Baire's Theorem in Probabilistic Modular spaces

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Abstract— **A real linear space X equipped with a probabilistic-valued function ρ defined on X is called a probabilistic modular space if it satisfies the following conditions:**
i. $\rho_x(0) = 0$,
ii. $\rho_x(t) = 1$ **for all** $t > 0$ **iff** $x = 0$,
iii. $\rho_{-x}(t) = \rho_x(t)$,
iv. $\rho_{\alpha x+\beta y}(s + t) \geq \rho_x(s) \wedge \rho_y(t)$ **for all** $x, y \in X$, **and** $\alpha, \beta, s, t \in {\mathcal{R}_0}^+$, $\alpha + \beta = 1$.

In this note, we give a modular formulation of Baire's Theorem in probabilistic modular spaces.

Keywords: ***Modular Space, Probabilistic Modular space, Baire's Theorem***

1 Introduction

A modular on a real linear space X is a real functional μ on X satisfying the following conditions:
1. $\mu(x) = 0$ iff $x = 0$,
2. $\mu(x) = \mu(-x)$,
3. $\mu(\alpha x + \beta y) \leq \mu(x) + \mu(y)$,
for all $x, y \in X$ and $\alpha, \beta \geq 0$, $\alpha + \beta = 1$.

Then, the vector subspace

$$X_\mu = \{x \in X : \mu(ax) \to 0 \text{ as a} \to 0\},$$

of X is called a modular space.

Such spaces were considered for instance in [1, 2, 3]. The notions of modular and probabilistic metric spaces [4] provide the inspiration for introducing a new concept that it is called a probabilistic modular space in this note. We aim to investigate the Baire's Theorem in such spaces.

2 Probabilistic Modulars

A function $f : \mathcal{R} \longrightarrow {\mathcal{R}_0}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathcal{R}} f(t) = 0$, and $\sup_{t\in\mathcal{R}} f(t) = 1$.

Definition 1 A pair (X, ρ) will be said a probabilistic modular space if X is a real vector space, ρ is a mapping from X into the set of all distribution functions(for $x \in X$, the distribution function $\rho(x)$ is denoted by ρ_x, and $\rho_x(t)$ is the value ρ_x at $t \in \mathcal{R}$) satisfying the following conditions:
1. $\rho_x(0) = 0$,
2. $\rho_x(t) = 1$ for all $t > 0$ iff $x = 0$,
3. $\rho_{-x}(t) = \rho_x(t)$,
4. $\rho_{\alpha x+\beta y}(s + t) \geq \rho_x(s) \wedge \rho_y(t)$ for all $x, y \in X$, and $\alpha, \beta, s, t \in {\mathcal{R}_0}^+$, $\alpha + \beta = 1$.

We say that (X, ρ) satisfies Δ_2-condition if there exists $c > 0$ (Δ_2-constant) such that $\rho_{2x}(t) \geq \rho_x(\frac{t}{c})$ for all $x \in X$ and $t > 0$.

Example 1 Suppose that X is a real vector space and μ is a modular on X. Define

$$\rho_x(t) = \begin{cases} 0 & t \leq 0 \\ \frac{t}{t+\mu(x)} & t > 0 \end{cases}$$

Then (X, ρ) is a probabilistic modular space.

Example 2 Suppose that X is a real vector space and μ is a modular on X. Define

$$\rho_x(t) = \begin{cases} 0 & t \leq \mu(x) \\ 1 & t > \mu(x) \end{cases}$$

Then (X, ρ) is a probabilistic modular space.

Definition 2 Let (X, ρ) be a probabilistic modular space.

- A sequence (x_n) in X is said to be ρ-convergent to a point $x \in X$ and denoted by $x_n \to x$ if for every $t > 0$ and $r \in (0, 1)$, there exists a positive integer k such that $\rho_{x_n-x}(t) > 1 - r$ for all $n \geq k$.
- A sequence (x_n) in X is called a ρ-Cuachy sequence if for every $t > 0$ and $r \in (0, 1)$, there exists a positive integer k such that $\rho_{x_n-x_m}(t) > 1 - r$, for all $m, n \geq k$.
- The modular space (X, ρ) is said to be ρ-complete if each ρ-Cauchy sequence in X is ρ-convergent to a point of X.

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- The ρ-closure of a subset E of X is denoted by $\overline{E}$ and defined by the set of all $x \in X$ such that there is a sequence (x_n) of elements of E such that $x_n \to x$. The subset E is ρ-dense in X if $\overline{E} = X$.

- For $x \in X$, $t > 0$, and $0 < r < 1$, the ρ-ball centered at x with radius r is defined by
$$B(x, r, t) = \{y \in X : \rho_{x-y}(t) > 1 - r\}.$$

- An element $x \in E$ is called a ρ-interior point of E if there are $r \in (0, 1)$ and $t > 0$ such that $B(x, r, t) \subseteq E$. We say that E is ρ-open in X if each element of E is a ρ-interior point.

Baire's Theorem. *Let (X, ρ) be a ρ–complete probabilistic modular space satisfying Δ_2-condition. If $(\mathcal{O}_n)$ is a sequence of ρ–open and ρ–dense subsets of X, then $\cap_1^\infty \mathcal{O}_n$ is ρ–dense in X.*

`Proof` First of all note that if $B(x, r, t)$ is a ball in X and y is an arbitrary element of it, then $\rho_{x-y}(t) > 1 - r$. Since $\rho_{x-y}(\cdot)$ is left continuous, there is $\epsilon_y > 0$ such that $\rho_{x-y}(\frac{t-\epsilon}{c}) > 1 - r$, for all $\epsilon > 0$ with $\frac{t-\epsilon}{c} > 0$ and $\frac{\epsilon}{c} \in (0, \epsilon_y)$, where c is the Δ_2-constant. If $0 < r' < r$, $\frac{\epsilon}{c} \in (0, \epsilon_y)$, and $z \in \overline{B}(y, r', \frac{\epsilon}{2c^2})$, then there exists a sequence (z_n) in $\overline{B}(y, r', \frac{\epsilon}{2c^2})$ such that $z_n \to z$ and hence we have

$$\begin{aligned} \rho_{z-y}(\tfrac{\epsilon}{c}) &\geq \rho_{(z-z_n)}(\tfrac{\epsilon}{2c^2}) \wedge \rho_{(y-z_n)}(\tfrac{\epsilon}{2c^2}), \\ &> 1 - r, \end{aligned}$$

for some $n \in \mathcal{N}$. Thus,

$$\begin{aligned} \rho_{x-z}(t) &\geq \rho_{2(z-y)}(\epsilon) \wedge \rho_{2(x-y)}(t - \epsilon), \\ &\geq \rho_{z-y}(\tfrac{\epsilon}{c}) \wedge \rho_{x-y}(\tfrac{t-\epsilon}{c}), \\ &> 1 - r, \end{aligned}$$

i.e., $\overline{B}(y, r', \frac{\epsilon}{2c^2}) \subseteq B(x, r, t)$. It implies that if A is a nonempty ρ–open set of X, then $A \cap \mathcal{O}_1$ is non-empty and ρ–open. Thus it contains a ball $B(x_0, r_0, \frac{t_0}{c})$. By induction, we choose $x_n \in X$, $r_n \in (0, 1)$, and $t_n > 0$ as follows: With x_i, r_i, and t_i if $i < n$, we see that

$$\mathcal{O}_n \cap B(x_{n-1}, r_{n-1}, \frac{t_{n-1}}{c}),$$

is non-empty and ρ-open, therefore we can choose x_n, r_n, t_n so that $0 < t_n < \frac{1}{n}$, $0 < r_n < \min\{r_{n-1}, \frac{1}{n}\}$ and

$$\overline{B}(x_n, r_n, \frac{t_n}{c}) \subseteq \mathcal{O}_n \cap B(x_{n-1}, r_{n-1}, \frac{t_{n-1}}{c}).$$

Now (x_n) is ρ-Cauchy. Because, if $0 < r < 1$, and $t > 0$, we can choose $k \in \mathcal{N}$ such that $2t_k < t$ and $r_k < r$. For $m, n \geq k$ we have $x_m, x_n \in B(x_k, r_k, \frac{t_k}{c})$ and

$$\begin{aligned} \rho_{x_m-x_n}(t) &\geq \rho_{x_m-x_n}(2t_k), \\ &\geq \rho_{x_k-x_n}(\tfrac{t_k}{c}) \wedge \rho_{x_k-x_n}(\tfrac{t_k}{c}), \\ &\geq 1 - r_k, \\ &> 1 - r. \end{aligned}$$

Since X is ρ-complete, $x_n \to x$ for some $x \in X$. But $x_n \in B(x_k, r_k, \frac{t_k}{c})$, for all $n \geq k$ and therefore

$$x \in \overline{B}(x_k, r_k, \frac{t_k}{c}) \subseteq \mathcal{O}_k \cap B(x_0, r_0, \frac{t_0}{c}) \subseteq \mathcal{O}_k \cap A,$$

for all k. Now, the fact that arbitrary ρ-open set A intersects $\cap \mathcal{O}_n$ and $B(z, r, t) \supseteq B(z, \frac{r}{n}, \frac{t}{n})$ for all n, completes the proof.

References

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