Proceedings of the World Congress on Engineering 2008 Vol II
WCE 2008, July 2 - 4, 2008, London, U.K.

Transforming the Expression of the Integral of a Vector Field Curl over a Cylindrical Surface into Another Surface Integral without Differential Operators

Andrei Nicolaide

***Abstract*—The integral of a curl of a vector field, defined in the paper, and extended over the lateral surface of a cylinder, will be transformed into a surface integral extended only over the bases of the cylinder, not containing differential vector operators. The transformation will be carried out using only the general formulae of vector analysis without resorting to other proofs of the field theory. The type of integral is an interesting one because it permits to obtain directly, without physical considerations, only mathematically, certain formulae useful in physics.**

***Index Terms*—Vector Analysis, Vector integral transformation, Field theory.**

I. Introduction

In the field theory many types of vector integral arise. In what follows, we shall consider the integral of a curl of a vector field, defined further on. This integral is extended over the lateral surface of the cylinder. In the application, which follows, this cylinder will represent a solenoid current sheet. The mentioned integral will be transformed into a surface integral not containing differential vector operators. The transformation will be carried out using only the general formulae of vector analysis [1]-[4], without resorting to other proofs of the field theory, what is not to be found in literature. The chosen integral is an interesting one because it permits to obtain directly, without physical considerations, only mathematically, certain useful formulae.

II. Definition of the Integral

In order to fix the ideas, we shall consider, as shown in Fig. 1, a right cylinder, with the symbols given in the figure. For the simplicity of the figure, we shall consider a cylinder of a circular or elliptic cross-section. This circumstance will not restrict the generality of the solution because all the relations will have a general use. The bases will be considered to be perpendicular to the cylinder generatrix.

The following symbols will be used: Σ_{cyl} – the closed surface of the cylinder, i.e., the lateral surface, Σ_{lat}, unified with the surface of each of both bases, S_{bases}; $\boldsymbol{r}$ – position vector having its origin at any source point denoted by P, placed at any point of the cylinder surface, and its arrow end at the field (observation) point denoted by N; $\boldsymbol{k}$ – unit vector of the cylinder axis or generatrix; $\boldsymbol{n}$ – unit normal to the surface element of the cylinder, at any point; $\boldsymbol{t}$ – unit vector of the tangent to any element of the lateral surface of the cylinder, and normal to its generatrix or axis; $f(r)$ – a harmonic function of the modulus of the position vector that has to satisfy the conditions to be continuous and have he partial derivatives of the first and second orders with respect to the space co-ordinates at each point of the volume bounded by the considered surface, except the observation points placed on the cylinder surface. The first derivative with respect to r will be denoted $f'(r)$.

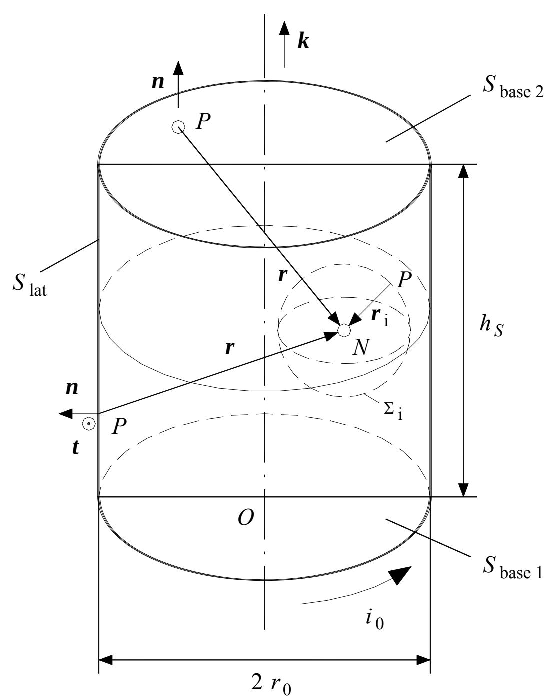

Fig. 1. A cylinder of finite length: P – source point, running point, in three positions; N – field point; S_{lat} – lateral surface of the solenoid cylinder; $S_{\mathrm{base\,1}}$, $S_{\mathrm{base\,2}}$, – surfaces of bases; Σ_{i} – small sphere surface surrounding the field point N. The field point is inside the cylinder but it may be also outside.

Manuscript received September 4, 2007.

A. Nicolaide is with the "Transilvania" University of Brasov, Bd. Eroilor Nr. 29, Brasov, Cod 500036, Romania (e-mail: nicolaide@leda.unitbv.ro).

We want to calculate the integral:

$$\boldsymbol{I} = \int_{S_{\text{lat}}} \text{curl}\left[\boldsymbol{t}\, f(r)\right] dS\,, \tag{1}$$

$$\boldsymbol{t} = \boldsymbol{k} \times \boldsymbol{n}\,, \tag{2}$$

where the curl has to be calculated at the field (observation) point *N*. Then, we can write:

$$\boldsymbol{I} = \int_{S_{\text{lat}}} \text{curl}\left[\boldsymbol{k} \times \boldsymbol{n}\, f(r)\right] dS\,, \tag{3}$$

or

$$\boldsymbol{I} = \int_{\Sigma_{\text{cyl}}} \text{curl}\left[\boldsymbol{k} \times \boldsymbol{n}\, f(r)\right] dS\,, \tag{4}$$

because the vector product is zero on both bases, so that the value of the integral will not change although it is extended over another, closed, surface. Hence, we have transformed the integral over an open surface into one over a closed surface, what will lighten the calculations.

Further, for simplicity we shall use, in several relations, instead of the scalar surface element, a vector surface element. Therefore, the integral to be calculated is:

$$\begin{aligned} \boldsymbol{I} &= \int_{\Sigma_{\text{cyl}}} \text{curl}\left[\boldsymbol{k} \times f(r)\, d\boldsymbol{S}\right], \\ d\boldsymbol{S} &= \boldsymbol{n}\, dS\,. \end{aligned} \tag{5}$$

We shall prove that under the definitions above, the following relationship is satisfied:

$$\begin{aligned} \boldsymbol{I} = \int_{S_{\text{lat}}} \text{curl}\left[\boldsymbol{k} \times f(r)\, d\boldsymbol{S}\right] &= m\, \boldsymbol{k}\, 4\pi \\ &- \int_{S_{\text{bases}}} f'(r) \frac{\boldsymbol{r}}{r} \left(\boldsymbol{k} \cdot d\boldsymbol{S}\right), \end{aligned} \tag{6}$$

where the constant *m* has the value zero if the field point *N* is outside the cylinder and has the value zero or unity, according to the form of function $f(r)$, if the field point is inside the cylinder.

III. Transformation of the Integral

We shall consider the vector quantity of (5), the curl of which has to be calculated, as composed of two factors: $\boldsymbol{k}$ and $f(r)\, d\boldsymbol{S}$. Taking into account that the first factor is constant; when differentiating at point *N*, we shall obtain [2], [3], [4, p. 229]:

$$\text{curl}\left[\boldsymbol{k} \times f(r)\, \boldsymbol{n}\right] = \boldsymbol{k}\, \text{div}\left[f(r)\, \boldsymbol{n}\right] - \left(\boldsymbol{k} \cdot \nabla\right)\left[f(r)\, \boldsymbol{n}\right]. \tag{7}$$

The right hand side contains a term that appears also in the expression of the gradient of the scalar product, one vector of which is constant, as follows [1]-[3], [4, p. 228]:

$$\begin{aligned} \text{grad}\left[\boldsymbol{k} \cdot f(r)\, \boldsymbol{n}\right] &= \left(\boldsymbol{k} \cdot \nabla\right) f(r)\, \boldsymbol{n} + \boldsymbol{k} \times \text{curl}\left[f(r)\, \boldsymbol{n}\right] \\ &= \left(\boldsymbol{k} \cdot \nabla\right)\left[f(r)\, \boldsymbol{n}\right] - \boldsymbol{k} \times \left[\boldsymbol{n} \times f'(r) \frac{\boldsymbol{r}}{r}\right]. \end{aligned} \tag{8}$$

Adding up side by side the equations (7) and (8), we shall obtain:

$$\begin{aligned} \text{curl}\left[\boldsymbol{k} \times f(r)\, \boldsymbol{n}\right] &= -\,\text{grad}\left[\boldsymbol{k} \cdot f(r)\, \boldsymbol{n}\right] \\ &+ \boldsymbol{k}\, \text{div}\left[f(r)\boldsymbol{n}\right] - \boldsymbol{k} \times \left[\boldsymbol{n} \times f'(r) \frac{\boldsymbol{r}}{r}\right]. \end{aligned} \tag{9}$$

From relations (4) and (9), it follows that:

$$\begin{aligned} \boldsymbol{I} &= \boldsymbol{I}_1 + \boldsymbol{I}_2 + \boldsymbol{I}_3\,, \\ \boldsymbol{I}_1 &= \int_{\Sigma_{\text{cyl}}} -\,\text{grad}\left[\boldsymbol{k} \cdot f(r)\right] d\boldsymbol{S}, \\ \boldsymbol{I}_2 &= \int_{\Sigma_{\text{cyl}}} \boldsymbol{k}\, \text{div}\left[f(r)\, d\boldsymbol{S}\right], \\ \boldsymbol{I}_3 &= \int_{\Sigma_{\text{cyl}}} -\,\boldsymbol{k} \times \left[d\boldsymbol{S} \times f'(r) \frac{\boldsymbol{r}}{r}\right]. \end{aligned} \tag{10}$$

We shall calculate the component produced by each of the three terms contained by the integral.

The first term yields [3], [4, p. 214]:

$$\begin{aligned} \boldsymbol{I}_1 &= -\int_{\Sigma_{\text{cyl}}} \text{grad}\left[\boldsymbol{k} \cdot f(r)\, d\boldsymbol{S}\right] \\ &= -\int_{\Sigma_{\text{cyl}}} f'(r) \frac{\boldsymbol{r}}{r} \left(\boldsymbol{k} \cdot d\boldsymbol{S}\right) = -\int_{S_{\text{lat}}} f'(r) \frac{\boldsymbol{r}}{r} \left(\boldsymbol{k} \cdot d\boldsymbol{S}\right) \\ &\quad - \int_{S_{\text{bases}}} f'(r) \frac{\boldsymbol{r}}{r} \left(\boldsymbol{k} \cdot d\boldsymbol{S}\right). \end{aligned} \tag{11}$$

In the integral extended over the lateral surface of the cylinder, the scalar product $\boldsymbol{k} \cdot d\boldsymbol{S}$ is zero because the two vectors are, at any point of this surface, perpendicular to each other. Hence:

$$\boldsymbol{I}_1 = -\int_{S_{\text{bases}}} f'(r) \frac{\boldsymbol{r}}{r} \left(\boldsymbol{k} \cdot d\boldsymbol{S}\right). \tag{12}$$

The second term yields [4, p. 228]:

$$\boldsymbol{I}_2 = \int_{\Sigma_{\text{cyl}}} \boldsymbol{k}\, \text{div}\left[f(r)\, d\boldsymbol{S}\right] = \boldsymbol{k} \int_{\Sigma_{\text{cyl}}} f'(r) \frac{\boldsymbol{r} \cdot d\boldsymbol{S}}{r}. \tag{13}$$

In order to simplify the calculation, we shall apply the theorem of Gauss and Ostrogradski for obtaining the transformation of a surface integral into a volume one. For applying the usual formulae containing differential operators, taking into account that the point *N* is fixed, we shall consider, when performing the calculation, the vector $\boldsymbol{r}' = -\boldsymbol{r}$, and, after the calculation has been performed, we shall return to vector $\boldsymbol{r}$. However, for conciseness reason, we shall not write this intermediary transformation.

In order to ensure the applicability of the mentioned theorem, we shall first separate, inside the cylinder, the singular point, that is the point *N*, corresponding to $\boldsymbol{r} = 0$, by surrounding it with a sphere, having the center at that point, and small enough, so that it does not touch the cylinder. Then, we have to calculate the integral:

$$\int_{\Sigma} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r} = \int_{\Sigma_{\mathrm{cyl}}} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r} + \int_{\Sigma_{\mathrm{i}}} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r}, \qquad (14)$$

$$\Sigma = \Sigma_{\mathrm{cyl}} \cup \Sigma_{\mathrm{i}} ,$$

where Σ_{i} represents the surface of the sphere, which surrounds the singular point above.

For the sphere surface, the unit normal is oriented, as in general, outwards the considered volume, that is, towards the sphere center, as $\boldsymbol{r}_{\mathrm{i}}$ in Fig. 1, and since $f(r)$ is assumed to be a harmonic function, we shall obtain:

$$\int_{\Sigma} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r} = \int_{V_\Sigma} \mathrm{div}\left[f'(r)\frac{\boldsymbol{r}}{r}\right] dv = \int_{V_\Sigma} \mathrm{div}\,\mathrm{grad}\, f(r)\, dv = 0, \qquad (15)$$

$$\Sigma = \Sigma_{\mathrm{cyl}} \cup \Sigma_{\mathrm{i}} .$$

Therefore, relations (14), (15) yield:

$$\int_{\Sigma_{\mathrm{cyl}}} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r} = -\int_{\Sigma_{\mathrm{i}}} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r}. \qquad (16)$$

Taking into account the symbols of (5), relations (13) and (16) yield:

$$\boldsymbol{I}_2 = -\boldsymbol{k}\int_{\Sigma_{\mathrm{i}}} f'(r)\frac{\boldsymbol{r}\cdot d\boldsymbol{S}}{r} = -\boldsymbol{k}\int_{\Sigma_{\mathrm{i}}} f'(r_{\mathrm{i}})\frac{\boldsymbol{r}_{\mathrm{i}}\cdot \boldsymbol{r}_{\mathrm{i}}}{r_{\mathrm{i}}^2}\, dS = \boldsymbol{k}\, m\, 4\pi , \qquad (17)$$

where the value of m is depending on the expression of $f(r)$. For the function $f(r)$ to be harmonic, it has to satisfy the Laplace equation.

From the Laplace equation, expressed in spherical co-ordinates, [2], [4, p. 244], it follows that this function has to be of the form r^n, apart from a multiplicative constant and an additive constant, and the exponent n can take only the values zero or –1. For $n = 0$, we shall obtain $\boldsymbol{I}_2 = 0$, hence $m = 0$. For $n = -1$, we shall obtain $m = 1$.

If the point N is outside the cylinder, the term $\boldsymbol{I}_2$ is zero.

The third term is:

$$\boldsymbol{I}_3 = -\int_{\Sigma_{\mathrm{cyl}}} \boldsymbol{k}\times\left[d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}}{r}\right]. \qquad (18)$$

We have [4, p. 232]:

$$-\int_{\Sigma} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}}{r} = -\int_{V_\Sigma} \mathrm{curl}\left[f'(r)\frac{\boldsymbol{r}}{r}\right] dv = -\int_{V_\Sigma} \mathrm{curl}\,\mathrm{grad}\, f(r)\, dv = 0, \qquad (19)$$

and:

$$-\int_{\Sigma} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}}{r} = -\int_{\Sigma_{\mathrm{cyl}}} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}}{r} - \int_{\Sigma_{\mathrm{i}}} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}_{\mathrm{i}}}{r_{\mathrm{i}}} = 0, \qquad (20)$$

hence:

$$-\int_{\Sigma_{\mathrm{cyl}}} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}}{r} = \int_{\Sigma_{\mathrm{i}}} d\boldsymbol{S}\times f'(r)\frac{\boldsymbol{r}_{\mathrm{i}}}{r_{\mathrm{i}}} = \int_{\Sigma_{\mathrm{i}}} \frac{\boldsymbol{r}_{\mathrm{i}}}{r_{\mathrm{i}}}\times f'(r)\frac{\boldsymbol{r}_{\mathrm{i}}}{r_{\mathrm{i}}}\, dS = 0. \qquad (21)$$

Therefore:

$$\boldsymbol{I}_3 = 0. \qquad (22)$$

The result will be given by summing up the three terms, according to relation (10), and we obtain:

$$\boldsymbol{I} = m\,\boldsymbol{k}\, 4\pi - \int_{S_{\mathrm{bases}}} f'(r)\frac{\boldsymbol{r}}{r}\left(\boldsymbol{k}\cdot d\boldsymbol{S}\right), \qquad (23)$$

which represents the resulting formula.

IV. Application to the Calculation of the Magnetic Field Strength Produced by a Solenoid Starting from the Biot-Savart-Laplace Formula

In order to fix the ideas, we shall consider that the cylinder shown in Fig. 1 represents an electric current sheet, hence a solenoid with the symbols given in the figure.

The current sheet round the cylinder will be considered infinitely thin and constant with time. The medium is assumed as vacuum. Under the preceding assumptions, the magnetic field strength produced by the electric current carried by the solenoid can be calculated by the Biot-Savart-Laplace formula, [4], as follows:

$$\boldsymbol{H} = \int_{\Gamma} \frac{i}{4\pi}\cdot\frac{d\boldsymbol{l}\times\boldsymbol{r}}{r^3}, \qquad (24)$$

where the following symbols have been used: Γ – helical curve, a very closed one, considered as the mean curve of a very thin and narrow strip, which forms the lateral surface of the cylinder; i – intensity of the electric current carried by the path corresponding to that curve; J_{0L} – linear current density of the current sheet; $dS = dh\, dl$ – area of an element of the lateral surface of the cylinder; Δh and dh –width of the strip along the generatrix of the cylinder; Δl and dl – element of length of curve Γ; $\boldsymbol{r}$ – position vector having its origin at any source point denoted by P, of the curve above, and its arrow end at the field (observation) point denoted by N; $\boldsymbol{k}$ **–** unit vector of the cylinder generatrix or axis.

The link between the vector representing an element of the lateral surface of the cylinder, and the current carried by the strip represented by curve Γ, can be expressed as follows:

$$\Delta S = (\Delta h)(\Delta l); \quad \Delta\boldsymbol{l} = (\boldsymbol{k}\times\boldsymbol{n})\Delta l; \quad i = J_{0L}\,\Delta h; \qquad (25)$$

and from relations (24) and (25), we get:

$$\boldsymbol{H} = \int_{S_{\text{lat}}} \frac{J_{0L}\left[(\boldsymbol{k}\times\boldsymbol{n})\times\boldsymbol{r}\right]dS}{4\pi r^3} = -J_{0L}\frac{1}{4\pi}\int_{S_{\text{lat}}}\frac{\left[\boldsymbol{r}\times(\boldsymbol{k}\times\boldsymbol{n})\right]dS}{r^3}. \qquad (26)$$

Let us consider the integrand of the integral (26) as composed of two factors: $\frac{\boldsymbol{r}}{r^3}$ and $\boldsymbol{t}=\boldsymbol{k}\times\boldsymbol{n}$. Then, taking into account the expansion of any term of the form $\operatorname{curl}\left[\boldsymbol{a}\,\varphi(r)\right]$, where $\boldsymbol{a}$ is a constant vector, the integral of (26) can be written [4, p. 228]:

$$\boldsymbol{H} = J_{0L}\frac{1}{4\pi}\int_{S_{\text{lat}}}\operatorname{curl}\left(\boldsymbol{k}\times\boldsymbol{n}\frac{1}{r}\right)dS\,. \qquad (27)$$

If we denote $f(r)=r^{-1}$ we can obtain the result by applying formula (6) or (23), and we obtain immediately:

$$\boldsymbol{H}(N) = m\,\boldsymbol{k}\,J_{0L} + \frac{1}{4\pi}J_{0L}\int_{S_{\text{bases}}}\frac{\boldsymbol{r}}{r^3}\boldsymbol{k}\cdot d\boldsymbol{S}\,;$$
$$J_{0L} = \frac{i_0}{h_S}; \qquad (28)$$
$$m=1,\quad \forall\, N\in V_{\text{cyl}}\setminus\Sigma_{\text{cyl}};\quad m=0,\quad \forall\, N\notin V_{\text{cyl}}\,,$$

where the symbols above have been used. We recall: $\boldsymbol{k}$ – unit vector along the cylinder generatrix or axis; $\boldsymbol{r}$ – position vector of observation (field) point *N* with respect to the origin (source) point *P*, placed at any point of the bases surfaces. Therefore: If the point *N* is outside the cylinder, the first term of the right-hand side is zero, and if it is inside, it is different from zero.

In order to verify the accuracy of the established formula we have compared the results obtained by formulae (24) and (28). Generally, it is the formula (24) that is used in literature. Results in closed form can be obtained only for circular cross-section, and only for points having particular positions. For this reason, the results in any case may be obtained only by numerical computation. In order to verify the accuracy of formula (28), we have used both formulae for the same case. If both formulae ensure a good accuracy, they have to give results very close to each other. Having compared the results for various points placed at various positions with respect to the cylinder, we have found that the greatest relative deviation

has been of the order of magnitude of 10^{-5}, even at points in the neighborhood of singular points. Having verified the results at points for which the solution of the last integral in closed form is known, errors of the same order have been found. For using and compare both formulae, we have prepared and used programs in Fortran language [5], [6].

In electrical engineering the calculation of magnetic field strengths is necessary. The generally used methods are combinations of physical and mathematical considerations. For certain cases (like for the thought introduction of a magnetic field state quantity, at a very small scale), a mathematical pure deduction is very useful. Moreover, the obtained formula is in many cases easier for computation than the usual ones.

References

[1] R. Bricard, *Le calcul vectoriel.* Collection Armand Colin. Librairie Armand Colin: Paris, 1932.

[2] N. E. Kochin, *Vector Calculus and Introduction to Tensor Calculus,* Seventh edition. Publishing House of the Academy of Sciences: Moscow, 1951.

[3] P. M. Morse, H. Feshbach, *Methods of Theoretical Physics. Part I.* McGraw-Hill Book Company: New York, 1953.

[4] A. Nicolaide, *Electromagnetics. General Theory of the Electromagnetic Field. Appendix 1, Vector Calculus.* Transilvania University Press: Brasov, 2002.

[5] A. Nicolaide, "Establishing a general formula of the magnetic field strength of a solenoid," Académie Roumaine, Rev. Roum. Sci. Techn. – Électrotechn. et Énerg., **49**, *3,* pp. 295-308, Bucharest, 2004.

[6] A. Nicolaide, Fortran computer programs SOLENBSL.F77, based on the Biot-Savart-Laplace formula, and SOLENC.F77, based on the transformed integral, for computing the magnetic field strength. Program package. Research and Modelling Electrotechnical Laboratory. "Transilvania" University of Brasov, 2006.