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Strange Attractors in Simple Control Systems

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Abstract— **We describe the limit sets in affine control systems, in affine differential systems with impulses. In contrast with the fixed points in ODE case, these sets may have a fractal structure. We give estimations of their Hausdorff dimension. For these systems we extend the notion of Shadowing Property and state this property in the neighborhoods of the attractors and repellers in control systems.**

Keywords: affine differential system, attractor, control system, impulses, shadowing.

1 Introduction

Let $\mathcal{U}$ denote the set of (simple) step functions $u : \mathbf{R} \to \mathbf{R}$, $u = \sum_{j=-\infty}^{+\infty} c_j 1_{[j,j+1)}$ with the sequence $(c_j)_{j\in\mathbf{Z}}$, taking values from a finite set $U = \{u_1, \dots, u_r\}$. For a given *control function* $u \in \mathcal{U}$ consider the control system

$$\dot{x}(t) = A(u(t))x + b(u(t)) \quad (t \in \mathbf{R} \setminus \mathbf{Z}, x \in \mathbf{R}^m). \qquad (1)$$

A continuous and piecewise differentiable function $x : \mathbf{R} \to \mathbf{R}^m$ is called a *solution* of the system (1) if it satisfies the system (1) on every interval $(j, j+1)$, $j \in \mathbf{Z}$. Given $\tau \in \mathbf{R}, x_0 \in \mathbf{R}^m, u \in \mathcal{U}$, denote by $\varphi(\cdot, \tau, x_0, u)$ the solution of the system (1) which satisfies the initial data $\varphi(t, \tau, x_0, u)|_{t=\tau} = x_0$. We will also use the notation $\psi_j(\cdot, \tau, x_0)$ to describe solution of the autonomous system

$$\dot{x}(t) = A(u_j)x + b(u_j), \qquad (2)$$

which satisfies the initial data $\psi_j(t, \tau, x_0)|_{t=\tau} = x_0$. Denote by $\Phi_j(t, \tau)$ the evolution operator of the system (2), i.e. $\Phi_j(t, \tau)x = \psi_j(t, \tau, x)$.

The dynamics of the control system (1) is generated by r affine differential systems with constant coefficients. The motion during the time $(j, j+1)$ is governed by one of these systems. At the moment $t = j+1$ the control function u switches the system to another equation, according to the value of u at this moment. As example, Fig. 1 represents the first three arcs on $[0, 3]$ of the trajectory of a system of the type (1) on $\mathbf{R}^2$, starting at the point $(1, 1)$ (each next arc on $[j, j+1]$ is drawn thicker than the previous one).

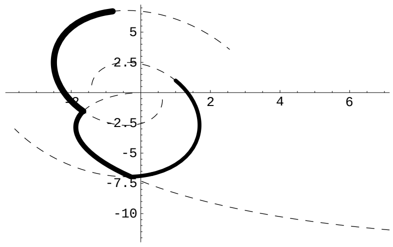

Figure 1: The first three arcs of a trajectory of a system (1).

If the control sequence $(c_n)_{n\in\mathbf{Z}}$ (and the function u) is periodic, the control system (1) admits at least one periodic orbit, which, depending on the spectra of matrices, may be attracting, or repelling. For general control sequences the behavior of the control system may drastically change; a strange attractor, or repeller, may arise.

A general question (see, e.g. [2]) is to describe the behavior of motions (trajectories) of the system (1) as t tends to $+\infty$.

2 Poincaré IFS

If one denotes by $x(n)$ the state of the system (1) at the moment $t = n$, then by a straightforward calculation we end up with a sequence of affine maps $F_{i_n} : x(n) \mapsto x(n+1)$, $n \in \mathbf{Z}$. Under the above assumptions, this sequence has a finite rank, say $\{F_1, F_2, \dots, F_r\}$, $F_j = \Phi_j(1, 0)$.

Let $\{\mathbf{R}^m; F_1, F_2, \dots, F_r\}$ denote the corresponding *Iterated Function System* (IFS) (see, e.g. [1]) generated by the control system (1). Going forward, we assume that all operators $e^{A(u_j)}$ $(1 \le j \le r)$ are either contracting, i.e.

$$||e^{A(u_j)}|| \le s < 1 \quad (1 \le j \le r), \qquad (3)$$

or expanding, i.e.

$$||e^{-A(u_j)}|| \le s < 1 \quad (1 \le j \le r). \qquad (4)$$

In this case the IFS $\{\mathbf{R}^m; F_1, F_2, \dots, F_r\}$ determines in $\mathbf{R}^m$ a global compact attractor or repeller K, which is the unique fixed point of the corresponding Nadler-Hutchinson operator F, $F(V) = \bigcup_{j=1}^{r} F_j[V]$ for any compact $V \subset \mathbf{R}^m$ [6]. This IFS plays a role similar to that

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of the Poincaré map for a periodic differential system. Hence, we call it the *Poincaré IFS* associated to the control system. The dynamics of the initial control system is determined to a large extent by the dynamics of the Poincaré IFS. That's why we will concentrate on the behavior of this IFS (more general cases are studied in [3], [4]).

Fix a natural number k. The sequence $(c_n)_{n\in\mathbf{Z}}$ is said to be *k-universal* if it contains every word of length k from the alphabet U. The sequence $(c_n)_{n\in\mathbf{Z}}$ is called *universal* if it is *k-universal* for every natural k.

If the sequence $(c_n)_{n\in\mathbf{Z}}$ is universal, then the orbit of each point in K is chaotic in the sense of Li-Yorke, in particular it is dense in K [1].

3 Attractors

The system (1) is nonautonomous and we can consider only its integral curves. Even in the case of periodic sequence $(c_n)_{n\in\mathbf{Z}}$ we cannot factorize the system to obtain a (autonomous) system on the direct product $S^1\times\mathbf{R}^m = [0,1)\times\mathbf{R}^m$. However, since the switch actions occur at the moments $t\in\mathbf{Z}$, we can obtain a "foliation" on the cylinder $S^1\times\mathbf{R}^m$ by factorization on time. This "foliation" consists of pieces of integral curves of the system (1).

Let us project the system (1) to the cylinder $S^1\times\mathbf{R}^m$, using the projection $\pi:(t,y)\mapsto(t \pmod 1, y)$.

We will say that a set $V\subset S^1\times\mathbf{R}^m$ is *invariant* (*positive invariant*) with respect to the system (1) if for every point $(\tau,x)\in V$ there is a natural k such that $(t \pmod 1, \varphi(t,\tau+k,x,u))\in V$ for $t\in\mathbf{R}$ $(t\geq\tau+k)$. In other words, V consists of pieces of integral curves of the systems (2).

By definition, such a set $V\subset S^1\times\mathbf{R}^m$ covers the whole base S^1 by projection. Denote by $(t,V_t)=\{(t,x)\in S^1\times\mathbf{R}^m \,|\, (t,x)\in V\}$ the *fiber* over the point $t\in[0,1)$. For convenience, we will identify this fiber with V_t. Moreover, in the rest of the paper the notation V_t for $t\in\mathbf{R}$ will mean $V_{t \pmod 1}$.

Theorem 1. *The set $V\subset S^1\times\mathbf{R}^m$ is positive invariant (invariant) with respect to the system (1) if and only if it satisfies the following conditions:*

1. $\Phi_j(t,0)V_0\subset V_t$ $\;(\bigcup_{j=1}^{r}\Phi_j(t,0)V_0=V_t)$ *for* $0\leq t<1$;
2. V_0 *is positive invariant (invariant) with respect to the Nadler-Hutchinson operator F, i.e.* $F(V_0)=\bigcup_{j=1}^{r}F_j[V_0]\subset V_0 \quad (F(V_0)=V_0)$.

Let the distance from point $x\in\mathbf{R}^m$ to compact $M\subset\mathbf{R}^m$ be $\varrho(x,M):=\min\{d(x,y)\,|\,y\in M\}$.

A bounded and closed subset $V\subset S^1\times\mathbf{R}^m$ is called an *attractor* of the system (1) if it is positive invariant and for every solution $\varphi(\cdot,\tau,x,u)$, $\varrho(\varphi(t,\tau,x,u),V_t)\to 0$ as $t\to+\infty$.

Recall that the attractor of a hyperbolic IFS is defined as the unique fixed point of the corresponding Nadler-Hatchinson operator. This attractor may be characterized as follows: it is a nonempty positive invariant under the action of each function of the IFS and is minimal with respect to these properties. Similarly, the nonempty compact subset $V\subset S^1\times\mathbf{R}^m$ will be an attractor of the system (1) if it is positive invariant with respect to the system and is minimal with respect to this property.

Theorem 2. *Assume that (3) holds. Then*

1. *for any control function $u\in\mathcal{U}$ there is an attractor $K^*\subset S^1\times\mathbf{R}^m$ of the system (1),*
$$K^*=\bigcup_{j=1}^{r}\{(t,\psi_j(t,0,x))\,|\,t\in[0,1),x\in K\},$$
where K is the attractor of the associated Poincaré IFS, with the Hausdorff dimension $DH(K^*)$, verifying the inequalities:
$$1<DH(K^*)\leq 1-\frac{\ln r}{\ln s};$$
2. *this attractor K^* does not depend on the control function u;*
3. *a point $\xi=(\tau,x_0)$ belongs to K^* if and only if there exists a control function $u\in\mathcal{U}$ such that*
$$\lim_{n\to-\infty}\varphi(\tau,n,x,u)=x_0$$
for any $x\in\mathbf{R}^m$.

When the condition (4) is satisfied one speaks about the repeller of the system (1).

Recall some notions. A set is called *totally disconnected* if for every point from this set the only connected component, containing this point, is the point itself. A set S is called *perfect* if it is closed and every point $p\in S$ is the limit of points $q_n\in S\setminus\{p\}$. A set is called a *Cantor set* if it is totally disconnected, perfect and compact.

Theorem 3. *Assume that the hypothesis of Theorem 2 are valid with $s<\frac{1}{r}$. Let the sequence $(c_n)_{n\in\mathbf{Z}}$ be universal. Then the attractor K^* is homeomorphic to the union of r direct products of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1<DH(K^*)<2$.*

Figure 2: Attractor K^* as an union of "tori".

Fig. 2 represents an attractor K^* of a system (1) on the cylinder $S^1 \times \mathbf{R}^m$ as union of "tori", having a triangle shape K in the section $t = 0$. Note that each such "torus" has a shape similar to that on Fig. 3.

Theorem 4. *If the sequence $(c_n)_{n\in\mathbf{Z}}$ is universal, then every trajectory of the system (1), starting in K^*, is chaotic as per Li-Yorke.*

Corollary. *If the sequence $(c_n)_{n\in\mathbf{Z}}$ is universal, then every trajectory of the system (1) is asymptotically chaotic according to Li-Yorke definition.*

4 Impulsive systems

A similar idea is applied in [5] to study the system of impulsive affine differential equations (see, e.g. [7])

$$\dot{x} = Ax + b \quad (x \in \mathbf{R}^m), \tag{5}$$

$$\Delta x\big|_{t=n} := x(n+0) - x(n-0) = C_{i_n} x(n) + d_{i_n} \qquad (n \in \mathbf{N}), \tag{6}$$

where A is a nonsingular matrix. The matrices C_{i_n} and the vectors d_{i_n} belong to some (finite or infinite) given sets.

Between any two consecutive kicks the motion of system obeys (5). At the moment $t = n$ the elements C_{i_n} and d_{i_n}, which determine the jump by (6), are chosen, say randomly. For convenience, we will consider that each solution of system (5)-(6) is right continuous at any moment $t \in \mathbf{N}$.

Assume that

$$||(E + C_n)e^A|| \le s < 1 \quad (1 \le n \le r). \tag{7}$$

Sometimes, if necessary, it is required that all operators $(E + C_n)$ $(1 \le n \le r)$ are invertible.

The results will be similar to the ones obtained above.

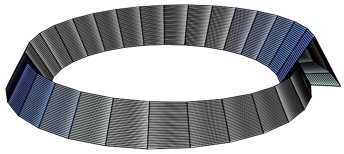

Figure 3: Attractor K^*, constructed on a compact K with a triangular shape.

Project again the system (5)-(6) on the cylinder $S^1 \times \mathbf{R}^m$.

Similarly, we associate to system (5)-(6) an IFS $\{\mathbf{R}^m; F_{i_1}, F_{i_2}, \ldots, \}$, $F_{i_n} : x(n) \mapsto x(n+1)$, consisting of affine contractions. This IFS determines in $\mathbf{R}^m$ a global compact attractor K, which is the unique fixed point of the corresponding Nadler-Hutchinson operator F. Assume that the sequence $(F_{i_n})_{n\in\mathbf{N}}$ consists of only r distinct maps.

Using [8], one can obtain the following result.

Theorem 5. *Assume that (7) holds and $s < \frac{1}{r}$. Then the attractor K is totally disconnected.*

Theorem 6. *Assume that (7) holds. Then there exists an attractor $K^* \subset S^1 \times \mathbf{R}^m$ of the system (5)-(6),*

$$K^* = \{(t, e^{tA}x + (e^A - E)A^{-1}b) \,|\, t \in [0,1), x \in K\},$$

with the Hausdorff dimension $DH(K^*)$, verifying the inequalities:

$$1 < DH(K^*) \le 1 - \frac{\ln r}{\ln s}. \tag{8}$$

Theorem 7. *In addition to the hypothesis in Theorem 5 assume that the sequence $(F_{i_n})_{n\in\mathbf{N}}$ is universal. Then the attractor K^* is homeomorphic to the direct product of a half-open interval and a Cantor set, its Hausdorff dimension verifying the inequalities: $1 < DH(K^*) < 2$.*

Theorem 8. *Assume that (7) holds and the sequence $(F_{i_n})_{n\in\mathbf{N}}$ is universal. Then every trajectory of the system (5)-(6) is asymptotically chaotic as per Li-Yorke.*

Fig. 3 represents an attractor K^* on the cylinder $S^1 \times \mathbf{R}^m$ with the section K for $t = 0$ of a triangular shape.

5 Linear oscillator

Let us consider, as an example, the linear oscillator with impulsive actions

$$\ddot{x} + c\dot{x} + kx = 0 \quad (c > 0, k > 0), \tag{9}$$

$$\Delta\dot{x}\big|_{t=n} := \dot{x}(n+0) - \dot{x}(n-0) = \xi_{i_n} \quad (n \ge 1). \tag{10}$$

Assume that the range of the sequence $(\xi_{i_n})_{n\ge1}$ contains only r distinct elements.

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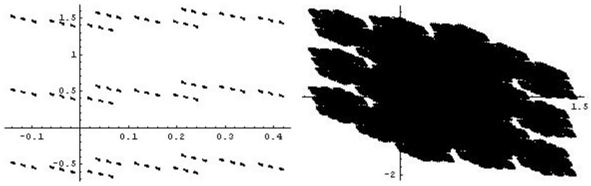

Figure 4: A totally disconnected (left) and a connected (right) attractor K.

Figure 5: Attractor K of an infinite IFS.

We can reduce the equations (9)-(10) to an affine system of impulsive differential equations (5)-(6) in the phase space $x_1 = x$, $x_2 = \dot{x}$. In this case we obtain similar results as in the previous theorems.

Theorem 9. *There exists an attractor* $K^* \subset S^1 \times \mathbf{R}^2_{(x,\dot{x})}$ *of the system (9)-(10). If*

$$2 \ln r < c < \frac{k}{\ln r} + \ln r$$

and the sequence $(\xi_{i_n})_{n \in \mathbf{N}}$ *is universal, then the attractor* K^* *is homeomorphic to the direct product of a half-open interval and a Cantor set and its Hausdorff dimension*

$$DH(K^*) = 1 + \frac{2\sqrt{2} \ln r}{\sqrt{2}c - \sqrt{c^2 - 4k + |c^2 - 4k|}}$$

verifies the inequalities: $1 < DH(K^*) < 2$.

Fig. 4 represents the respective attractors $K \subset \mathbf{R}^2_{(x,\dot{x})}$ for distinct parameter values for two impulsive differential equations (9)-(10): on the left for $c = 5/2$, $k = 2$ and $r = 3$ (K is totally disconnected), on the right for $c = 1$, $k = 5/4$ and $r = 3$ (K is connected).

Remark. If the value set of sequence $(\xi_{i_n})_{n \geq 1}$ is infinite but bounded, then system (9)-(10) admits an attractor as well.

Fig. 5 represents the attractor K (of an infinite IFS) for an impulsive differential equation (9)-(10) with $c = 2$, $k = 2$, where values $(\xi_{i_n})_{n \geq 1}$ are randomly chosen from $[0, 1]$.

6 Shadowing Property

We will say that a function $\psi : [0, +\infty) \to \mathbf{R}^m$ is a *δ-pseudo-solution* of the equation (1) if:

- it is piecewise differentiable and

$$\|\dot{\psi}(t) - A(u(t))\psi(t) - b(u(t))\| < \delta \ \ (t \notin \mathbf{N});$$

- $\|\Delta\psi\big|_{t=n}\| < \delta$ $(n \in \mathbf{N})$.

We will say that a *δ-pseudo-solution* ψ is *ε-shadowed* by the solution φ if there exists a differentiable parametrization $\gamma : [0, +\infty) \to [0, +\infty)$, $|\dot{\gamma} - 1| < \varepsilon$, $\gamma(n) = n$ for $n \in \mathbf{N}$, and $\|\varphi(t) - \psi(\gamma(t))\| < \varepsilon$ for $t \geq 0$.

Theorem 10 (Shadowing Property). *Let* K^* *stand for the attractor (repeller) of the equation (1). Then there exists a neighborhood* $V(K^*)$ *in* $[0, 1) \times \mathbf{R}^m$ *such that for any* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that any δ-pseudo-solution in* $V(K^*)$ *is ε-shadowed by an actual solution.*

7 Conclusions

For simplest control systems defined by affine differential equations and with simple control functions, including impulsive control, we give conditions for the existence of strange attractors or repellers. We prove the Shadowing Property near these limit sets and thus we obtain a theoretical justification for computer simulation of control and impulsive systems.

All calculations and graphic objects have been done using the Computer Algebra System *Mathematica*.

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