Algorithms for Differential Equations with Oscillatory Solutions

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Abstract— **In this paper, an Exponentially weighted Legendre-Gauss Tau Method (ELGT) for solving ordinary differential equations (ODEs) with oscillatory solutions is developed. An algorithm for ELGT is build up by combining three apparantly different numerical techniques: The classical Legendre-Gauss spectral Tau Method, exponential fitting and piecewise coefficients perturbation methods. Numerical examples illustrating the efficiency and the high accuracy of my established results are presented.**

Keywords: oscillatory functions, Legendre polynomials, interpolation

1 Introduction

The solution of second order ODEs has played a fundamental role in the evolution of mathematical physics, starting with the eigenvibrations of a string, and culminating in the atomic vibrations of Schrödinger wave equations. While the solution of this class of ODEs cannot be given in a closed form except in special cases, it is possible to obtain accurate approximate solutions by means of numerical procedures with high degree of accuracy. But a challenging problem continues to face numerical analysists and computational physicists is the approximation of ODEs with highly oscillatory solutions. In the past there has been much interest in standard numerical methods such as Numerov, Runge-Kutta or de Vogelaere (see [6]). But due to the unsatisfactory performance of those standard methods in detecting the strong oscillations exhibited by the solutions, efforts have concentrated on modern techniques that have proven to be highly accurate and more effective in approximating this class of ODEs. Among those techniques are procedures based on piecewise coefficients perturbation methods and on exponential fitting (see [4] and [5]).

The present work is a contribution to this line of research. The specific aim of this paper is to develop an algorithm that combines three apparantly different techniques: Legendre-Gauss Tau Method (LGT), exponential fitting and coefficients perturbation methods.

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Section 2 is intended to give a brief description of LGT. We try to get insight into the behavior of this method by solving a simple ODE with a highly oscillatory solution. The unsatisfactory numerical results suggest that a combination of the coefficients perturbation method with LGT could result in a modified version of LGT that can be more effective in detecting the sharp variations in the oscillatory solutions. The main features of LGT are recalled in section 3. Section 4 is devoted to develop a modified LGT, called Exponentially weighted Legendre-Gauss Tau Method (ELGT), and to formulate its algorithms. Section 5 is concerned with analysing the error of ELGT and to propose a reference correction procedure that allows to increase the degree of accuracy. Numerical examples supporting our results will be given in Section 6. In the last section, ELGT is extended to solve nonlinear problems and to present some illustrative examples.

2 Legendre-Gauss Tau Method

LGT was invented by Lanczos [7] and later developed by Ortiz [9] and by Gottlieb and Orszag [3] to treat problems with different degrees of complexities.

2.1 The Main Features of LGT

Let us consider the initial value problem (IVP),

$$(Dy)(x) := \sum_{i=0}^{\nu} P_i(x)\frac{d^i y}{dx^i} = f(x), \quad x \in [a,b], \quad (1)$$

$$y^{(k)}(a) = \alpha_k \in \mathbf{R}, \quad k = 0,1,\cdots,\nu-1, \quad (2)$$

where $\{P_i(x), i = 0,1,\cdots,\nu\}$ are continuous functions with $P_\nu(x)$ not vanishing in $I := [a,b]$.

LGT seeks an approximation y_N for y of the form

$$y_N = \sum_{i=0}^{N+\nu-1} a_i L_i(x),$$

where $\{a_i; i = 0,1,\ldots,n+\nu-1\}$ are determined by

- imposing the supplementary conditions (2) on y_N,

$$y_N^{(k)}(a) = \alpha_k; \quad k = 0,1,\cdots,\nu-1,$$

- and, either, by an orthogonal projection of the residual $R_N(x) := Dy_N(x) - f(x)$ against subspace

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$$\text{span}\{L_0(x), L_1(x), \ldots, L_{N-1}(x)\},$$

$$\int_a^b R_N(t)(t)L_k(t)dt = 0, \quad l = 0, 1, 2, \ldots, N-1,$$

- or, by forcing $R_N(x)$ to vanish at the N LG points $\{z_i; i = 1, 2, \ldots, N\} \subset I$,

$$R_N(z_i) = f(z_i), \quad i = 1, 2, 3, ..., N.$$

In the piecewise version of LGT we consider a partition $a = x_0 < x_1 < \ldots < x_M = b$ of $[a, b]$; $h_i = x_i - x_{i-1}$, and we use LGT(N) to solve the following M IVPs,

$$\begin{aligned} &(Dy_i)(x) = f(x), \quad x \in [x_{i-1}, x_i], \; i = 1, 2, \ldots, M, \\ &y_i^{(k)}(x_{i-1}) = y_{i-1}^{(k)}(x_{i-1}), \; y_1^{(k)}(x_0) = \alpha_k, \quad k = 0, 1. \end{aligned} \tag{3}$$

Throughout, LGT(M,N) will stand for piecewise LGT. When $M = 0$, LGT(0,N)=LGT(N).

2.2 Numerical Experiment

To see the performance of LGT(M,N), let us apply it to the following IVP,

$$\begin{aligned} &y^{''} + 4x^2 y = 2\cos x^2, \quad x \in [0, 40], \\ &y(0) = 0, \quad y^{'}(0) = 0, \end{aligned} \tag{4}$$

whose the exact solution, $y = \sin x^2$, is highly oscillatory for large x (see Figure 1). The exact errors at some $\{x_i, \; i = 0, 1, \ldots, 800\}$ committed by LGT(M,N) with $M = 800$, $h = 0.05$ and $N = 2$ are listed in Table 1.

i	x_i	err(x_i)	err$'$(x_i)	$\sqrt{\text{err}^2 + \text{err}'^2}$
0	0.05	6.51E -10	-8.68E -9	8.70E -9
100	5.	1.18E -4	5.42E -4	5.55E -4
200	10	6.07E -3	7.99E -2	8.02E -2
300	15	2.67E -2	2.01	2.01
400	20	-2.14E -1	9.48	9.48
500	25	-7.70E -1	-3.00E+1	3.00E+1
600	30	-8.67E -1	1.17E+1	1.17E+1
700	35	-3.84E+5	9.81E+8	9.81E+8
800	40.	-2.91E+5	-7.19E+8	7.19E+8

Table 1: LGT(800,2) error for $y = \sin x^2$ in [0,40].

It is clearly seen, that the accuracy of LGT(M,N) deteriorates as we approach the end point.

In this paper we develop a modified LGT by introducing in the desired approximate solution exponential weights of the form $e^{\omega x}$ in a way that, for suitably chosen frequencies ω, those weights will detect the strong oscillations throughout the domain of integration. The main tool to achieve this goal will be the piecewise perturbation method that will be presented in the next section.

3 Piecewise Coefficients Perturbation

This technique has been essentially devised to approximate second order ODE of the form

$$\begin{aligned} (Dy)(x) := \quad &y^{''} + b(x)y = 0, \quad x \in [a, b], \\ &y(a) = \alpha_0, \; y^{'}(a) = \alpha_1. \end{aligned} \tag{5}$$

The basic idea of the piecewise coefficient pertrubation method (PPM) is as follows: Consider a partition $a = x_0 < x_1 < \ldots < x_M = b$ of [a,b], and on each $[x_{i-1}, x_i]$, $i = 1, 2, \ldots M$, replace $b(x)$ by an approximation $\tilde{b}_i(x)$ in a way that the following M IVPs:

$$\begin{aligned} &y_{0i}^{''} + \tilde{b}_i(x)y_{0i} = 0, \; x \in [x_{i-1}, x_i], \\ &y_{0i}^{(\ell)}(x_{i-1}) = y_{0,i-1}^{(\ell)}(x_{i-1}), \quad y_{00}^{(\ell)}(x_0) = \alpha_\ell, \\ &i = 1, 2, \ldots, M, \; \ell = 0, 1, \end{aligned} \tag{6}$$

can be solved *analytically.* For each $i = 1, 2, \ldots, M$, the accuracy of y_{0i} (called *reference*) can be increased by adding *corrections* $\{y_{ki}(x); k = 1, 2, \ldots\}$ that are defined by the following sequence of IVPs

$$\begin{aligned} &y_{ki}^{''} + \tilde{b}_i(x)y_{ki} = \delta b_i(x)y_{k-1,i}, \\ &y_k^{(\ell)}(x_{i-1}) = 0, \; k \geq 1, \quad \ell = 0, 1, \end{aligned} \tag{7}$$

where $\delta b_i(x) := \tilde{b}_i(x) - b(x)$. We call a PPM mth approximation of y on subinterval $[x_{i-1}, x_i]$, the finite sum

$$Y_{mi} := y_{0i} + y_{1i} + \ldots + y_{mi}. \tag{8}$$

When $\tilde{b}(x)$ is constant, the method is called *CP-Method.* When $\tilde{b}(x)$ is linear, it is called *LP-method.*

3.1 Strucrure of CP-Method Residual

In this section indices i will be spressed and X will designate x_i. Adding up the reference equation (6) and the first m correction equations (7), we find that

$$\begin{aligned} &Y_m^{''} + b(x)Y_m = -\delta b \, y_m, \quad x \in [X, X+h], \\ &Y_m(X) = \eta_0, \quad Y_m^{'}(X) = \eta_1, \end{aligned} \tag{9}$$

where $\{\eta_0, \eta_1\}$ are generic values available from the approximation computed on subinterval $[x_{i-2}, x_{i-1}]$.

Comparing (9) with the given IVP (5), we observe that Y_m is the exact solution of a perturbed version of the original one where the perturbation occurs in the right hand side as a residual of the form

$$R(x) = -\delta b \, y_m. \tag{10}$$

In particular, if $\tilde{b}(x) \equiv \bar{b}$ is constant with $\omega = \sqrt{-\bar{b}}$, then

- the CP-reference y_0 is given as

  $$y_0(x) = p_{10}\mathrm{e}^{\omega x} + p_{20}\mathrm{e}^{-\omega x},$$

  where $\{p_{10}, \; p_{20}\}$ are constants fixed in terms of the initial conditions associated with (6),

- the kth CP-correction has the form

  $$y_k(x) = p_{1k}(x)\mathrm{e}^{\omega x} + p_{2k}(x)\mathrm{e}^{-\omega x},$$

  for some polynomials $\{p_{1k}, \; p_{2k}\}$ that involve two constants fixed in terms of the initial conditions $y_k(X) = y_k^{'}(X) = 0$,

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- the CP-approximant $Y_m := y_0 + y_1 + \ldots + y_m$ can be written as,

$$Y_m = P_{m1}(x)\mathrm{e}^{\omega x} + P_{m2}(x)\mathrm{e}^{-\omega x},$$

- the CP-residul (10) takes the form

$$R(x) = -(\delta b\, p_{1,m})\mathrm{e}^{\omega x} - (\delta b\, p_{2m})\mathrm{e}^{-\omega x}. \qquad (11)$$

This structure of CP-residual will be very constructive in assuring the close dependence between the error function and the quality of perturbation measured by $\delta b(x)$. Subsequently this will allow to propose a technique that reduces the error substantially.

3.2 Analyzing the CP-Error

Let $e_m(x) := y(x) - Y_m(x)$ denote the mth error function. The difference between (5) and (9), taking into account (11), yields the error equation,

$$e_m^{''} + b(x)e_m = (\delta b\, p_{1m})\mathrm{e}^{\omega x} + (\delta b\, p_{2m})\mathrm{e}^{-\omega x},$$
$$e_m(X) = \epsilon_m, \;\; e_m^{'}(X) = \epsilon_m^{'}$$

where $x \in [X, X+h]$. $e_m(x)$ is formally represented as

$$e_m(x) = \int_X^x G^*(x,t)\delta b(t)dt + G(x,X)\epsilon_m^{'} + G_x(x,X)\epsilon_m. \qquad (12)$$

$G(x,t)$ being the Green function associated with D and

$$G^*(x,t) := G(x,t)\left[p_{1,m}(t)\mathrm{e}^{\omega t} + p_{2,m}(t)\mathrm{e}^{-\omega t}\right].$$

For the local truncation error (l.t.e.), let $\epsilon_m = \epsilon_m^{'} = 0$ and take norms in (12), to get, for some constant $\kappa = \kappa(\omega)$.

$$\|e_m\| \le \kappa\|\delta b\|, \quad \text{where } \|G^*\| \le \kappa.$$

As far as CP-method is concerned, in the uniform norm $\|.\|_\infty$ the smallest $\|\delta b\|_\infty$ is realized when $\bar{b}$ is the best zeroth approximation of $b(x)$,

$$\bar{b} = b(X + \frac{h}{2}), \qquad \omega = \sqrt{-b(X + \frac{h}{2})}.$$

Alternatively, for the L^2-norm, $\|.\|_2$, the smallest $\|\delta b\|_2$ is achieved if $\bar{b}$ is the best zeroth approximation of $b(x)$ in $L^2[X, X+h]$,

$$\bar{b} = \int_0^1 b(X + ht)dt.$$

Hence, whether $\|.\|_\infty$ or $\|.\|_2$ is adopted, we have

$$\delta b(x) = L_{1,h}(x) \times \text{function of } x.$$

We conclude that the residual (11) can be written as

$$R(x) = L_{1,h}(x)\tau_1(x)\mathrm{e}^{\omega x} + L_{1,h}(x)\tau_2(x)\mathrm{e}^{-\omega x}. \qquad (13)$$

This result suggests that there could be a PPM version other than CPM that would lead to a residual whose the same structure as (13), except that the coefficients of the exponentials $\mathrm{e}^{\pm\omega x}$ must be multiples of higher order Legendre polynomial, $L_{N,h}(x)$ say. In other words, we wish to find a method whose the residual is of the form

$$R_N(x) = L_{N,h}(x)\tau_1(x)\mathrm{e}^{\omega x} + L_{N,h}(x)\tau_2(x)\mathrm{e}^{-\omega x}. \qquad (14)$$

Next section demonstrates that LGT can be extended to achieve this goal.

4 Exponentially Weighted LGT

In this section, two cases will be investigated:

4.1 Case 1: $y^{''} + a(x)y^{'} + b(x)y = 0$

For each $\omega \in \mathbf{C}$, associate to $Du := u^{''} + a(x)u^{'} + b(x)u$ the auxiliary operator D_ω defined as

$$D_\omega u := u^{''} + (2\omega + a(x))u^{'} + (\omega^2 + a(x)\omega + b(x))u.$$

We can now, by means of operators D_ω, give a new characterisation for the exact solution of 2nd order ODE:

Theorem 1. *The exact solution of*

$$Dy := y^{''} + a(x)y^{'} + b(x)y(x) = 0 \qquad (15)$$

is expressible as a linear combination of $\{\mathrm{e}^{\omega_1 x}, \mathrm{e}^{\omega_2 x}\}$,

$$y = \phi_1(x)\mathrm{e}^{\omega_1 x} + \phi_2(x)\mathrm{e}^{\omega_2 x},$$

where frequencies $\{\omega_1, \omega_2\}$ *are the (real of complex) roots of the quadratic equation*

$$\omega^2 + a(\bar{X})\omega + b(\bar{X}) = 0, \quad \bar{X} = X + h/2,$$

and where $\{\phi_1(x), \phi_2(x)\}$ *are exact solutions of*

$$D_{\omega_j}\phi = \phi_j^{''} + (2\omega_j + a(x))\phi_j^{'} + (\omega_j\delta a + \delta b)\phi = 0. \qquad (16)$$

The proof of Theorem 1 is based on this technical lemma:

Lemma 1. *For any constants* $\{\omega_i,\ c_i;\ i = 1, 2\}$,

$$\{D_{\omega_i}\phi_i = 0, i = 1, 2\} \;\Rightarrow\; D[c_1\phi_1 e^{\omega_1 x} + c_2\phi_2 e^{\omega_2 x}] = 0.$$

Theoretically, $y(x)$ can be found, once $\{\phi_1, \phi_2\}$ are available, and the constants c_1 and c_2 in $y = c_1\phi_1(x)e^{\omega_1 x} + c_2\phi_2(x)e^{\omega_2 x}$ are fixed according to the given initial conditions. Analytically, solving (16) is not easier, however, than solving the original problem (15). But, computationally, numerical methods that approximate the smooth solutions of (16) could be more successful than approximating (15) directly, specially when $y(x)$ exhibits sharp variations. Next I will propose an algorithm for LGT that can effectively generate approximations $\{\tilde{\phi}_1, \tilde{\phi}_2\}$ for $\{\phi_1, \phi_2\}$ defined by (16) and subsequently construct an approximation $\tilde{y} = c_1\tilde{\phi}_1 e^{\omega x} + c_2\tilde{\phi}_2 e^{-\omega x}$ for y.

I will refer to this procedure by ELGT(M,N) where M indicates the number of steps and N is the number of Legendre-Gauss points in each subinterval $[x_{i-1}, x_i]$.

ALGORITHM 1 – Follows is an ELGT(M,N) algorithm that approximates IVPs of the from

$$y'' + a(x)y' + b(x)y(x) = 0, \quad x \in [a, b],$$
$$y(a) = \alpha_0, \quad y^{')}(a) = \alpha_1,$$

1. construct a partition $a = x_0 < x_1 < \ldots < x_M = b$ of $[a, b]$; set $h_i = x_i - x_{i-1}$.
2. provide $\{z_k\}_{k=1}^N$, the N LG points in [0,1],
3. for $i = 1, 2, \ldots, M$ repeat (a)-(d)
   (a) compute $\{\omega_{1i}, \omega_{2i}\}$ for $[x_{i-1}, x_i]$ by solving $\omega^2 + a(\bar{x}_i)\omega + b(\bar{x}_i) = 0,$, $\bar{x}_i = x_{i-1} + \frac{h_i}{2}$,
   (b) construct $\phi_{N,i,1} = \sum_{j=0}^N a_{ji} L_{ji}(x)$ whose the coefficients $\{a_{ji}\}$ of satisfy the linear system,
   $$(D_{\omega_{1i}} \phi_{N,i,1})(x_{i-1} + h_i z_k) = 0$$
   $$\phi_{N,i,1}(x_{i-1}) = 1, \quad k = 1, 2, ..., N,$$
   (c) construct $\phi_{N,i,2} = \sum_{j=0}^N b_{ji} L_{ji}(x)$ whose the coefficients $\{b_{ji}\}$ satisfy the linear system,
   $$(D_{\omega_{2i}} \phi_{N,i,2})(x_{i-1} + h_i z_k) = 0,$$
   $$\phi_{N,i,2}(x_i) = -1, \quad k = 1, 2, ..., N,$$
   (d) construct $y_{Ni} = c_{1i}\phi_{N,i,1}e^{\omega_{1i}x} + c_{2i}\phi_{N,i,2}e^{\omega_{2i}x}$; $\{c_{1i}, c_{2i}\}$ are fixed by left-end conditions
   $$y_{Ni}^{(\ell)}(x_{i-1}) = y_{N,i-1}^{(\ell)}(x_{i-1}), \quad \ell = 0, 1.$$

Let us identify now the residual resulting from ELGT(M,N).

THEOREM 2. *ELGT(M,N) approximant $y_{N,i}$ produces a residual of the form*

$$R_N(x) = L_{N,i}(x)\tau_1(x)e^{\omega_1 x} + L_{N,i}(x)\tau_2(x)e^{\omega_1 x} \qquad (17)$$

which is identical to (17) for $a(x) \equiv 0$.

Proof - Parts (b) and (c) imply respectively that

$$(D_{\omega_{1i}}\phi_{N,i,1})(x) = L_{Ni}(x) \times \rho_1(x),$$
$$(D_{\omega_{2i}}\phi_{N,i,2})(x) = L_{Ni}(x) \times \rho_2(x).$$

Therefore, if D is operated on $y_{N,i}$ given in (d) we get

$$\begin{aligned} Dy_{N,i} &= D[c_{1i}\phi_{N,i,1}e^{\omega_{1i}x} + c_{2i}\phi_{N,i,2}e^{\omega_{2i}x}], \\ &= c_{1i}L_{Ni}(x)\rho_1(x)e^{\omega_{1i}x} + c_{2i}L_{Ni}(x)\rho_2(x)e^{\omega_{2i}x}. \end{aligned}$$

4.2 Case 2: $y'' + a(x)y' + b(x)y = f(x)$

Let us extend ELGT to nonhomogenous 2nd order ODE

$$Dy = y'' + a(x)y' + b(x)y = f(x), \ x \in [a, b], \quad (18)$$
$$y(a) = \alpha_0, \ y'(a) = \alpha_1.$$

The general solution of (18) is written as

$$y = \text{const}_1 u_1(x) + \text{const}_2 u_2(x) + Y(x),$$

where $\{u_1, u_2\}$ are two particular solutions of $Dy = 0$ and $Y(x)$ is a particular solution of $Dy = f$.
The ELGT solution takes the form,

$$y_N = c_1\phi_{1,N}(x)e^{\omega_1 x} + c_2\phi_{2,N}(x)e^{\omega_2 x} + Y_N(x),$$

where c_1 and c_2 are fixed by the initial conditions. To generate this approximation, replace, in Algorithm 1, step (d) by (d)-(e):

ALGORITHM 2 – Algorithm 1 +

(d) compute the coefficients $\{c_{ji}, \ j = 0, 1, \ldots, N+1\}$ of
$$Y_{N,i} = \sum_{j=0}^{N/2} c_{ji}L_{ji}(x)e^{\omega_{1i}x} + \sum_{j=0}^{N/2} c_{N/2+1+j,i}L_{ji}(x)e^{\omega_{2i}x}$$
by solving
$$(DY_{N,i})(x_{i-1} + h_i z_k) = f(x_{i-1} + h_i z_k),$$
$$Y_{N,i}(x_{i-1}) = 0, \ Y'_{N,i}(x_{i-1}) = 1, \ k = 1, 2, \ldots, N,$$
(e) compute $y_{Ni} = c_{1i}\phi_{N,i,1}e^{\omega_{1i}x} + c_{2i}\phi_{N,i,2}e^{\omega_{2i}x} + Y_{N,i}$ where $\{c_{1i}, c_{2i}\}$ are fixed by left-end conditions
$$y_{Ni}^{(\ell)}(x_{i-1}) = y_{N,i-1}^{(\ell)}(x_{i-1}), \quad \ell = 0, 1.$$

5 Error Analysis

5.1 Exactness of ELGT(M,N)

DEFINITION 1 *Call ELGT(M,N) exact for a function $u(x)$ if ELGT(M,N) produces $u(x)$ exactly when applied to some equation $u'' + a(x)u' + b(x)u(x) = f(x)$, whose u is the exact solution.*

Based on this definition, it is obvious to see that,

THEOREM 3. *By its very construction, ELGT is exact for functions of the form $\{x^k e^{\omega x}; \ k = 0, 1, 2, \ldots\}$, $\omega \in \mathbf{C}$.*

Proof. For any ω and k, $y := x^k e^{\omega x}$ satisfies exactly ODE

$$y'' - \omega^2 y = k(k-1)x^{k-2}e^{\omega x} + 2\omega x^{k-1}e^{\omega x}. \qquad (19)$$

Thus, Theorem 3 holds true because (19) has constant coefficients.

5.2 Error Estimation of ELGT(N)

Reconsider the nonhomogenous 2nd order ODE (18). Omit indices i and let $[X, X + h] \equiv [x_{i-1}, x_i]$ and $Y_N \equiv y_{Ni}$, the ELGT(M,N) approximant in $[X, X + h]$.

DEFINITION 2. *Call $Y_N := y_{Ni}$ Reference.*

Let $e_N(x) := y(x) - Y_N(x)$ be the error function in $[X, X + h]$. In this section we develop a correction procedure that allows to improve the accuracy of the reference.

In [2], I gave an infinite series representation $e(x)$, recalled in Theorem 4. In order to formulate it we need to introduce the following recursions: For all $k \geq 2$ let

$$\begin{aligned} a_{k+1}(x) &:= a_k^{'}(x) + b_k(x) - a(x)a_k(x), \\ b_{k+1}(x) &:= b_k^{'}(x) - b(x)a_k(x), \end{aligned}$$

with

$$\begin{aligned} &a_0(x) := 0, \quad a_1(x) := 1, \quad a_2(x) := -a(x), \\ &b_0(x) := 1, \quad b_1(x) := 0, \quad b_2(x) := -b(x). \end{aligned}$$

Let $F(x) := f(x) - R_N(x)$ where $R_N(x)$ is given by (17).

THEOREM 4. *If $a(x)$ and $b(x)$ belong to $\mathcal{C}^\infty[X, X+h]$ then, for $\ell = 0, 1$*

$$e^{(\ell)}(x) = \sum_{k \geq 0} \frac{1}{k!} \left\{ \mathcal{F}_{k\ell}(x) + (x - X)^k \Delta_{k+\ell}(X) \right\}$$

for all $x \in [X, X+h]$, where

$$\begin{aligned} &\Delta_k(X) = [a_k(X)e^{'}(X) + b_k(X)e(X)], \\ &\mathcal{F}_{k\ell}(x) := \int_X^x a_{k+\ell}(t)(x-t)^k F(t)dt, \quad \mathcal{F}_k := \mathcal{F}_{k0}. \end{aligned}$$

Consequently, the exact solution of (18) has the expansion

$$y^{(\ell)}(x) = Y_N^{(\ell)}(x) + \delta_0^{(\ell)}(x) + \delta_1^{(\ell)}(x) + \delta_2^{(\ell)}(x) + \ldots$$

where $\{\delta_k, \ \delta_k^{'}\}$ are called corrections and given by

$$\delta_k^{(\ell)}(x) = \frac{1}{k!} \left\{ \mathcal{F}_{k\ell}(x) + (x - X)^k \Delta_{k+\ell}(X) \right\}.$$

Notation.

- ELGT(M,N,K) stands for ELGT(M,N) with K+1 corrections $\{\delta_0, \delta_1, \ldots, \delta_K\}$.
- Accordingly, ELGT(M,N,K) approximation is
$$Y_{N,K} = Y_N + \delta_0 + \delta_1 + \ldots + \delta_K.$$
- The error function of ELGT(M,N,K):
$$e_{N,K} := y(x) - Y_{NK}(x).$$

THEOREM 5. *Under the above assumptions and notations, we have*

$$e_{N,K}(X+h) = \begin{cases} O(h^{2N+1}) & \text{if } k \leq N, \\ O(h^{2N+d+1}) & \text{if } k > N \text{ with } d\text{=N-k}. \end{cases}$$

Proof. The accuracy of Y_{NK} is measured by the order of δ_{K+1} in terms of h because $e_{N,K} = \delta_{K+1} + \delta_{K+2} + \ldots$.

Assume $X = 0$. Find the order of $\delta_k(x)$ at the left-end point $x = h$. Then $\delta_k(h)$ reduces to

$$\delta_k(h) = \frac{1}{k!}\mathcal{F}_k(h).$$

Analyse $\mathcal{F}_k(h)$:

$$\begin{aligned} \mathcal{F}_k(h) &= \int_0^h a_k(t)(h-t)^k R(t)dt \\ &= \int_0^h a_k(t)(h-t)^k [\tau_1(t)e^{\omega_1 t} L_{Ni}(t) \\ &\quad + \tau_2(t)e^{\omega_2 t} L_{Ni}(t)]dt = \Im_1 + \Im_2. \end{aligned}$$

For $\Im \in \{\Im_1, \Im_2\}$:

$$\begin{aligned} \Im &= \sum_{j=0}^r \tau_j \int_0^h a_k(t)(h-t)^k t^j e^{\omega t} L_{Ni}(t)dt \\ &\sim \tau_0 \int_0^h a_k(t)(h-t)^k e^{\omega t} L_{Ni}(t)dt \\ &\sim \tau_0 \left\langle a_k(t)(h-t)^k e^{\omega t} \,\middle|\, L_{Ni} \right\rangle \\ &\sim \begin{cases} \tau_0 O(h^N) = O(h^{2N+1}) \text{ if } k \leq N, \\ \tau_0 O(h^k) = O(h^{N+k+1}) \text{ if } k > N. \end{cases} \end{aligned}$$

The last assertion follows from Lemma 2. Thus,

$$\Im_1 \text{ and } \Im_2 \sim \begin{cases} O(h^{2N+1}) \text{ if } k \leq N, \\ O(h^{2N+d+1}) \text{ if } k > N \text{ with } d = N - k. \end{cases}$$

LEMMA 2.

1. *If $f(t) = \sum_{m=0}^\infty f_m L_{m,h}(t)$, $t \in [0, h]$, then $f_m = O(h^m)$.*
2. *If, further, $f(t) = (h-t)^k g(t)$, then*
$$f_m = \begin{cases} O(h^m) \text{ if } k \leq m, \\ O(h^k) \text{ if } k > m. \end{cases}$$

6 Numerical Examples

EXAMPLE 1. The IVP

$$\begin{aligned} &y^{''} + y = 0.001\cos(x), \quad x \geq 0, \\ &y(0) = 1, \quad y^{'}(0) = \omega, \end{aligned}$$

has the exact solution $y(x) = \cos(x) + 0.0005x\sin(x)$. This problem can be solved exactly by ELGT(M,N,0) for $N \geq 2$ and arbitrary M.

EXAMPLE 2. The IVP

$$\begin{aligned} &y^{''} - 2y^{'} + 101y = \tfrac{1}{500}\mathrm{e}^x(\cos(10x) - \tfrac{1}{25}x\sin(10x)), \ x \geq 0, \\ &y(0) = 0, \quad y^{'}(0) = 10, \end{aligned}$$

whose $y(x) = \mathrm{e}^x(\sin(10x) + \frac{1}{1000}x^2\cos(10x))$ is the exact solution, can be solved exactly by ELGT(M,N,0) for $N \geq 4$ and any M.

EXAMPLE 3. $z(x) = \mathrm{e}^{ix}(1 - 0.005ix) \in \mathbf{C}$ is the exact solution of IVP,

$$\begin{aligned} &z^{''} + z = 0.001\mathrm{e}^{ix}, \\ &z(0) = 1, \quad z^{'}(0) = 0.9995i. \end{aligned}$$

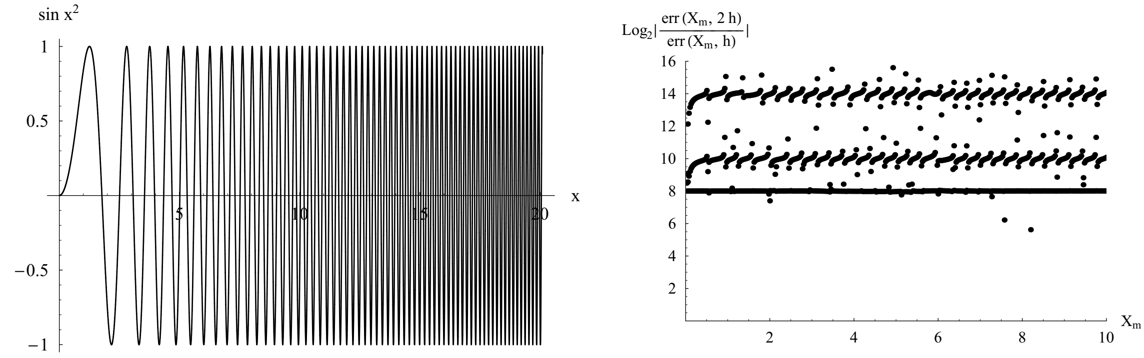

Figure 1: Left (Eq. 4): Plot of $\sin x^2$ in [0,20]. Right (Example 5): Plot of $\log_2 \left|\frac{\text{global err}[400,4,P]}{\text{global err}[800,4,P]}\right|$ at points x_m for ELGT(M,N,P) with M=400,800, N=4 and P=0(bottom), 6(middle), 10(top). Note that for P>2N, order[ELGT(M,N,P)] is N+P.

It can be reproduced exactly by ELGT(M,N,0) for $N \geq 2$ and arbitrary M.

Example 4. Reconsider IVP (4) and let us solve it by ELGT(800,2). The errors of ELGT(800,2) along those of classical LGT(800,2) are listed Table 2. One can easily appreciate the significant improvement throughout the interval of integration and at right end point.

k	x_k	ω_k	$er(x_k)$	$er'(x_k)$	$\sqrt{\text{er}^2+\text{er}'^2}$
1	0.05	0.05i	-1.26E-9	-7.71E-8	-7.71E-8
100	5.00	9.95i	1.70E-6	4.48E-5	4.48E-5
200	10.00	19.95i	-3.01E-5	1.33E-3	1.33E-3
300	15.00	29.95i	-3.43E-4	4.26E-3	4.28E-3
400	20.00	39.95i	-1.01E-3	-2.45E-2	2.45E-2
500	25.00	49.95i	3.89E-4	-1.35E-1	1.35E-1
600	30.00	59.95i	5.55E-3	4.23E-2	4.27E-2
700	35.00	69.95i	-2.26E-3	4.54E-1	4.53E-1
800	**40.00**	79.95i	**-9.92E-3**	**-6.42E-1**	**6.42E-1**
Classical LGT(800,2) result:					
800	**40.00**		**-2.91E+5**	**-7.19E+8**	**7.19E+8**

Table 2: (Example 4) ELGT(800,2) error in approximating $\sin x^2$ in [0,40] compared to classical LGT(800,2).

Example 5. Consider the nonhomogenous linear IVP

$$y''+4x^2y=(4x^2-\omega^2)\sin(\omega x)-2\sin(x^2), 0\leq x\leq 10, \quad (20)$$
$$y(0)=1, \quad y'(0)=\omega,$$

with exact solution $y(x)=\sin(\omega x)+\cos(x^2)$. Let $\{0 = x_0 < x_1 \ldots < x_M = 10\}$ be a uniform partition of [0,10]. We applied ELGT(M,N,P) with M=400, 800, N=4 and P=0, 6, 10. The global errors at x_m are displayed Table 3. Note that the last 3 columns assure that for P>2N, the order of ELGT(M,N,P) is N+P.

Variations of ELGT error in terms N. To see the variations of the ELGT error at points $\{x_m;\ m=1,2,\ldots,M\}$ with respect to N, the order of $L_N(x)$, we report in Table 4 the exact errors when (20) is solved by ELGT(M,N)

		Method		order		
i	x_i	ELGT(100,4,0) ELGT(100,4,6) ELGT(100,4,10)	ELGT(200,4,0) ELGT(200,4,6) ELGT(200,4,10)	h^8	h^{10}	h^{14}
60	3	1.77E-09	5.83E-12	8		
		7.39E-11	6.97E-14		10	
		7.53E-15	4.88E-19			14
80	4	2.41E-09	9.81E-12	8		
		1.83E-10	1.52E-13		10	
		5.16E-15	4.90E-19			13
100	5	1.28E-09	5.66E-12	8		
		1.60E-10	7.89E-14		11	
		1.12E-14	1.52E-18			13
160	8	1.03E-06	3.69E-09	8		
		1.35E-07	1.63E-10		10	
		2.94E-10	2.52E-14			14
180	9	3.59E-06	1.25E-08	8		
		7.59E-07	8.20E-10		10	
		3.12E-09	2.21E-13			14
200	10	6.25E-06	2.12E -08	8		
		2.17E-06	2.11E-09		10	
		1.46E-08	9.10E-13			14

Table 3: (Example 5) Comparison of the global error at some x_m obtained by ELGT(M,N,P) for M=100, 200, N=4 and P=0, 6, 10.

with $M = 10$ (i.e. $h = 1$), and $N = 10, 12, \ldots, 20$. To explain the dependence between the global error and N, we plot in Fig. 2 the pairs $\{(N, \ln|er_N(x_m)|); N = 10, 12, \ldots, 20\}$, for each m= $1, 2, \ldots$,M, (M=10 and 20). It is observed that the error decays exponentially with respect to N, in accordance with an earlier result given in [1], i.e.

$$\text{error of ELGT(M,N)} \sim \frac{1}{N!c_N^N}$$

where c_N^N is the leading coefficient of $L_N(x)$ in [0,1].

7 Nonlinear Differential Equations

Consider nonlinear ODEs of the form

$$y'' = f(x, y, y'), \quad x \in [a, b], \quad (21)$$

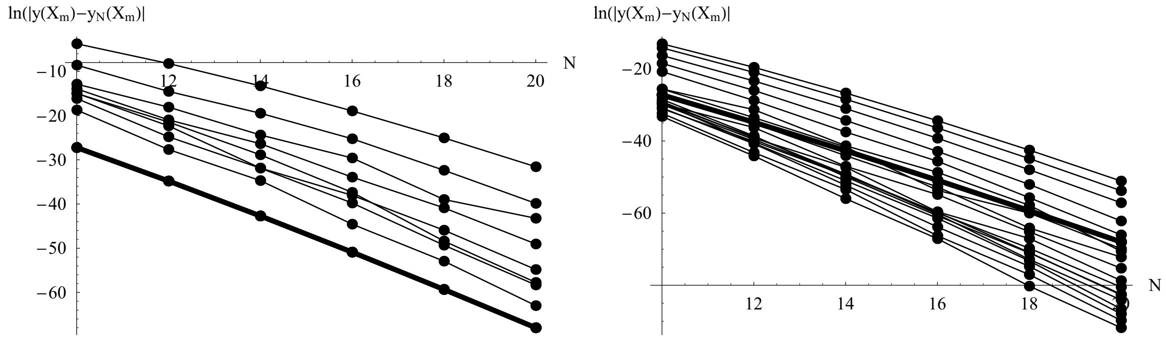

Figure 2: (Example 5.) The order of ELGT(M,N) vis-avis N: Variations with N of $\ln|y(x_m) - y_N(x_m)|$ compared to the (heavy) plot of $\ln|\frac{1}{N!c_N^N}|$, N=10, 12, . . . , 20 and m=1,2,...,8 and M=10 (left), M=20 (right).

x_m	$\mathrm{er}_N(x_m)$					
	N=10	N=12	N=14	N=16	N=18	N=20
1	7.34E-9	-9.62E-13	-8.55E-16	-4.45E-20	1.03E-23	4.45E-28
2	-9.81E-8	-1.70E-11	1.43E-14	5.34E-18	3.83E-22	-4.76E-26
3	3.19E-7	2.01E-10	1.45E-14	-3.03E-17	-1.14E-20	-1.57E-24
4	-3.05E-7	-4.79E-10	-2.90E-13	-5.93E-17	-9.77E-22	8.48E-26
5	-8.71E-7	7.85E-10	3.56E-12	1.86E-15	-1.79E-18	-5.04E-22
6	2.54E-6	1.40E-8	-2.58E-11	-1.42E-13	1.18E-17	1.69E-19
7	-1.87E-4	4.89E-7	-3.51E-9	1.09E-11	8.77E-15	4.90E-18
8	2.34E-2	-2.70E-4	1.72E-6	-5.94E-9	1.33E-11	-1.95E-14
9	1.30E-1	1.03E-4	-2.30E-5	2.89E-7	-1.74E-9	6.02E-12
10	-4.98E1	-4.31E-1	9.78E-3	-1.26E-4	9.98E-7	-5.03E-9

Table 4: (Example 5.) Exact error $\mathrm{er}_N(x_m)$ at some points x_m for $h = 1$ for $N = 10, 12, ..., 20$.

$$y(a) = \alpha_0, \;\; y^{'}(a) = \alpha_1.$$

To approximate the exact solution y by means of ELGT we follow the following procedure:

Take a partition $a = x_0 < x_1 < \ldots < x_M = b$ of [a,b] and set $h = x_i - x_{i-1}$.

On each $[x_{i-1}, x_i]$, apply ELGT iteratively to linear IVPs,

$$y_k^{''} - f_{y'} y_k^{'} - f_y y_k = f(x, y_{k-1}, y_{k-1}^{'}) - f_{y'} y_{k-1}^{'} - f_y y_{k-1} + O(e^2),$$

where $e(x) = y - y_{k-1}$, and $\{f_y,\ f_{y'},\ f_{yy},\ f_{y'y'}, f_{yy'}\}$ are evaluated at $(y_{k-1}, y_{k-1}^{'})$.

For small h, one can safely drop $O(e^2)$.

For each cycle, construct y_k. Repeat the process until a prescribed convergence tolerance ϵ is satisfied.

Example 6. Consider the nonlinear problem [5]

$$y^{''} + y + y^3 = (\cos x + \epsilon\ \sin(10x))^3 - 99\epsilon \sin(10x),\ x \geq 0,$$
$$y(0) = 1,\ y^{'}(0) = 10\epsilon,$$

whose the exact solution is $y(x) = \cos x + \epsilon \sin(10x)$. I solved this problem over $[0, 200]$ and measured the error at $x{=}100$ and $x{=}200$. The results are given in Table 3.

M	h	x_m	ELGT[M,4,0]	ELGT[M,4,6]
400	0.5	100.0	1.96E-5	1.60E-6
800	0.25	100.0	2.31E-8	6.20E-10
1600	0.125	100.0	6.90E-11	2.74E-13
400	0.5	200.0	9.04E-6	8.31E-7
800	0.25	200.0	1.16E-8	4.68E-10
1600	0125	200.0	3.36E-11	2.18E-13

Table 5: (Example 6.) The error at the mid-point and end-point of $[0, 200]$ obtained by ELGT[M,4,K] for $M = 400, 800, 1600$ and $K = 0, 6$.

Example 7. The following nonlinear problem was studied in Papeorgiou et al [10]

$$y^{''} + 100y = \sin y, \quad x \geq 0,$$
$$y(0) = 0,\ y^{'}(0) = 1.$$

Exact $y(x)$ is not available, but exact $y(20\pi) = 0.000392823991$. We applied ELGT in $[0, 20\pi]$ and reported the errors at $x = 20\pi$ in Table 6.

ELGT(M,4)		Papageorgiou *et al* [10]		Tsitouras and Simos [13]	
M steps	Error	M steps	Error	F. Ev.	Error
100	2.66E-4	2400	5.63E-4		
200	9.13E-6	4800	9.29E-6		
400	6.04E-8	7200	8.24E-7	12000	2.6E-11
800	2.69E-10			14000	3.6E-11
1600	1.50E-12			16000	5.8E-12

Table 6: (Example 7.) The error at $x = 20\pi$ for the ELGT and [10].

Example 8. (Duffing problem). The exact solution of

$$y^{''} + y + y^3 = F\cos(\Omega x), \quad x \geq 0,$$
$$y(0) = y_G(0),\ y^{'}(0) = 0,$$

is given by $y_G(x) = \sum_{i=0}^{\infty} \alpha_{2i+1} \cos((2i+1)\Omega x)$, where $\alpha_1 = 0.200179477536$, $\alpha_3 = 0.246946143\text{E}{-3}$, $\alpha_5 = 0.304014\text{E}{-6}$, $\alpha_7 = 0.374\text{E}{-9}$, $\alpha_9, \alpha_{11}, \ldots < 10^{-12}$.

I solved this problem in $[0, 20\pi]$ with $F = 0.002$ and $\Omega = 1.001$ for M=50, 100 and 200. The maximum errors over $[0, 20\pi]$ were computed and listed in Table 7. My results are compared to those obtained by the five stage method introduced in [10]. I also solved the Duffing problem in

ELGT(M,4)		Papageorgiou *et al* [10]	
M steps	Error	M steps	Error
50	5.41E-06	600	4.31E-07
100	2.25E-08	1200	6.56E-09
200	9.15E-11	2400	1.05E-10

Table 7: (Example 8.) Maximum errors over $[0, 20\pi]$ for ELGT compared to those given in [10].

interval $[0, 300]$ with several number of steps. The results are listed in Table 8.

Error at $x = 300$			
M steps	ELGT	Ixaru and Vanden Berghe [5]	Simos [13]
300	6.25E-7	1.10E-3	1.70E-3
600	2.78E-9	5.42E-5	1.88E-4
1200	1.31E-11	1.86E-6	1.37E-5
2400		6.19E-8	8.70E-7
4800		2.40E-9	5.41E-8

Table 8: (Example 8.) The Euclidean norm of the global error at the end-point $x = 300$ of [0,300] for ELGT and EFERKM of Ixaru and Vanden Berghe [5] and by Simos's method given in Simos [13], [14].

8 Conclusions and Future Work

In this paper, an exponentially weighted Legendre-Gauss Tau Method for approximating ODEs with strongly oscillatory solution is developed. ELGT involves some weights with frequencies $\{\omega\}$ being the roots of the quadratic equation associated with the constant reference equation. The new method is capable of detecting the sharp variations of the function throughout a considerably large interval of integration. The accuracy of ELGT can be measured either in terms of the step size h or in terms of N, the degree Legendre polynomial L_N. In the former we obtain an error of order $O(h^{2m})$ and the latter results in error of order $O(\frac{1}{N!})$.

ELGT needs to be tested on Sturm-Liouville problems, both regular and irregular. A comprehensive treatment for systems of nonlinear ODEs is not presented here, but it is possible using Alekseev-Góbner lemma.

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