# Exact Weight Perfect Matching of Bipartite Graph is NP-Complete *

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*Abstract*—**This paper proves that the complexity of exact weight perfect matching problem is NP-complete by reduction from the good perfect matching problem. Following this result, the other two open problems DNA sequence analysis and discrete min-max assignment problems are proven to be NP-complete.**

***Keywords: computational complexity, color coding, exact perfect matching, good perfect matching, knapsack problem***

## 1 Introduction

Throughout this paper we only consider the finite simple undirected graph $G = (V, E)$, i.e. the graph has no multi-edges and self loops. If each edge of $G$ is assigned a color, then $G$ is an edge-colored graph $G(C)$, where $C$ is a color partition $C = \{c_1, c_2, \ldots, c_k\}$. Let $c_i(e_j)$ denotes the edge $e_j$ assigned in color $c_i$ and $c_i(E)$ means the subset of edges all in color $c_i$. A *matching* in $G$ is a subset of pairwise non-adjacent edges; that is, no two edges share a common vertex. A *perfect matching* $M$ is that every vertex meets exactly one memeber of $M$. A *good matching* is that all the edges in $M$ are of different colors. Let $w(e_i)$ denotes the weight edge $e_i$, $e_i \in E$. And $W(S) = w(e_1) + w(e_2) + \ldots + w(e_s)$ denotes the sum of weights of all the edges $S \subseteq E$.

It is well known that the perfect matching or assignment problem in *weighted/unweighted* bipartite graphs could be solved efficiently [7]. But for the following question, it becomes more difficult:

**Name.** Exact Weight Perfect Matching ($EWPM$ for short)

**Input.** An edge weight bipartite graph and a positive integer $\alpha$.

**Question.** Does there exist a perfect matching $M$ with $W(M) = \alpha$ ?

The $EWPM$ becomes a more and more famous open problem in recent years. For instance, Jacek proved that it is equivalent to the *DNA sequencing analysis problem* in polynomial time [2]. Vladimir proved that it is equivalent to the *discrete mini-max assignment problem* with a fixed number of scenarios in polynomial time [4].

**Input.** A complete bipartite graph with bipartition X U Y; a set of $k$ cost scenarios that are specified by non-negative integer edges costs $b_1, \ldots, b_k : X \times Y \to N$; a bound $B$.

**Question.** Find a perfect matching $M \subseteq X \times Y$ with $b_i(M) \leq B$ for $i = 1, \ldots, k$.

In this paper, we study this problem by two known NP-complete problems. The first one is the *good perfect matching problem.*

**Name.** Good Perfect Matching ($GPM$ for short)

**Input.** A edge colored bipartite graph.

**Question.** Does there exist a perfect matching $M$ with each edge in $M$ has different color?

Good prefect matching problem is a special case of *maximum labeled perfect matching problem* in [9]. And Cameron [3] proved that

**Theorem 1** *[3] The good prefect matching problem in bipartite graph is NP-complete.*

But Cameron showed that the restricted $GPM$ problem could be solved in $O(n^2)$ when the graph is complete bipartite graph and there are no more than two edges in the same color existing in this complete bipartite graph [3].

Another related NP-complete problem is *0-1 knapsack problem*, which can be expressed as follows:

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**Name.** $0-1$ Knapsack Problem

**Input.** A vector of positive integers $A = \{a_1, a_2, \ldots, a_n\}$ and an integer $T$.

**Question.** Does there exist a binary vector $X$ satisfying $\sum_{i=1}^{n} a_i x_i = T$?

In this paper, we only consider a simple $0-1$ Knapsack problem: the sequence of $A$ is superincreasing if $\sum_{j=1}^{i-1} a_j \leq a_i - 1$ $(i = 2, \ldots, n)$. This case is known to be solvable in polynomial time [10].

All graph theoretical terms that do not defined in this paper can be found in [5]. We refer to [6] for definitions linked to complexity and knapsack problem.

## 2 Color coding

Itai studied the restricted matching problem with a single restriction $|M \cap c_1(E)| \leq r$[8]. His approach is to encode the edges with binary coding, thus the restricted matching problem can be solved in polynomial time by solving the minimum cost maximum flow problem.

But the binary coding is so greedy that it lose a lot of useful information in this solution. For instance, if there is a $K_{2,2}$ graph with 1 *red* edge, 1 *green* edge and 2 *blue* colors and we encode *red* edge as 1, and other edges value as 0. Since the perfect matching $M$ of $K_{2,2}$ only include two edges, if $W(M) = 1$, it shows that there must be a *red* edge in $M$. However, it could not tell us the color of the another edge in $M$.

Therefore, it is necessary to distinguish all colors from the perfect matching efficiently. Let us consider the natural primary colors: *Red, Blue* and *Green.* For any given color, it is always composed of these primary colors; Conversely, if any value of primary colors is given, an unique color can be obtained. Similarly, if each edge in a perfect matching $M$ is colored by the primary colors, then color of $M$ can be viewed as a mixed color; on the other hand, if the mixed color of a $M$ is given, we can get the color of every edge in $M$.

Now suppose that there exists $k$ different colors in graph $G$. Let $w(e_i)$ equals to the weight of edge $e_i$ and $c(e_j)$ the color of edge $e_j$, we encode each weight of edge as a superincreasing number by the following approach

$$we_i = \begin{cases} 1, & \text{if } c(e_i) = c_1; \\ |c_1(E)| + 1, & \text{if } c(e_i) = c_2; \\ \ldots, & \ldots \\ \sum_{j=1}^{k-1} |C_j(E)| + 1, & \text{if } c(e_i) = c_k; \end{cases} \qquad (1)$$

It is obvious that the color coding is an one-to-one map. We can thus obtain the color of an edge if the weight of the edge is given.

Finally, let us taken an example to show how to distinguish the color from a weight perfect matching. Given a red-blue edge color $K_{3,3}$ graph, let $e_r$ be an edge in color *red* and $e_b$ an edge in color *blue.* Then according to Equation (1), we have $w(e_r) = 1$ and $w(e_b) = 3$, suppose the $W(M) = 3$, then the matching $M$ includes 3 red edges.

## 3 Exact weight perfect matching problem is NP-complete

In this section the $EWPM$ problem is proven to be NP-complete by reducing the good perfect matching problem. In other words, $EWPM$ problem is harder than $GPM$ problem. The main theorem is as follows:

**Theorem 2** *The exact weight perfect matching problem of bipartite graph is NP-complete.*

*Proof*: Given a perfect matching $M$ of bipartite graph and an integer $k$, it is easy to check that the sum of weight edges in $M$ equals to the value of $k$, thus $EWPM$ is NP.

Now let us reduce the good perfect matching problem to $EWPM$ of a bipartite graph $G$.

Suppose that an edge colored bipartite graph $G$ has $2n$ vertices, and the colors set $C$ is $\{c_1, c_2, \ldots, c_k\}$, where $(k \geq n)$. A good perfect matching $M$ of $G$ consists of a set of edges $\{e_1, e_2, \ldots, e_n\}$.

Let's encode each edge with color coding according to Equation (1). Thus the set of edges in $M$ are labeled with a set of weights $\{w(e_1), w(e_2), \ldots, w(e_n)\}$. The sum of weights of $M$ is

$$W(M) = w(e_1) + w(e_2) + \ldots + w(e_n). \qquad (2)$$

where $w(e_i)$ satisfies the superincreasing condition.

Thus Equation (2) could be rewritten as the following equation:

$$W(M) = x_1 w(c_1) + x_2 w(c_2) + \ldots + x_k w(c_k), \qquad (3)$$

where $x_i \in \{0, 1\}$ and $w(c_i)$ satisfies the superincreasing condition.

Obviously, Equation (3) shows that it is a *0-1 knapsack problem* . Thus if there exists a good perfect matching $M$ in $G$, then there exists an exact weight perfect matching satisfying Equation (3) and if an edge in $M$ is labeled with color $c_i$, then we can get $x_i = 1$, otherwise $x_i = 0$.

Conversely, if there exists an exact weight perfect matching $W(M)$ in $G$, then it could solve Equation (3) in polynomial time according to the complexity of superincreasing knapsack problem [10]. □

According to Theorem 8 in [2], we can give the following corollaries:

**Corollary 1** *DNA-SEQ problem is NP-completed.*

Since Valdimir had proven that the discrete min-max assignment problem with a fixed number of scenarios is reducible to the exact weight perfect matching problem in polynomial time [4], a corollary is obtained according to Theorem 2.

**Corollary 2** *Discrete min-max assignment problem with a fixed number of k scenarios (* $k \geq 2$ *) is NP-completed.*

Valdimir also shown that the *Discrete min-max regret assignment problem* is equivalent to the *exact perfect matching problem* in polynomial time [4].

**Corollary 3** *Discrete min-max regret assignment problem is NP-completed.*

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