Krylov subspace

$$\begin{aligned} &K_m((s_0 I - A)^{-1}, (s_0 I - A)^{-1} B) \\ &= \{(s_0 I - A)^{-1} B, (s_0 I - A)^{-2} B \dots (s_0 I - A)^{-m} B\} \end{aligned}$$

The rationale for selection of this subspace is that it results in matching the first  $m$  moments of the original and reduced systems.  $s_0$  is the point in the complex plane about which moments are matched. However, the Krylov-based reduction methods are preferable to direct moment matching techniques as the methods avoid the numerical problems arising in explicit moment matching.

For nonlinear systems, model reduction is not as straightforward as for linear systems. In this contribution, we look at some of the properties of linear systems with a view to adapting the perturbative representation of the nonlinear system so that a reduction strategy similar to that for linear systems can be applied to it.

#### IV. SCALE INVARIANCE PROPERTY

Consider the behaviour of a linear system when  $u \rightarrow \alpha u$ . In this case, the output also changes as  $y \rightarrow \alpha y$ . We term this the scale invariance property which holds for linear systems. The result is that the Krylov-based reduction method is unaffected when  $u \rightarrow \alpha u$ . Similarly, if  $x \rightarrow \beta x$ , the reduction process is unaffected. However, nonlinear systems are not scale invariant. For example, consider the perturbative system under a rescaling of the input. I.e. consider  $u \rightarrow \alpha u$ . The  $\underline{Bu}$  term of (6) transforms as:

$$\underline{Bu} \rightarrow \alpha \begin{bmatrix} 1 & & & \\ & \alpha & & \\ & & \alpha^2 & \\ & & & \ddots \end{bmatrix} \underline{Bu} \quad (11)$$

It is evident from (11) that the scale invariance property does not hold. To enable application of linear theory to (6) would require that  $\underline{Bu} \rightarrow \alpha \underline{Bu}$  which is not the case as evident from (11). Consequently, linear model reduction techniques may not be applied directly to the perturbative representation and hence, a modification is required. To this end, a parameter  $\mu$  is introduced with a view to explicitly accounting for the scale dependence of the nonlinear system. The role of  $\mu$  is to bear the nonlinear properties of the system throughout the reduction process. Consider (6) and (7). The  $\underline{Bu}$  term can be rewritten as:

$$\underline{Bu} = D \underline{BU}$$

where

$$D = \text{diag}(1 \quad \mu \quad \mu^2 \quad \dots)$$

with

$$\underline{U} = [u_1 \quad \mu^{-1}u_2 \quad \mu^{-2}u_3 \quad \mu^{-2}u_4 \quad \dots]$$

for any nonzero function  $\mu$ .

If when  $u_1(t) \rightarrow \alpha u_1(t)$ ,  $\mu$  transforms as:

$$\mu \rightarrow \alpha \mu$$

then  $\underline{U}$  transforms as:

$$\underline{U} \rightarrow \alpha \underline{U} \quad (12)$$

It transforms in the same manner as the input to a linear system. The property in (12) is very important as it shows that to enable application of linear systems theory to (6), then the proper input to (6) is actually  $\underline{U}$  and not  $\underline{u}$ .

An estimate for  $\mu$  may be determined as follows: If  $\mu = 0$ , then the system in (6) is linear. Thus,  $\mu$  must be proportional to the output due to the nonlinearity  $y - y_1$  where  $y_1 = Cx_1$  is the output from the linear part of (6). For the purposes of practical implementation of the reduction scheme, it is convenient to take  $\mu$  as a constant parameter. Hence, the following is deemed an appropriate choice for  $\mu$ :

$$\mu = \frac{\overline{|y - y_1|}}{T \overline{|u|}} \quad (13)$$

where the bar denotes the average value of the waveform over the time interval  $[0 T]$  for which the behaviour of the system is under examination, provided  $\overline{u} \neq 0$ . An exact optimal value for  $\mu$  for a particular model parameterisation may be chosen from computer simulations for a particular 'test input' that is close to the inputs for which the system is designed.  $\mu$  is then determined by minimising an error function using the Nelder-Mead algorithm. A suitable error function is the following:

$$\text{err} = \frac{\sqrt{\sum (f_{ex} - f_{red})^2}}{N} \quad (14)$$

$f_{ex}$  is the output from the exact model and  $f_{red}$  is the output from the reduced model.  $N$  is the number of samples taken of  $f_{ex}$  to compute the error.

However, for most practical cases, the estimate in (13) suffices. Obviously, the reduced model approximates the input-output behaviour of the system locally. No conclusions however, can be drawn about its global behaviour.

#### V. KRYLOV REDUCTION OF PERTURBATIVE REPRESENTATION

The reduction process for the perturbative representation proceeds as follows: Let the approximation in (2) involve  $K$  terms. The dimension of the system representation in (6) is thus  $NK$ . Suppose it is required to reduce the system to dimension  $k$ . The Krylov space for the first-order response  $x_1$  in (4) and (6) is formed as

$K_1 = (A_1^{-1}, A_1^{-1}B)$  where  $s_0$  is set to zero. An orthogonal projection matrix,  $V_1$  for the first-order system is formed from  $K_1$  I.e.  $\hat{x}_1 = V_1 x_1$ . Now, the second-order system in (4) and (6) is formed as:

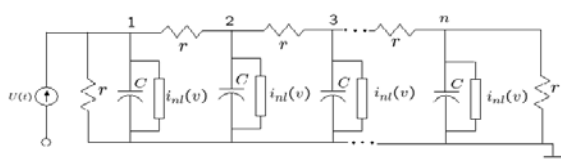
$$\begin{aligned} \dot{x}_2 &= A_1 x_2 + \mu A_2 (\hat{x}_1 \otimes \hat{x}_1) \\ &= A_1 x_2 + \mu A_2 (V_1 \otimes V_1) (x_1 \otimes x_1) \\ &= A_1 x_2 + \mu A_2 (V_1 \otimes V_1) \hat{u}_2 \\ &= A_1 x_2 + B_2 \hat{u}_2 \end{aligned} \quad (15)$$

This differs from the standard second-order system such as that presented by Phillips [4]. In the standard version,

$\mu=1$ . However, results in section 6 will show that inclusion of the novel proposal for  $\mu$  achieves greater accuracy.

The Krylov space for the linear system in (15) is then formed as  $K_2 = (A_2^{-1}, A_2^{-1}B_2)$ . An orthogonal projection matrix,  $V_2$ , is formed from  $K_2$  and this matrix is used to reduce the second order system. The procedure for reducing the higher order terms in (4) and (6), i.e.  $x_3 \dots$ , in the perturbative system is similar.

### VI. ILLUSTRATIVE EXAMPLE



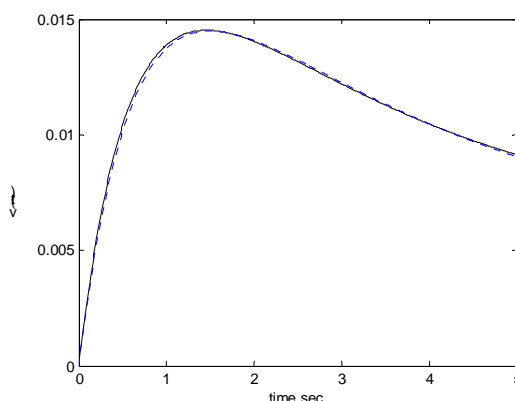
**Fig. 1** Nonlinear circuit

The circuit employed is the nonlinear ladder shown in Fig. 1. The number of nodes in the system is  $n=30$ . The ladder in represents a heat flow model [7]. The voltage at the  $m$ th node represents the temperature on a rod at a distance proportional to  $m$ . The (input) voltage at node 1 represents the heat source. The nonlinearities represent the dependence of the conductivity on the temperature. The average voltage at all nodes is taken as the output and this represents the average temperature of the rod. Varying the circuit parameters corresponds to different spatial or environment conditions [7]. The nonlinear resistor introduces a quadratic nonlinearity at each node:

$$i_{nl}(v) = gv^2 \tag{16}$$

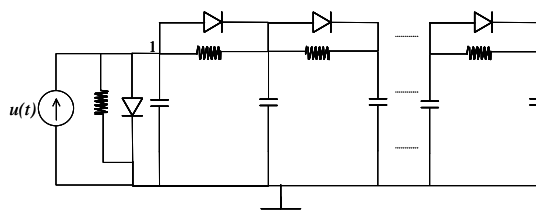
for  $v>0$ . The parameters are  $C=r=1$ . The strength of the nonlinearity is varied by varying  $g$ .

The dimension of the original state-space is  $n=30$ . The perturbative representation (6) contains two terms, i.e.  $K = 2$ . The reduction process is performed from the representation (6) of order  $nK = 60$  to a representation of order  $k = 6$ . The value is  $\mu = 1.6443$ . Fig. 2 shows the result for an exponential input  $e^{-t}$  from the reduced perturbative model for  $g=0.1$  superimposed on the result from a full nonlinear model of Fig. 1. The root mean square error between this result and that computed from a full model is 0.0026. The reduced model is equally accurate for other input signals. In order to confirm the value of inclusion of  $\mu$ , the root mean square error is 0.0042 when  $\mu=1$ .



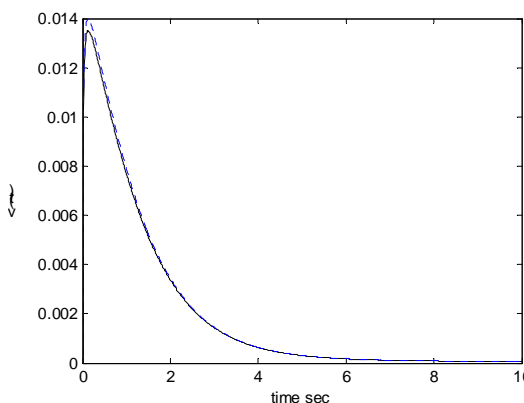
**Fig. 2** Reduced perturbative model  $g=0.1$   
Solid line=full model Dashed line= reduced model

As a second example, consider the 30-section nonlinear RC ladder shown in Fig. 3.



**Fig. 3** Nonlinear RC ladder

The reduction process described in Section 5 is applied. The system is reduced from a dimension of 60 to a dimension of 6. The value of  $\mu$  is determined from (13) as 0.6747. With this value of  $\mu$ , the RMS error is 0.0037. With the standard approach of [4], the RMS error is 0.0076.



**Fig. 4** Reduced perturbative model  
Solid line=full model Dashed line= reduced model

### VII. CONCLUSIONS

The paper has investigated Krylov model reduction of a perturbative representation. The input-output mapping for nonlinear systems depends on inputs and is not scale-independent. To explicitly account for this

dependence, a parameter is introduced into the perturbative representation. Results in this contribution indicate that inclusion of the parameter leads to greater accuracy. Previous work [8] has shown that the same approach also leads to improved accuracy in balanced truncation model reduction.

The principal advantage of the perturbative representation compared to the bilinear representation is the much smaller size of the former ( $nK$ ) compared to the size of the latter ( $\sim n^K$ ), which results in reduced computational cost.

#### ACKNOWLEDGMENT

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