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A Novel Approach to Calculate Stable Densities

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Abstract—**This paper concerns the calculation of general stable densities via asymptotic series in continuous parameterization. Up to now, series have been presented in discontinuous one and their convergence regions had not been characterized. Here, series are represented in continuous parameterization and their convergence regions will be derived in two closed form formula. Getting ready these series in continuous parameterization yields more spacious convergence region than discontinuous one. Also, resolves the computation problems in adjacent of parameters boundary values particularly α=1and large values of |x|. The series can compute accurately density function on their convergence regions. Thus, computing on the base of series can be done for much more areas of real line, x's and almost over the whole area of parameter space quite faster and more accurate than other conventional methods. For areas beyond convergence region which consist only limited areas, conventional approach can be employed. Techniques suggested in this paper are very useful in evaluation of Fisher information matrix which plays main role in ML estimation.**

Index Terms— **Stable Distributions, Asymptotic Series, Parameterization of stable family, Fast Fourier Transform, Inversion Formula.**

I. INTRODUCTION

In recent few years, there has been a great interest in α-stable distributions for modeling impulsive data. Many observations collected from signal processing have long or heavy tails. A wide range of modeling areas that encompasses signal processing, signal interpolation, channel noise modeling have received treatment with alpha-stable distributions, see [4], [5], [9]-[11]. The major difficulty in the use of stable family is the lack of closed formula for their density function. There are many works in this context, density function calculation. However, asymptotic series, as a novel method have been applied for computing density function only for symmetric case so that explicit formulas for their convergence regions in which series are applicable for density computing has not been proposed yet. These are important obstacles because most real-life signals are skewed and in the other hand, series convergence region had not been determined by the closed form formula. Generally, stable distributions are defined based on their characteristics function and pdf is derived by using inversion formula from characteristics via:

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-itx}\,\varphi(t)\,dt \qquad (1)$$

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Where $\varphi(t)$ is characteristics function which involves a much of numerical complexity, direct method, see [2], [5]. There have been several efforts to compute stable densities. A few of them are worth mentioned. Direct method is the most common method offered by Nolan, 1997 in which evaluations have been performed by STABLE program. This approach challenging numerical problems and works poorly in the following cases:
1-when α is near 1
2-For very small and large values of |x|

Also, in 1999 he then offered an algorithm for computing general stable densities of $\alpha > 0.75$. McCulloch, 1998 had been extended an algorithm for computing $S\alpha S$ pdf for $\alpha > 0.85$. Matsui and Takemura, 2006 proposed some formulas for computing of $S\alpha S$ pdf, first and second derivatives of which on computing asymptotic series caused by straightforward computations using these series. This work is structured as follow: In section 2 we present asymptotic series in continuous parameterization, S0. Section 3 deals with the numerical considerations using asymptotic series comparing with STABLE program. Section 4, conclusion, describes characteristics of asymptotic series as a powerful pdf calculating approach. Ultimately, series convergence region computation methods have been given in appendixes (A) and (B).

II. ASYMPTOTIC SERIES AND MAIN THEOREM

Asymptotic series is known for stable densities except for $\alpha = 1$, see [8], [9]. We have:

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{1}{\pi(x-\mu)}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(\alpha k+1)}{\Gamma(k+1)}\left(\frac{x-\mu}{r\,\sigma}\right)^{-\alpha k}\sin\left(\frac{k\pi(\alpha+\gamma)}{2}\right) \qquad 0<\alpha<1 \quad (2)$$

$$f(x;\alpha,\beta,\mu,\sigma)=\frac{1}{\pi(x-\mu)}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(k/\alpha+1)}{\Gamma(k+1)}\left(\frac{x-\mu}{r\,\sigma}\right)^{k}\sin\left(\frac{k\pi(\alpha+\gamma)}{2\alpha}\right) \qquad 1<\alpha<2 \quad (3)$$

Where $\gamma=-(2/\pi)\arctan(\eta)$, $r=(1+\eta^2)^{\frac{1}{2\alpha}}$ and $\eta=-\beta\tan(\pi\alpha/2)$ .In fact, the equation 1 is an asymptotic series for all $0<\alpha<2$ and it converge when $|x-\mu|$ gets large. Also, equation 2 is an asymptotic series for all $0<\alpha<2$ and it converges when $|x-\mu|$ tends to get 0, see [9]. Therefore for analyzing the series behavior it is suitable the series to be introduced in terms of $|x-\mu|$ instead of $x-\mu$. The following lemma which is based on reflexive property of stable family makes it possible to present the equation 1 and 2 as functions of $|x-\mu|$ instead of $x-\mu$.

Lemma 1**:** if $X \sim S(\alpha,\beta,\sigma,\mu)$ then:

$$f(x;\alpha,\beta,\mu,\sigma)=f(|x-\mu|;\alpha,\operatorname{sgn}(x-\mu)*\beta,\sigma)$$

Proof: We know stable densities belong to location family then: $f(x-\mu;\alpha,\beta,\sigma) = f(x;\alpha,\beta,\mu,\sigma)$ and according to the chief

Fact about this family that is named reflection property, hence:

$$f(x;\alpha,\beta,\mu,\sigma) = f(-x;\alpha,-\beta,-\mu,\sigma) \quad (4)$$

Then if,

$x-\mu > 0 \Rightarrow \operatorname{sgn}(x-\mu)*\beta = \beta$ Therefore:

$$f(|x-\mu|;\alpha,\operatorname{sgn}(x-\mu)*\beta,\sigma) = f(x-\mu;\alpha,\beta,\sigma) = f(x;\alpha,\beta,\mu,\sigma)$$

If $x-\mu < 0 \Rightarrow \operatorname{sgn}(x-\mu)*\beta = -\beta$ Therefore:

$$f(|x-\mu|;\alpha,\operatorname{sgn}(x-\mu)*\beta,\sigma) = f(-x+\mu;\alpha,-\beta,\sigma) = f(-x;\alpha,-\beta,-\mu,\sigma) = f(x;\alpha,\beta,\mu,\sigma)$$

Where, the last equality is implied by equation (4). It is clear that $f(x;\alpha,\beta,\mu,\sigma) = f(|x-\mu|;\alpha,\operatorname{sgn}(x-\mu)*\beta,\sigma)$ where sgn(.) is sign function. Consequently, asymptotic series can be presented for every real value as follow:

$$f(x;\alpha,\beta,\mu,\sigma) = \frac{1}{\pi|x-\mu|} \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(\alpha k+1)}{\Gamma(k+1)}\left|\frac{x-\mu}{\sigma r}\right|^{-\alpha k}\sin\left(\frac{k\pi(\alpha+\gamma^*)}{2}\right) \quad (5)$$

$$f(x;\alpha,\beta,\mu,\sigma) = \frac{1}{\pi|x-\mu|} \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(k/\alpha+1)}{\Gamma(k+1)}\left|\frac{x-\mu}{\sigma r}\right|^{k}\sin\left(\frac{k\pi(\alpha+\gamma^*)}{2\alpha}\right) \quad (6)$$

Where, $\gamma^* = -(2/\pi)\arctan(\eta)\times\operatorname{sgn}(x-\mu)$ .Therefore, It is enough to replace $x-\mu$ and β respectively with $|x-\mu|$ and $\operatorname{sgn}(x-\mu)*\beta$. It can be shown that the length of series convergence regions tends to infinity as $\alpha\to 1$. To resolve this issue, it is suitable asymptotic series to be used in a continuous parameterization, see [6], [9]. It is suffice to replace $x-\mu$ by $x-\mu-\zeta$, where $\zeta = -\beta\,\sigma\tan(\pi\alpha/2)$ .This change has a drastic effect on convergence interval length in adjacent of $\alpha = 1$. It can be shown that the length of series convergence regions tends to zero as $\alpha\to 1$. Also, two series behave homogonously and uniformly in adjacent of $\alpha = 1$. Below, the theorem 1 offers asymptotic series for stable random variables in continuous parameterization, S0.

Theorem1. Suppose that $X \sim S_0(\alpha,\beta,\sigma,\mu)$ then asymptotic series for X density function are as following:

$$f(x;\alpha,\beta,\mu,\sigma) = \frac{1}{\pi|x-\mu-\zeta|} \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(\alpha k+1)}{\Gamma(k+1)}\left|\frac{x-\mu-\zeta}{\sigma r}\right|^{-\alpha k}\sin\left(\frac{k\pi(\alpha+\gamma^{**})}{2}\right) \quad (7)$$

$$f(x;\alpha,\beta,\mu,\sigma) = \frac{1}{\pi|x-\mu-\zeta|} \lim_{n\to\infty}\sum_{k=1}^{n}\frac{(-1)^{k-1}\Gamma(k/\alpha+1)}{\Gamma(k+1)}\left|\frac{x-\mu-\zeta}{\sigma r}\right|^{k}\sin\left(\frac{k\pi(\alpha+\gamma^{**})}{2\alpha}\right) \quad (8)$$

Where, $\gamma^{**} = -(2/\pi)\arctan(\eta)*\operatorname{sgn}(x-\mu-\zeta)$. Notice that pdf at $x = \mu+\zeta$ is evaluated by series 8. Method of deriving the convergence regions for series 7 and 8 has been brought in appendixes (A) and (B) in which the results are explicit formulas for these series convergence regions. For ease, a general shape of series regions for symmetric case has been given in Fig.1.

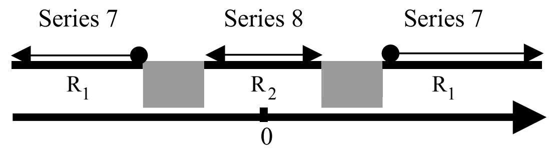

Fig.1. Series 7 and 8 convergence regions, R_1 and R_2 for symmetric case

Where Ω denotes the real line, x's and density must be evaluated via other methods in areas which are colored in Fig.1. Keep in mind that these areas length is very small for all the parameter space values.

III. NUMERICAL CONSIDERATION

It must be emphasized here that all researchers have agreement on this matter that computing pdf via asymptotic series will approximate the pdf more reliable and accurate than other fashion approaches in their convergence region especially for large values of x, of course we will show its accuracy for some cases of parameter values. The methods used in [2], [3] have been only discussed for special cases $\alpha > 1$ and symmetric, of which relative errors are considerably greater than our proposed algorithm and also much time-consuming.

SαS Densities and Fisher information matrix evaluation using expansion series have been studied in [1]. But regions of which series have been applied are much narrower than ours and even so, we acknowledge that the most reliable approach for comparing is the STABLE program which is accessible via address “http://academic2.american.edu/~jpnolan/stable”. Be conscious that for $\alpha = 1$, series are not defined. Proposing series in continuous parameterization helps us to approximate pdf for all α 's that are considerably close to 1, because by a straightforward investigation it is found that the lower bound and upper bound of series 8 tend to meet their counterparts in series 7. For example, in the case $\alpha = .99$ and $\beta = .99$ the values -126.6 and -124.2 also, -.85 and .7 get together. Table I shows this fact for three choices of β and four values of α in neighborhood of 1.

Table I.
Series 7 and 8 convergence regions, R_1 and R_2 in vicinity of $\alpha = 1$ and some values of β

α	β	R_1	R_2
.99	0	$(-\infty,-.94]\cup[.94,\infty)$	(-.31, .31)
	0.5	$(-\infty,-4.7]\cup[1.2,\infty)$	(-62,-.39)
	0.99	$(-\infty,-126.6]\cup[.7,\infty)$	(-124, -.85)
.995	0	$(-\infty,-2.9]\cup[2.9,\infty)$	(-.47, .47)
	0.5	$(-\infty,-28.6]\cup[1.3,\infty)$	(-126, -1.2)
	0.99	$(-\infty,-53.1]\cup[1.1,\infty)$	(-250, -1.4)
1.005	0	$(-\infty,-3.1]\cup[3.1,\infty)$	(-.53, .53)
	0.5	$(-\infty,-2.7]\cup[130,\infty)$	(-.19, 127)
	0.99	$(-\infty,-2.9]\cup[255,\infty)$	(-.42, 252)
1.01	0	$(-\infty,-3]\cup[3.08,\infty)$	(-.05, .05)
	0.5	$(-\infty,-2.9]\cup[66.6,\infty)$	(1.1, .62.6)
	0.99	$(-\infty,3.3]\cup[129.4,\infty)$	(65, 125.4)

More investigations show that offering the series in continuous parameterizations is reliable for all α's even greatly close to 1 which resolve the problem 1 of STABLE.

Series 7 is not valid for $\alpha = 2$, since tails of pdf decays exponentially it can not be showed by a polynomial. Also due to same reason (exponentially growth) we can't consider the series in this study for the case $|\beta| = 1$ (totally skew). Computing pdf via asymptotic expansions is often very suitable for cases in which STABLE can't work proficiently, see [1]. Analyzing the series more, yields the number of summation terms (n) so that the magnitude of sum of residual terms would be at most 10E-26. Tables II and III show minimum and maximum of absolute relative error (ARE) of closed form Cauchy $f(\alpha = 1, \beta = 0)$ and $f(\alpha = 0.5, \beta = 0)$, using series 7 and STABLE, with 500 points from 1 to 500 and 499 points from 2 to 500 by step one respectively. ARE is defined as $\left| f - \hat{f} \right| / f$ where, $\hat{f}$ is either STABLE based computation or series 7 based.

Table II.
Minimum and maximum of ARE comparison with
$f(\alpha = 0.5, \beta = 0)$

ARE	minimum of ARE	maximum of ARE
STABLE	0.11366E-15	0.43913E-14
Series 7	0	0.15566E-82

Table III.
Minimum and maximum of ARE comparison with
Cauchy case, $f(\alpha = 1, \beta = 0)$

ARE	minimum of ARE	maximum of ARE
STABLE	0.11366E-15	0.11163E-11
Series 7	0	0.126217E-27

It is obvious that series 7 is more accurate than STABLE. For large values of x, as born out in [1]-[3] asymptotic series approximate the pdf more accurately and rapidly than STABLE. For the purpose of simplicity, computations have been performed by Maple 9 software and standard case is chosen in computations. Determined convergence regions, terms number (n) and constant $C(\alpha)$ have been given in Table IV.

Table IV.
Algorithm specifications for using series 7 and 8

series 7	series 8	α
$n = 150, C = \alpha$	$n = 2, C = 0$	[.1, .2]
$n = 150, C = \alpha$	$n = 5, C = 0$	(.2,.3]
$n = 150, C = \alpha$	$n = 10, C = 0$	(.3, .5]
$n = 150, C = \alpha$	$n = 100, C = 0$	(.5, .8]
$n = 150, C = \alpha$	$n = 150, C = 0$	(.8,.9]
$n = 150, C = \alpha$	$n = 150, C = \alpha / 2$	(.9, .97]
$n = 200, C = 2\alpha$	$n = 200, C = \alpha / 2$	(.97,.995)
$n = 200, C = 2\alpha$	$n = 200, C = 2\alpha$	(1.005, 1.03]
$n = 200, C = 2\alpha$	$n = 200, C = \alpha / 2$	(1.03,1.1]
$n = 150, C = \alpha$	$n = 150, C = \alpha$	(1.1,1.5]
$n = 150, C = \alpha$	$n = 150, C = \alpha$	(1.5, 1.8]
$n = 150, C = \alpha$	$n = 150, C = 2\alpha$	(1.8,1.995]

To show the proficiency of these series, it has been executed series 7 for parameter space in standard case. Table V shows maximum relative error (RE) of series 7 comparing with STABLE program. Since, the STABLE has 10 decimal places of accuracy, it is mentioned in STABLE user guide it is evident that results are very pleasing.

Table V.
Series 7 maximum RE comparison with STABLE

α \ β	0	0.5	0.99
0.1	0.188×10^{-14}	0.171×10^{-15}	0.167×10^{-14}
0.5	0.220×10^{-14}	0.254×10^{-14}	0.381×10^{-15}
0.99	0.728×10^{-13}	0.168×10^{-15}	0.536×10^{-13}
1.01	0.465×10^{-13}	0.759×10^{-12}	0.221×10^{-9}
1.2	0.192×10^{-14}	0.370×10^{-14}	0.427×10^{-15}
1.5	0.651×10^{-13}	0.464×10^{-14}	0.368×10^{-14}
1.99	0.580×10^{-8}	0.259×10^{-13}	0.185×10^{-12}

Also, for more clarifications it has been shown in Table IV the series 7 and 8 convergence regions, R_1 and R_2 in standard $S\alpha S$ case. It is evident that above series can be very good technique for computing the pdf in a rapid and accurate discipline. As it shows series 7 undertakes more share for this aim so that, approximately compute the pdf almost for all real line when for example those α which are lass than 1.

Table VI.
Series 7 and 8 convergence regions, R_1 and R_2 in standard
$S\alpha S$ case

α	R_2	R_1
0.1	$[-0.2\times10^{-30}, 0.2\times10^{-30}]$	$(-\infty, -0.1]\cup[0.1, \infty)$
0.2	$[-0.6\times10^{-13}, 0.6\times10^{-13}]$	$(-\infty, -0.2]\cup[0.2, \infty)$
0.3	$[-0.15\times10^{-6}, 0.15\times10^{-6}]$	$(-\infty, -0.3](\cup[0.3, \infty)$
0.4	$[-0.5\times10^{-5}, 0.5\times10^{-5}]$	$(-\infty, -.4]\cup[.4, \infty)$
0.5	$[-0.001, 0.001]$	$(-\infty, -0.51]\cup[.51, \infty)$
0.6	$[-0.016, 0.016]$	$(-\infty, -.62]\cup[.62, \infty)$
0.7	$[-0.07, 0.07]$	$(-\infty, -.78]\cup[.78, \infty)$
0.8	$[-0.21, 0.21]$	$(-\infty, 1.03]\cup[1.03, \infty)$
0.9	$[-0.39, 0.39]$	$(-\infty, -1.41]\cup[1.41, \infty)$
0.99	$[-0.31, 0.31]$	$(-\infty, -1.91]\cup[1.91, \infty)$
1.01	$[-0.95, 0.95]$	$(-\infty, -3.08]\cup[3.08, \infty)$
1.1	$[-1.21, 1.21]$	$(-\infty, -3.93]\cup[3.93, \infty)$
1.2	$[-1.48, 1.48]$	$(-\infty, -3.96]\cup[3.96, \infty)$
1.3	$[-2.59, 2.59]$	$(-\infty, -5.43]\cup[5.43, \infty)$
1.4	$[-3.92, 3.92]$	$(-\infty, -7.26]\cup[7.26, \infty)$
1.5	$[-5.47, 5.47]$	$(-\infty, -9.47]\cup[9.47, \infty)$
1.6	$[-5.59, 5.59]$	$(-\infty, -12.08]\cup[12.08, \infty)$
1.7	$[-7.36, 7.36]$	$(-\infty, -15.1]\cup[15.1, \infty)$
1.8	$[-9.27, 9.27]$	$(-\infty, -18.51]\cup[18.51, \infty)$
1.9	$[-11.27, 11.27]$	$(-\infty, -22.32]\cup[22.32, \infty)$
1.99	$[-11.14, 11.14]$	$(-\infty, -26.1]\cup[26.1, \infty)$
1.999	$[-11.39, 11.39]$	$(-\infty, -26.49]\cup[26.49, \infty)$

IV. CONCLUSIONS

In this work we have proposed a new algorithm for computing general alpha-stable densities via asymptotic series. Formerly, these series have been applied for density approximation only for symmetric case. Since most real-life signals in telecommunication are skewed we generalized their use for all of the skew parameter space. Also, despite lots of scientific and technical studies which have been done concerning computation density and its derivatives via these series, their convergence region has not yet determined. Here, we have determined their convergence region by two closed form formula, but in continuous parameterization. The fully success of using these series in continuous parameterization has not yet practically proved.

We have found proposing series in continuous parameterization yields the more extensive convergence region than discontinuous one. The Series can approximate the density very well in their convergence region. Studies show that these series are convergent over most of the real line of observations, x and for almost all of parameter space in continuous parameterization. Therefore, those can be applicable for almost all of parameter values and x's.

Complexity of technique which we suggest is very little, where it does not involve numerical complexities which other approaches do. Also, they are very truthful and accurate in computations. Thus, computing pdf can be done very precise and sharply for a wide area of parameter space, especially for large and very small value of |x|. Since, stable densities have heavy tails, the extreme values are likely for happening and tail of densities are more important for ML estimation and tail fractiles are generally more important than central ones.

ML estimations depend highly on Quasi-Newton which is complicated approach. Fashion methods have not reliable estimation for ML estimations. Many advantages can be found via these series in computation of Fisher information matrix, as the core of ML estimators. Since, derivative and integration have no effect on convergence region length, thus these series can approximate distribution function and pdf derivatives with respect to parameter in their convergence regions as well as pdf. In future, we will pursue this work for finding a combined technique to evaluate stable densities and distribution functions.

APPENDIX A
SERIES 7 CONVERGENCE REGION

We know that,

$$\left|f(x;\alpha,\beta,\mu,\sigma)\right| \le \frac{1}{\pi\left|x-\mu-\zeta\right|}\sum_{k=1}^{\infty}\frac{\Gamma(\alpha k+1)}{\Gamma(k+1)}\left|\frac{x-\mu-\zeta}{\sigma\, r}\right|^{-\alpha k} = g(x;\alpha,\beta,\mu,\sigma)$$

Let $g_k(x;\alpha,\beta,\mu,\sigma)$ is general sentence of above series then, $g_k(x;\alpha,\beta,\mu,\sigma)=\dfrac{\Gamma(\alpha k+1)}{\pi\,\Gamma(k+1)}\dfrac{\left|x-\mu-\zeta\right|^{-k\alpha-1}}{\left|\sigma\, r\right|^{-k\alpha}}$

Using ratio test, then:

$$\left|R\right|=\left|\frac{g_{k+1}(x;\alpha,\beta,\mu,\sigma)}{g_k(x;\alpha,\beta,\mu,\sigma)}\right|=\frac{\Gamma(\alpha k+\alpha+1)}{(k+1)\Gamma(\alpha k+1)}\left|\frac{x-\mu-\zeta}{\sigma\, r}\right|^{-\alpha}$$

And $\left|R\right|<1 \Rightarrow x:\left|x-\mu-\zeta\right|>\underbrace{\sigma\, r\left(\frac{\alpha\,\Gamma(\alpha k+\alpha)}{\Gamma(\alpha k+1)}\right)^{\frac{1}{\alpha}}}_{U} \quad \alpha\neq 1$

Is convergence region of series 7. To prepare superior computation of pdf, it is convenient to construct above formula as following:

$$R_1=(-\infty,-U+\mu+\zeta-C(\alpha)]\cup[U+\mu+\zeta+C(\alpha),\infty)$$

Where $C(\alpha)$ is an arbitrary factor that depends just on α , It is added to lower bound and subtracted from upper bound.

APPENDIX B
SERIES 8 CONVERGENCE REGION

Similar to appendix (A),

$$\left|f(x;\alpha,\beta,\mu,\sigma)\right| \le \frac{1}{\pi\ \left|x-\mu-\zeta\right|}\sum_{k=1}^{\infty}\frac{\Gamma(k/\alpha+1)}{\Gamma(k+1)}\left|\frac{x-\mu-\zeta}{\sigma\ \ r}\right|^{k} = g(x;\alpha,\beta,\mu,\sigma)$$

Let $g_k(x;\alpha,\beta,\mu,\sigma)$ be general term for above series then,

$$g_k(x;\alpha,\beta,\mu,\sigma)=\frac{\Gamma(k/\alpha+1)}{\pi\ \Gamma(k+1)}\frac{\left|x-\mu-\zeta\right|^{k-1}}{\left|\sigma\ r\right|^{k}}$$

Employing ratio test, accordingly:

$$\left|R\right|=\left|\frac{g_{k+1}(x;\alpha,\beta,\mu,\sigma)}{g_k(x;\alpha,\beta,\mu,\sigma)}\right|=\frac{\Gamma(k/\alpha+1/\alpha)}{\alpha\ \Gamma(k/\alpha+1)}\left|\frac{x-\mu-\zeta}{\sigma\, r}\right|$$

And $\left|R\right|<1 \Rightarrow x:\left|x-\mu-\zeta\right|<\underbrace{\sigma\, r\frac{\alpha\Gamma(k/\alpha+1)}{\Gamma(k/\alpha+1/\alpha)}}_{L} \quad \alpha\neq 1$.

Is convergence region of series 8. To provide more accurate pdf approximation, $C^*(\alpha)$ is subtracted from upper and added to lower bound then, it will be corrected as follows:

$$R_2=\left[-L+C^*(\alpha)+\mu+\zeta\ ,\ L-C^*(\alpha)+\mu+\zeta\right]$$

REFERENCES

[1] Matsui, M. and Takemura, A., "Some improvements in numerical evaluation of symmetric stable density and its derivatives". Communications in statistics, theory and methods, Vol.35, 2006, pp.149-172.

[2] McCulloch, J.H.,"Numerical approximation of symmetric stable distribution and density". In: Adler, R. J., Feldman, R., Taqqu, M. S. (Eds.), A practical guide to heavy tails: statistical techniques and applications. Birkhäuser, Boston, Besel, Berlin, 1998, pp.480-500.

[3] Menn C. and Rachev S.,"Calibrated FFT-based density approximations of α-stable distributions". Computational statistics and data analysis, vol.50 (8), 2006, pp.1891-1904.

[4] C. L. Nikias and M. Shao, *"Signal processing with alpha-stable distributionsand applications".* New York: Wiley, 1995.

[5] Nolan, J. P., "Numerical calculations of stable densities and distribution functions", Communication statistics-stochastic models, Vol. 13, 1997, pp. 759-774.

[6] Nolan, J. P.,"Parameterization and modes of stable distributions". Statistics and probability letters, Vol. 38,1998, pp. 38,187-195.

[7] Nolan, J. "Maximum likelihood estimation and diagnostics for stable distributions". Birkhauser, Boston, 2001, pp. 379-400.

[8] Samorodnitsky,G. & Taqqu,M.S., *"Stable non-Gaussian random processes". Stochastic models and infinite variance*, Chapman& Hall, 1994.

[9] G. A. Tsihrintzis and C. L. Nikias, "Performance of optimum and suboptimum receivers in the presence of impulsive noise modeled as an alpha-stable process". *IEEE Trans. Comm,* vol. 44, 1995, pp. 904–914.

[10] Uchaikin, V.V. and Zolotarev, V.M.,*"Chance and Stability: Stable Distributions and their Applications".* VSP, Utrecht, 1999.

[11] V. Zolotarev. *"One-dimensional stable distributions".* American Mathematical Society, Providence, RI, 1986.