Common Fixed Point Theorems of Gregus Type in a Complete Metric Space

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Abstract—**In this paper, we consider the existence of common fixed points for a compatible pair of self maps of Gregus type defined on a complete linear metric space. We also establish the existence of common fixed points for these pair of compatible maps of type (B) and consequently type (A). The results are generalisations and extensions of the results of several authors.**

Keywords: linear metric space, common fixed point, affine map, compatible maps, compatible maps of type (A), compatible maps of type (B)

1 Introduction

Fisher and Sessa [5] generalised Gregus' fixed point Theorem [6] by proving the following.

THEOREM 1.1 [5]. *Let T and I be two weakly commuting self maps of a closed convex subset K of a Banach X satisfying the inequality*

$$\|Tx-Ty\| \leq a\|Ix-Iy\|+(1-a)max\{\|Tx-Ix\|, \|Ty-Iy\|\}$$

for all $x, y \in K$, where $0 < a < 1$. If I is linear and nonexpansive in K such that $TK \subseteq IK$, then T and I have a unique common fixed point.

If I is an identity map, we have an immediate generalisation of the Gregus fixed point Theorem [6]. In [12], the author generalised the Gregus fixed point Theorem to when the underlying space is a complete metrisable locally convex space. Diviccaro et al. [3] generalised the Gregus fixed point Theorem to when $0 < a < \frac{1}{2^{p-1}}$, $p \geq 1$. Mukherjee and Verma [11] generalised Theorem 1.1 by replacing the linearity of I with a more general condition that I is affine, while Jungck [7] generalised it further by replacing the commutativity and nonexpansiveness assumptions in the Theorem with compatibility and continuity respectively. Murthy et al. [10] improved on the result by replacing nonexpansiveness, linearity and weak commutativity with continuity and compatibity of type (A). Sing et al. [17] recently generalised the Theorem 1.1 to set-valued mappings.

This paper deals with a triple generalisation of the Theorem of Fisher and Sessa (Theorem 1.1). First, the underlying space is generalised from Banach spaces to complete linear metric spaces, which include Banach spaces and complete metrisable locally convex spaces. Second, we introduce $d(Tx, Iy)$ and $d(Ty, Iy)$ to the distances under consideration. Thirdly, the linearity of I is replaced with a more general assumption that I is affine. Furthermore, in section 3, we prove the existence of common fixed points with the same pair of maps which are compatible of type (B) and consequently true for comaptible maps of type (A).

The Theorem will also be a partial generalisation of the recently proved result by the author and Akewe [13] which states thus:

THEOREM 1.2 [13]. *Let K be a closed convex subset of a complete linear metric space (X, d) and $T : K \to K$ a mapping that satisfies $d(Tx, Ty) \leq ad(x, y) + bd(x, Tx) + cd(y, Ty) + ed(y, Tx) + fd(x, Ty)$ for all $x, y \in K$ where $0 < a < 1$, $b \geq 0$, $c \geq 0$, $e \geq 0$, $f \geq 0$ and $a + b + c + e + f = 1$. Then T has a unique fixed point.*

Since the four points $\{x, y, Tx, Ty\}$ determine six distances in X, in view of Theorem 1.2, it is logical and natural to extend Theorem 1.1 to include $d(Tx, Iy)$ and $d(Ty, Ix)$

Definition 1. Two mappings T and I defined on a metric space (X, d) into itself is said to be weakly commuting if

$$d(TIx, ITx) \leq d(Ix, Tx)$$

for all $x,\ y\ \in X$ [16]. Two commuting maps are weakly commuting but the converse is generally not true [5].

In 1986, Jungck [7] introduced a generalisation of weakly commuting maps which he termed compatible maps.

Definition 2 [7]. Two selfmaps T and I are said to be

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compatible if

$$lim_{n\to\infty}F(ITx_n - TIx_n) = 0$$

whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty}Tx_n = lim_{n\to\infty}Ix_n = t$ for some $t \in X$.
It is clear that every weakly commuting pair of maps is compatible but the converse is not true [7].
Definition 3. A mapping $I : K \to K$ is called affine (where K is a convex set) if $I(ax + by) = aIx + bIy$ for all $x, y \in K$ and $a, b \geq 0$ with $a + b = 1$.
A linear map is an affine map but the converse need not be true as we will show later with an example.

The following Theorem will be needed for our result.

THEOREM 1.3 [1]. *A linear topological space X is metrisable if and only if it has a countable base of neighbourhoods of zero. The topology of a linear metric space can always be defined by a real-valued function $F : X \to \Re$, called $F-norm$ such that for all $x, y \in X$ and scaler K, we have*
(1). $F(x) \geq 0$,
(2). $F(x) = 0 \Rightarrow x = 0$,
(3). $F(x + y) \leq F(x) + F(y)$,
(4). $F(\lambda x) \leq F(x)$ for all $\lambda \in K$ with $|\lambda| \leq 1$,
(5). If $\lambda_n \to 0$, and $\lambda_n \in K$, then $F(\lambda_n x) \to 0$

Henceforth, unless otherwise indicated, F shall denote an F-norm if it is characterising a linear metric space. Observe that an F-norm will be a norm if it is defining a normed linear space.

This is our Gregus type condition which we will call condition (*):

$$\begin{aligned} F(Tx - Ty) \quad \leq \quad & aF(Ix - Iy) + (1 - a) \\ & max\{F(Tx - Ix), F(Ty - Iy), \\ & cF(Tx - Iy), bF(Ty - Ix)\} \end{aligned}$$

for all $x, y \in K$ where $0 < a < 1$, $0 \leq b < 1$, and $0 \leq c \leq \frac{1}{2}$.
Observe that condition (*) is equivalent to the following:

$$\begin{aligned} F(Tx - Ty) \quad \leq \quad & aF(Ix - Iy) + (1 - a) \\ & max\{F(Tx - Ix), F(Ty - Iy), \\ & \tfrac{F(Tx-Iy)}{2}, bF(Ty - Ix)\} \end{aligned}$$

for all $x, y \in K$ where $0 < a < 1$ and $0 \leq b < 1$.
We will use the second version of condition (*) throughout this paper.
Observe that if $b = c = 0$, and X is restricted to Banach spaces, condition (*) reduced to the pair of maps considered by Fisher and Sessa in [5] (Theorem 1.1).

We now prove our main theorem. Our technique which was originally due to Gregus [6] has been used by several authors, e.g. see [2-13].

2 Main Results

PROPOSITION 2.1. *Let T and I be selfmaps of X which are compatible and satisfying condition (*). If I is continuous, then $Tv = Iv$ for some $v \in X$ if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : F(Ix - Tx) \leq \frac{1}{n}\}$.*
Proof. Suppose $Tv = Iv$ for some $v \in X$. Then $v \in K_n$ for all n and thus $Tv \in TK_n \subseteq \overline{TK_n}$ for all n. Hence $Tv \in A$ so that A is nonempty.
Conversely, suppose $A \neq \phi$. If $v \in A$ for each n, then there is a $y_n \in TK_n$ such that $F(v - y_n) < \frac{1}{n}$. Hence for each n, there is a $x_n \in K_n$ such that $y_n = Tx_n$ and $F(v - Tx_n) < \frac{1}{n}$ for all n and so $Tx_n \to v$ as $n \to \infty$. Since $x_n \in K_n$, we have $F(Ix_n - Tx_n) \leq \frac{1}{n}$. Thus,

$$lim_{n\to\infty}Ix_n = lim_{n\to\infty}Tx_n = v. \tag{1}$$

Since T and I are compatible mappings, we have

$$F(ITx_n - TIx_n) \to 0 \ as \ n \to \infty. \tag{2}$$

Since I is continuous, from (1), we have that

$$I^2x_n, \ TIx_n, \ ITx_n \to Iv \ as \ n \to \infty. \tag{3}$$

Taking $x = v$ and $y = Ix_n$ in (*), we have

$$\begin{aligned} F(Tv - TIx_n) \quad \leq \quad & aF(Iv - IIx_n) + (1 - a) \\ & max\{F(Tv - Iv), F(TIx_n - I^2x_n), \\ & \tfrac{F(Tv-I^2x_n)}{2}, bF(TIx_n - Iv)\} \end{aligned}$$

As $n \to \infty$, and using (3), we have

$$\begin{aligned} F(Tv - Iv) \quad \leq \quad & aF(Iv - Iv) + (1 - a)max \\ & \{F(Tv - Iv), F(Iv - Iv), \\ & \tfrac{F(Tv-Iv)}{2}, bF(Iv - Iv)\} \\ = \quad & (1 - a)F(Tv - Iv) \end{aligned}$$

Thus $Tv = Iv$ since $1 - a < 1$.

THEOREM 2.2.- *Let T and I be two compatible self maps of a closed convex subset K of a complete linear metric space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point.*
PROOF.- Suppose $x = x_o$ is an arbitrary point in X and x_1, x_2, x_3 are chosen in K such that
$Ix_1 = Tx$, $Ix_2 = Tx_1$, $Ix_3 = Tx_2$.
This is possible since $TK \subseteq IK$. For $r = 1, 2, 3$ and using

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condition (*), we have

$$\begin{aligned}F(Tx_r - Ix_r) &= F(Tx_r - Tx_{r-1})\\ &\leq aF(Ix_r - Ix_{r-1}) + (1-a)\\ &\quad max\{F(Tx_r - Ix_r),\\ &\quad F(Tx_{r-1} - Ix_{r-1}),\\ &\quad \tfrac{F(Tx_r - Ix_{r-1})}{2}, bF(Tx_{r-1} - Ix_r)\}\\ &\leq aF(Ix_r - Ix_{r-1}) + (1-a)\\ &\quad max\{F(Tx_r - Ix_r),\\ &\quad F(Tx_{r-1} - Ix_{r-1}),\\ &\quad \tfrac{F(Tx_r - Ix_r + Tx_{r-1} - Ix_{r-1})}{2}\}\end{aligned}$$

It is easy to check that in all cases $F(Tx_r - Ix_r) \leq F(Tx_{r-1} - Ix_{r-1})$ and consequently

$$F(Tx_r - Ix_r) \leq F(Tx - Ix) \qquad (4)$$

for $r = 1, 2, 3$. In view of (4), and the condition (*), we have

$$\begin{aligned}F(Tx_2 - Tx) &\leq aF(Ix_2 - Ix) + (1-a)\\ &\quad max\{F(Tx_2 - Ix_2), F(Tx - Ix),\\ &\quad \tfrac{F(Tx_2 - Ix)}{2}, bF(Tx - Ix_2)\}\\ &\leq aF(Tx_1 - Ix_1 + Tx - Ix)\\ &+ (1-a)max\{F(Tx - Ix),\\ &\quad \tfrac{F(Tx_2 - Tx) + F(Tx - Ix)}{2},\\ &\quad bF(Ix_1 - Tx_1)\}\end{aligned}$$

On further computing, we have

$$F(Tx_2 - Tx) = F(Tx_2 - Ix_1) \leq \frac{3a+1}{a+1}F(Tx - Ix) \quad (5)$$

Let us define a point z by

$$z = \frac{1}{2}x_2 + \frac{1}{2}x_3.$$

By the convexity of K, $z \in K$ and by the fact that I is affine, we have

$$Iz = \frac{1}{2}Ix_2 + \frac{1}{2}Ix_3 = \frac{1}{2}Tx_1 + \frac{1}{2}Tx_2.$$

Thus

$$\begin{aligned}F(Tz - Iz) &\leq \tfrac{1}{2}F(Tz - Tx_1) + \tfrac{1}{2}F(Tz - Tx_2)\\ &\leq \tfrac{1}{2}\{aF(Iz - Ix_1) + (1-a)\\ &\quad max\{F(Tz - Iz), F(Tx_1 - Ix_1),\\ &\quad \tfrac{F(Tz - Ix_1)}{2}, bF(Tx_1 - Iz)\}\\ &+ \tfrac{1}{2}\{aF(Iz - Ix_2) + (1-a)\\ &\quad max\{F(Tz - Iz), F(Tx_2 - Ix_2\\ &\quad \tfrac{F(Tz - Ix_2)}{2}, bF(Tx_2 - Iz)\} \qquad (**)\end{aligned}$$

In view of (5), we have

$$\begin{aligned}F(Iz - Ix_1) &\leq \tfrac{1}{2}F(Tx_1 - Ix_1) + \tfrac{1}{2}F(Tx_2 - Ix_1)\\ &\leq \tfrac{1}{2}F(Tx - Ix) + \tfrac{3a+1}{2(a+1)}F(Tx - Ix)\end{aligned}$$

Hence

$$F(Iz - Ix_1) \leq \frac{2a+1}{a+1}F(Tx - Ix) \qquad (6)$$

Similarly, in view of (4)

$$F(Iz - Tx_1) \leq \frac{1}{2}F(Ix_3 - Ix_2) \leq \frac{1}{2}F(Tx - Ix) \qquad (7)$$

Using (5), we have

$$\begin{aligned}F(Tz - Ix_1) &= F(Tz - Iz + Iz - Ix_1)\\ &\leq F(Tz - Iz) + F(\tfrac{1}{2}Tx_1 - \tfrac{1}{2}Ix_1)\\ &+ F(\tfrac{1}{2}(Tx_2 - \tfrac{1}{2}Ix_1)\\ &\leq F(Tz - Iz) + \tfrac{1}{2}F(Tx - Ix)\\ &+ \tfrac{3a+1}{2(a+1)}F(Tx - Ix)\\ &\leq F(Tz - Iz) + \tfrac{2a+1}{a+1}F(Tx - Ix)\end{aligned}$$

Thus

$$F(Tz - Ix_1) \leq F(Tz - Iz) + \frac{2a+1}{a+1}F(Tx - Ix) \quad (8)$$

It is also easy to show that

$$F(Tx_1 - Iz) \leq \frac{1}{2}F(Ix_2 - Tx_2) \leq \frac{1}{2}F(Tx - Ix) \quad (9)$$

Similarly, it is easy to check that

$$F(Tz - Ix_2) \leq (Tz - Iz) + \frac{1}{2}F(Tx - Ix) \qquad (10)$$

and

$$F(Tx_2 - Iz) \leq \frac{1}{2}F(Tx - Ix) \qquad (11)$$

It therefore follows from (4) and (**), that

$$\begin{aligned}F(Tz - Iz) &\leq \tfrac{a}{2}(\tfrac{2a+1}{a+1})F(Tx - Ix) + \tfrac{a}{2}(\tfrac{1}{2}F(Tx - Ix))\\ &+ \tfrac{1-a}{2}max\{F(Tz - Iz), F(Tx_1 - Ix_1),\\ &\quad \tfrac{F(Tz - Ix_1)}{2}, bF(Tx_1 - Iz)\}\\ &+ \tfrac{1-a}{2}max\{F(Tz - Iz), F(Tx_2 - Ix_2),\\ &\quad \tfrac{F(Tz - Ix_2)}{2}, F(Tx_2 - Iz)\}\\ &\leq \tfrac{5a^2+3a}{4(a+1)}F(Tx - Ix) + (1-a)max\\ &\quad \{F(Tz - Iz), F(Tx - Ix), \tfrac{F(Tz - Ix_1)}{2},\\ &\quad bF(Tx_1 - Iz),\\ &\quad \tfrac{F(Tz - Ix_2)}{2}, bF(Tx_2 - Iz)\}\\ &\leq \tfrac{5a^2+3a}{4(a+1)}F(Tx - Ix) + (1-a)max\\ &\quad \{F(Tz - Iz), F(Tx - Ix), \tfrac{F(Tz - Iz)}{2}\\ &+ \tfrac{2a+1}{2(a+1)}F(Tx - Ix), \tfrac{bF(Tz - Iz)}{2}\\ &+ \tfrac{bF(Tx - Ix)}{4}\}\end{aligned}$$

Since $\frac{F(Tz-Iz)}{2} + \frac{2a+1}{2(a+1)}F(Tx - Ix) > \frac{bF(Tz-Iz)}{2} + \frac{bF(Tx-Ix)}{4}$, we now consider the following possibilities. Suppose that maximum is $F(Tz - Iz)$, then

$$\begin{aligned}F(Tz - Iz) &\leq \tfrac{5a^2+3a}{4(1+a)}F(Tx - Ix) + (1-a)F(Tz - Iz)\\ &\leq \tfrac{5a+3}{4(1+a)}F(Tx - Ix)\end{aligned}$$

Suppose the maximum is $F(Tx - Ix)$

$$\begin{aligned}F(Tz - Iz) &\leq (\tfrac{5a^2+3a}{4(a+1)} + 1 - a)F(Tx - Ix)\\ &= \tfrac{a^2+3a+4}{4(a+1)}F(Tx - Ix)\end{aligned}$$

Suppose the maximum is $\frac{F(Tz-Iz)}{2}+\frac{2a+1}{2(a+1)}F(Tx-Ix)$, then

$$\begin{aligned} F(Tz-Iz) &\leq \tfrac{5a^2+3a}{4(a+1)}F(Tx-Ix)+(1-a) \\ &\quad \{\tfrac{F(Tz-Iz)}{2}+\tfrac{2a+1}{2(a+1)}F(Tx-Ix)\} \end{aligned}$$

After computation, it yields

$$F(Tz-Iz)\leq \frac{a^2+5a+2}{2(a+1)^2}F(Tx-Ix)$$

In view of the three possibilities, it is clear that

$$F(Tz-Iz)\leq \lambda F(Tx-Ix)$$

where

$$\lambda = max\{\frac{5a+3}{4(1+a)},\frac{a^2+3a+4}{4(a+1)},\frac{a^2+5a+2}{2(a+1)^2}\}<1.$$

Hence, since x is arbitrary, we have

$$inf\{F(Tx-Ix):x\in K\}=0.$$

Each of the sets

$$K_n=\{x\in K:F(Tx-Ix)\leq \frac{1}{n}\}$$

and

$$H_n=\{x\in K:F(Tx-Ix)\leq \frac{a+1}{an}\}$$

for $n=1,2,..$ must therefore be nonempty and clearly

$$...\overline{K_n}\subseteq ..\overline{K_2}\subseteq \overline{K_1}.$$

Thus each of the sets $\overline{TK_n}$, where $\overline{TK_n}$ denotes the closure of TK_n, is nonempty for $n=1,2,...$ and

$$...\overline{TK_n}\subseteq ..\overline{TK_2}\subseteq \overline{TK_1}$$

For an arbitrary $x,y\in K_n$,

$$\begin{aligned} F(Tx-Ty) &\leq aF(Ix-Iy)+(1-a) \\ &\quad max\{F(Tx-Ix),F(Ty-Iy), \\ &\quad \tfrac{F(Tx-Iy)}{2},bF(Ty-Ix)\} \end{aligned}$$

Case 1: Suppose $F(Tx-Ix)$ or $F(Ty-Iy)$ is the maximum. Then

$$\begin{aligned} F(Tx-Ty) &\leq a\{F(Ix-Tx)+F(Tx-Ty) \\ &+ F(Ty-Iy)\}+\tfrac{1-a}{n} \\ &\leq \tfrac{a+1}{n}+aF(Tx-Ty) \end{aligned}$$

and so,

$$F(Tx-Ty)\leq \frac{a+1}{(1-a)n} \tag{12}$$

Case 2: Suppose $\frac{1}{2}F(Tx-Iy)=\frac{1}{2}\{F(Tx-Ix)+F(Ix-Iy)\}$ is the maximum, then,

$$\begin{aligned} F(Tx-Ty) &\leq aF(Ix-Iy)+\tfrac{1-a}{2}F(Tx-Ix) \\ &+ \tfrac{1-a}{2}F(Ix-Iy) \\ &\leq \tfrac{a+1}{2}F(Ix-Iy)+\tfrac{1-a}{2n} \\ &\leq \tfrac{a+1}{2}\{F(Ix-Tx)+F(Tx-Ty) \\ &+ F(Ty-Iy)\}+\tfrac{1-a}{2n} \end{aligned}$$

Hence on further computing, in view of the fact that $x,y\in K_n$, we have

$$F(Tx-Ty)\leq \frac{3-a}{2n(1-a)} \tag{13}$$

Case 3: Suppose $bF(Ty-Ix)=b\{F(Ty-Iy)+F(Iy-Ix)\}$ is the maximum. Similarly, as in case 2, it is clear that

$$F(Tx-Ty)\leq \frac{3-a}{bn(1-a)} \tag{14}$$

Combining equations (12), (13) and (14), it is clear that

$$F(Tx-Ty)\leq max\{\frac{a+1}{n(1-a)}\frac{3-a}{2n(1-a)},\frac{3-a}{bn(1-a)}\}$$

goes to 0 as $n\to 0$ for all $x,y\in K_n$. Thus

$$lim_{n\to\infty}diam(TK_n)=lim_{n\to\infty}diam(\overline{TK_n})=0.$$

Hence, by the well known Cantor's intersection Theorem, we have that the intersection $\cap_{n=1}^{\infty}\overline{TK_n}$ contains exactly one point v.
Thus from Proposition 2.1, we have that

$$Tv=Iv.$$

We claim that v is a common fixed point of T and I. If we take $x=v$ and $y=x_n$ in (*), we have

$$\begin{aligned} F(Tv-Tx_n) &\leq aF(Iv-Ix_n)+(1-a)max \\ &\quad \{F(Tv-Iv),F(Tx_n-Ix_n), \\ &\quad \tfrac{F(Tv-Ix_n)}{2},bF(Tx_n-Iv)\} \end{aligned}$$

As $n\to\infty$ and using (1) and (3), we have

$$\begin{aligned} F(Tv-v) &\leq aF(Tv-v)+(1-a)max\{F(Tv-Iv), \\ &\quad F(v-v), \\ &\quad \tfrac{F(Tv-v)}{2},bF(v-Tv)\} \\ &= a+(1-a)bF(Tv-v)<F(Tv-v) \end{aligned}$$

Thus $Tv=v$ since $a+(1-a)b<1$. Hence

$$Tv=Iv=v.$$

Thus v is the common fixed point of T and I. The uniqueness follows from the condition $(*)$.

COROLLARY 2.3.- *Let T and I be two compatible self maps of a closed convex subset K of a complete metrisable locally convex space X satisfying the condition (*). If I is affine and continuous on X such that $TK\subseteq IK$, then T and I have a unique common fixed point.*

COROLLARY 2.4.- *Let T and I be two compatible self maps of a closed convex subset K of a Banach space X satisfying the condition (*). If I is affine and continuous on X such that $TK\subseteq IK$, then T and I have a unique* *common fixed point.*

EXAMPLE. Let X be the set of all real (or complex) valued functions continuous on $(-\infty,\infty)$ and denoted $C(-\infty,\infty)$. It forms a locally convex space under the topology defined by the seminorms $p_n(x) = sup_{-n\leq t\leq n}|f(t)|$, $(n = 1, 2, ...)$. This space is Haursdoff, metrisable and complete (thus complete linear metric space)but not normable [15, pp75,236]. Suppose $K{=}C[1,3]$ is the space of all real (or complex) valued functions continuous on $[1,3]$ with the same metric topology. Let
$Tf = \frac{f}{f+1} + \frac{1}{2}$ and $If = \frac{f}{2} + \frac{1}{2}$.
It is easy to check that I is affine and continuous.
$TK = C[1,\frac{5}{4}] \subseteq IK = C[1,2]$.

$$\begin{aligned} F(TIf, ITf) &= \tfrac{f+1}{f+3} + \tfrac{1}{2} - \tfrac{f}{2f+2} - \tfrac{1}{4} \\ &= \tfrac{f+1}{f+3} - \tfrac{f}{2f+2} + \tfrac{1}{4} \\ &\leq \tfrac{f^2-2f+1}{2(f+1)} \leq \tfrac{f^2-x}{2(f+1)} = d(If,Tf), f \geq 1 \end{aligned}$$

Therefore, T and I are compatible.
Since $Tf - Tg \leq \frac{f-g}{(f+1)(g+1)}$, then, for $n \geq 1$, we have
$F(Tf - Tg) \leq \frac{1}{4}F(f-g) \leq \frac{1}{2}\frac{F(f-g)}{2} = \frac{1}{2}F(If - Ig)$
If $a = \frac{1}{2}$ in the condition (*), in view of the fact that all the $f, g \geq 1\ \forall\ x, y \in K$, then,

$$\begin{aligned} &\tfrac{F(f-g)}{2} + \tfrac{1}{2}max\{F(Tf - If), F(Tg - Ig), \\ &\tfrac{1}{2}F(Tf - Ig), F(Tg - If)\} \\ \leq\ &\tfrac{F(g-f)}{2} + \tfrac{F(f-g)}{2} = F(If - Ig) \end{aligned}$$

Thus all the conditions of our Theorem are satisfied. It is easy to see that $f(K) = 1$ is the common fixed point of T and I. It should also be noted that the map I, though affine, is not linear.

A combination of Proposition 2.1 and Theorem 2.2 yield the following Theorem.

THEOREM 2.5.- *Let T and I be two compatible self maps of a closed convex subset K of a complete linear metric space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : F(Ix - Tx) \leq \frac{1}{n}\}$.*

3 Compatible mappings of Type (A), Compatible maps of type (B) and common fixed points

Definition [9]. Two maps I and T are said to be compatible of type (A) if

$$lim_{n\to\infty}F(ITx_n, T^2x_n) = 0 \qquad (15)$$

and

$$lim_{n\to\infty}F(TIx_n, S^2x_n) = 0, \qquad (16)$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.
Weakly commuting maps are compatible of type (A) but the converse is not true [9]. However, compatible maps and compatible maps of type (A) are independent [9].
Definition 5 [15]. Two maps I and T said to be compatible of type (B) if
$lim_{n\to\infty}F(ITx_n, T^2x_n) \leq$
$\frac{1}{2}[lim_{n\to\infty}F(ITx_n, It) + lim_{n\to\infty}F(It, I^2x_n)]$
and
$lim_{n\to\infty}F(TIx_n, I^2x_n)$
$\leq \frac{1}{2}[lim_{n\to\infty}F(TIx_n, Tt)$
$+ lim_{n\to\infty}F(Tt, T^2x_n)]$,
whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Compatible maps of type (A) are compatible maps of type (B) but the converse is not true [15]. However, compatibility, compatibility of type (A) and compatibility of type (B) are all equivalent if I and T are continuous [14].

PROPOSITION 3.1[14]. *Suppose two maps I and T are compatible of type (B) such that $\lim_{n\to\infty} Ix_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. If I is continuous at t, then $lim_{n\to\infty}T^2x_n = It$.*

The proof of the following proposition is essentially the same as Proposition 2.1 in view of Proposition 3.1.

PROPOSITION 3.2. *Let T and I be selfmaps of X which are compatible maps of type (B) and satisfying condition (*). If I is continuous, then $Tv = Iv$ for some $v \in X$ if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : F(Ix - Tx) \leq \frac{1}{n}\}$.*

The proof of the following Theorem follows the same argument as that of Theorem 2.2 and Proposition 3.2.

THEOREM 3.3.- *Let T and I be two compatible self maps of type (B) defined on a closed convex subset K of a complete linear metric space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point in X.*

COROLLARY 3.4.- *Let T and I be two compatible self maps of type (B) defined on a closed convex subset K of a complete locally convex space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point in X.*

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COROLLARY 3.5.- *Let T and I be two compatible self maps of type (B) defined on a closed convex subset K of a Banach space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point in X.*

THEOREM 3.5.- *Let T and I be two compatible self maps of type (B) defined on a closed convex subset K of a complete linear metric space X satisfying the condition (*). If I is affine and continuous on X such that $TK \subseteq IK$, then T and I have a unique common fixed point if and only if $A = \cap\{\overline{TK_n} : n \in N\} \neq \phi$, where $K_n = \{x \in X : F(Ix - Tx) \leq \frac{1}{n}\}$.*

REMARK. Babu and Prasad [2] proved similar results for Ciric's contraction type condition which is defined thus: there exists real number a, b, c with $0 < a < 1$, $b \geq 0$, $a + b = 1$, $0 \leq c < \eta$ such that

$$\begin{aligned} \|Tx - Ty\| &\leq amax\{\|Ix - Iy\|, c[\|Ix - Ty\| \\ &+ \|Iy - Tx\|]\} + bmax\{\|Ix - Tx\|, \\ & \|Iy - Ty\|\} \end{aligned}$$

for all $x, y \in X$, where $\eta < \frac{1}{2}$. However, we note that their results are restricted to Banach spaces and our Gregus type condition (*) is independent of their Ciric type condition.

OPEN PROBLEM

Jungck and Rhoades [8] defined I and T to be weakly compatible if they commute at their concident points; i.e., if $Iu = Tu$ for some $u \in X$, then $ITu = TIu$. Can we replace the compatibility of type (B) of I and T in our Theorems with a more general weakly compatibility?.

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