On Multidimensional Analog of Kendall-Takacs Equation and its Numerical Solution

Gh. Mishkoy, member of the Academy of Sciences of Moldova *

Abstract—**A class of priority queueing systems with non-zero switching times, referred to as generalized priority systems, is considered and discussed. Analytical results regarding the distribution of busy periods and various auxiliary characteristics are presented. The algorithms of the numerical solution of recurrent functional equations which appear in the analysis of the characteristics of such systems are developed. Some examples are considered and numerical solution are presented.**

Keywords: numerical algorithms, priority, switchover time.

1 Introduction

The mathematical models of Queueing Systems, in particular, Priority Queueing Systems, play an important role in the analysis, modeling and design of various modern networks and their components [1]. Below we will present and discuss some results regarding a class of priority queueing systems with non-zero switching time, referred to as generalized priority systems [2]. This class of systems appeared as a result of mathematical formalization and consideration of the switchover times between priority classes and strategies in the free states. The assumption of non-zero switchings of the service process allows one to take into consideration the various time losses and delays existing in real time systems. Its consideration and analysis is very important from the applied point of view, especially at a stage of modeling and designing. It has appeared that generalized priority systems are more adecuate to real processes taking place in real time systems than classical priority systems. On the other hand, the consideration of the switchover times and their formalization inevitably leads to the appearance of a number of new important features in the elaboration of priority queueing models. Among these features we shall point out the appearance of new priority disciplines more flexible than traditional ones. From the theoretical point of view the elaboration and study of the generalized priority systems correspond to the intrinsic logic of the development of the Priority Queueing Systems Theory. Namely, the results regarding generalized models are, naturally, more general and contain as particular cases many of the results already obtained in the theory of priority systems and in the classical Queueing Theory (see, for example [3]-[5]). However, the obtained theoretical results have a rather complicated mathematical structure. For example, presented below the multidimensional analog of famous Kendall (Kendall-Takacs) equation represents a system of n recurrent functional equations in the terms of Lapace-Stieltjes Transforms (LTS). To use these results in application it is necessary not only to solve the named system for any priority class (that itself is a complicated problem, as the given systems has no exact analytical solution), but also to invert numerically the obtained solution. Except for that, the mentioned system is involved in the core of other important characteristics of the evolution of the generalized systems. The evaluation of such characteristics potentially makes use of many advanced fields and techniques of modern mathematics. In what follows we present the algorithms of LST numerical evaluation of busy periods and related characteristics.

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2 Preliminary results. Kendall-Takacs equation

Let's consider the queueing systems $M|G|1$ with an exhaustive service. Denote by λ the parameter of input Poisson flow, by $B(x) = P\{B < x\}$ the distribution function of service, by $\beta(s) = \int_0^\infty e^{-sx} dB(x)$ and by $\beta_1 = \int_0^\infty x dB(x)$ its LST and first moment, respectively. Consider the busy period. Denote by $\Pi(x)$ the distribution function of the busy period, by $\pi(s)$ and π_1 the LST of $\Pi(x)$ and its first moment, respectively.

The following result is known as Kendall-Takacs functional equation.

The LST of the busy period is determined in the unique way from the functional equation

$$\pi(s) = \beta(s + \lambda - \lambda\pi(s)) \tag{1}$$

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If $\lambda\beta_1 < 1$, *then*

$$\pi_1 = \frac{\beta_1}{1-\lambda\beta_1} \tag{2}$$

Formulas (1) and (2) are referred to in most standard textbooks on Queueing Theory (see, for example[6],[7]).

3 Multidimensional analog of Kendall-Takacs equation

Let's consider a queuing system $M_n|G_n|1$ with n priority classes of messages.The moments of i-messages appearance represent a Piosson flow with parameter λ_i and the service times are random variables with the distribution functions $B_i(x)$, $i = 1,\dots,n$. The priority classes are numbered in the decreasing number of priorities, namely, it is assumed that i-messages have a higher priority than j-messages if $1 \le i,j \le n$. It is also assumed that the server needs some time to switching the service process from the queue i to queue j. The length of ij-switching is considered a random variable with the distribution function C_{ij}, $1 \le i,j \le n$, $i \ne j$. More details regarding the classification and nomenclature for the priority queuing systems with switchover times are presented in [2]. In what follows consider the preemptive priority discipline. Namely, suppose that the incoming messages of the higher priority than those presented in the system interrupt both the servicing and the switching. Regarding the further evolution of the interrupted servicing and switching three preemptive schemes will be considered:

P11 - "resume", "resume". The interrupted switching and the interrupted message (after the system is free of the higher priority messages) will be continued from the time point it was interrupted at;

P12 - "resume", "repeat again". The interrupted switching will be continued, the interrupted message will be serviced again;

P21 - "repeat again", "resume". The interrupted switching will be realized again, the interrupted message will be continued.

Also suppose that the switching C_{ij} depends only on index j, C_{ij}=C_j. The strategy in the free state is considered "reset". Denote by $\Pi_k(x)$ the distribution function of the busy period with the messages of the priority not less than k, $\Lambda_k = \lambda_1 + \ldots + \lambda_k$, $\Lambda = \Lambda_n$, $\Lambda_0 = 0$, $\beta_i(s)$, $c_j(s)$, $\pi(s)$,$\dots$, $\pi_k(s)$ are the LST of the distribution functions $B_i(x)$, $C_j(x)$, $\Pi(x)$,$\dots$,$\Pi_k(x)$, respectively.

The following results hold.

A. Preemptive priority policy P11

The LST $\pi(s) = \pi_n(s)$ *of the busy period for the generalized system with preemptive priority scheme P11 is the solution of the following system of the functional equations.*

$$\pi_k(s) = \frac{\Lambda_{k-1}}{\Lambda_k}\pi_{k-1}(s+\lambda_k)+\frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)])-$$
$$-\pi_{k-1}(s+\lambda_k))\nu_k(s+\lambda_k[1-\pi_{kk}(s)])+$$
$$+\frac{\lambda_k}{\Lambda_k}\nu_k(s+\lambda_k[1-\pi_{kk}(s)])\pi_{kk}(s), \tag{3}$$
$$\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]), \tag{4}$$
$$\nu_k(s) = c_k(s+\Lambda_{k-1}[1-\pi_{k-1}(s)]), \tag{5}$$
$$h_k(s) = \beta_k(s+\Lambda_{k-1}[1-\pi_{k-1}(s)\nu_k(s)]). \tag{6}$$

B. Preemptive priority policy P12

The LST $\pi(s) = \pi_n(s)$ *of the busy period for the generalized system with preemptive priority scheme P12 is the solution of the following system of the functional equations.*

$$\pi_k(s) = \frac{\Lambda_{k-1}}{\Lambda_k}\pi_{k-1}(s+\lambda_k)+\frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)])-$$
$$-\pi_{k-1}(s+\lambda_k))\nu_k(s+\lambda_k[1-\pi_{kk}(s)])+$$
$$+\frac{\lambda_k}{\Lambda_k}\nu_k(s+\lambda_k[1-\pi_{kk}(s)])\pi_{kk}(s),$$
$$\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]),$$
$$\nu_k(s) = c_k(s+\Lambda_{k-1}[1-\pi_{k-1}(s)]),$$
$$h_k(s) = \beta_k(s+\Lambda_{k-1})\{1-\frac{\Lambda_{k-1}}{s+\Lambda_{k-1}}[1-\beta_k(s+$$
$$+\Lambda_{k-1})]\pi_{k-1}(s)\nu_k(s)\}^{-1}.$$

C. Preemptive priority policy P21

The LST $\pi(s) = \pi_n(s)$ *of the busy period for the generalized system with preemptive priority scheme P21 is the solution of the following system of the functional equations.*

$$\pi_k(s) = \frac{\Lambda_{k-1}}{\Lambda_k}\pi_{k-1}(s+\lambda_k)+\frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s+\lambda_k[1-\pi_{kk}(s)])-$$
$$-\pi_{k-1}(s+\lambda_k))\nu_k(s+\lambda_k[1-\pi_{kk}(s)])+$$
$$+\frac{\lambda_k}{\Lambda_k}\nu_k(s+\lambda_k[1-\pi_{kk}(s)])\pi_{kk}(s),$$
$$\pi_{kk}(s) = h_k(s+\lambda_k[1-\pi_{kk}(s)]),$$
$$\nu_k(s) = c_k(s+\Lambda_{k-1})\{1-\frac{\Lambda_{k-1}}{s+\Lambda_{k-1}}[1-c_k(s+$$
$$+\Lambda_{k-1})]\pi_{k-1}(s)\}^{-1},$$
$$h_k(s) = \beta_k(s+\Lambda_{k-1})\{1-\frac{\Lambda_{k-1}}{s+\Lambda_{k-1}}[1-\beta_k(s+$$
$$+\Lambda_{k-1})]\pi_{k-1}(s)\nu_k(s)\}^{-1}.$$

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Remark 1. The functions $\nu_k(s)$, $h_k(s)$, $\pi_{kk}(s)$ can be viewed as auxiliary functions but yet they all have a clear informative meaning. Thus, $\nu_k(s)$ and $h_k(s)$ is LST of the complete switching to priority class k and complete service time of k-message, respectively.

Remark 2. The system of functional equations presented above (refered below as generalized system) can be viewed as the generalization of Kendall-Takacs equation(1). Namely, in [3] it is shown that for C_j=0 and the number of priority classes n=1, equation (1) follows from the mentioned system.

4 Numerical solution of generalized system

The analysis of generalized models has shown that many of important characteristics of their evolution, such as traffic intensity, queue length distribution, steady state conditions etc., involve the mentioned system of equations in their analytical expressions. Evaluating the enumerated characteristics leads to the study of the elaboration of methods and algorithms for numerical solution of generalized systems type (3)-(6). In what follows we present the algorithms of LST numerical solution of the k-busy periods $\pi_k(s)$ as well as the k-cycle of switching $\nu_k(s)$, the k-cycle of service $h_k(s)$, and kk-periods $\pi_{kk}(s)$.

A. Algorithm for the numerical solution of the system with priority policy P11-"resume", "resume".

Algorithm P11.

Input: r, s^*, $\epsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$.

Output:$\pi_k(s^*)$

Description: *IF (k==0) THEN* $\pi_0(s^*) := 0$;*RETURN* $k := 1$; $q := 1$; $\Lambda_0 := 0$;

Repeat

inc(q); $\Lambda_q := \Lambda_{q-1} + \lambda_q$;

Until $q == r$;

Repeat

$$\nu_k(s) := c_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)]);$$

$$h_k(s^*) := \beta_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)\nu_k(s^*)]);$$

$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$;

Repeat

$\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)})$;

inc(n);

Until $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \epsilon$;

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^* + \lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*)) - \pi_{k-1}(s^* + \lambda_k))\nu_k(s^* + \lambda_k[1 - \pi_{kk}(s^*)]) + \frac{\lambda_k}{\Lambda_k}\nu(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*))\pi_{kk}(s^*);$$

inc(k);

Until $k == r$;

End of Algorithm P11.

B. Algorithm for the numerical solution of the system with priority policy P12-"resume", "repeat again".

Algorithm P12.

Input: r, s^*, $\epsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$.

Output: $\pi_k(s^*)$

Description:*IF (k==0) THEN* $\pi_0(s^*) := 0$; *RETURN* $k := 1$; $q := 1$; $\Lambda_0 := 0$;

Repeat

inc(q);

$\Lambda_q := \Lambda_{q-1} + \lambda_q$;

Until $q == r$;

Repeat

$$\nu_k(s) := c_k(s^* + \Lambda_{k-1}[1 - \pi_{k-1}(s^*)]);$$

$$h_k(s^*) := \beta_k(s + \Lambda_{k-1})\{1 - \frac{\Lambda_{k-1}}{s^* + \Lambda_{k-1}}[1 - \beta_k(s^* + \Lambda_{k-1})]\pi_{k-1}(s^*)\nu_k(s^*)\}^{-1};$$

$\pi_{kk}^{(0)}(s^*) := 0$; $n := 1$;

Repeat

$\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)})$;

inc(n);

Until $|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \epsilon$;

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^* + \lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*)) - \pi_{k-1}(s^* + \lambda_k))\nu_k(s^* + \lambda_k[1 - \pi_{kk}(s^*)]) + \frac{\lambda_k}{\Lambda_k}\nu(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*))\pi_{kk}(s^*);$$

inc(k);

Until $k == r$;

End of Algorithm P12.

C. Algorithm for the numerical solution of the system with priority policy P21-"repeat again", "resume".

Algorithm P21.

Input: r, s^*, $\epsilon > 0$, $\{\lambda_k\}_{k=1}^r$, $\{\beta_k(s)\}_{k=1}^r$, $\{c_k(s)\}_{k=1}^r$.
Output: $\pi_k(s^*)$

Description: *IF (k==0)THEN* $\pi_0(s^*) := 0$; *RETURN*

$k := 1;q := 1;\ \Lambda_0 := 0;$

Repeat

inc(q); $\Lambda_q := \Lambda_{q-1} + \lambda_q$;

Until $q == r$;

Repeat

$$\nu_k(s) := c_k(s^* + \Lambda_{k-1})\{1 - \frac{\Lambda_{k-1}}{s^* + \Lambda_{k-1}}\cdot$$

$$\cdot [1 - c_k(s^* + \Lambda_{k-1})]\, \pi_{k-1}(s^*)\}^{-1};$$

$$h_k(s^*) := \beta_k(s + \Lambda_{k-1})\{1 - \frac{\Lambda_{k-1}}{s^* + \Lambda_{k-1}}[1 - \beta_k(s^*+$$

$$+\Lambda_{k-1})]\pi_{k-1}(s^*)\nu_k(s^*)\}^{-1};$$

$$\pi_{kk}^{(0)}(s^*) := 0, n := 1;$$

Repeat

$\pi_{kk}^{(n)}(s^*) := h_k(s^* + \lambda_k - \lambda_k \pi_{kk}^{(n-1)});$

$inc(k)$;

Until $k == r$;

$|\pi_{kk}^{(n)}(s^*) - \pi_{kk}^{(n-1)}(s^*)| < \epsilon;$

$$\pi_k(s^*) := \frac{\Lambda_{k-1}\pi_{k-1}(s^* + \lambda_k)}{\Lambda_k} + \frac{\Lambda_{k-1}}{\Lambda_k}(\pi_{k-1}(s^*+$$

$$+\lambda_k - \lambda_k\pi_{kk}(s^*)) - \pi_{k-1}(s^* + \lambda_k))\nu_k(s^* + \lambda_k[1 - \pi_{kk}(s^*)])$$

$$+\frac{\lambda_k}{\Lambda_k}\nu(s^* + \lambda_k - \lambda_k\pi_{kk}(s^*))\pi_{kk}(s^*);$$

$inc(k)$;

Until $k == r$;

End of Algorithm P21.

Remark 3. The Algorithms P11—P21 are convergent. The LST of k-busy period $\pi_k(s)$ and auxiliary characteristics ($\nu_k(s)$, $h_k(s)$ and $\pi_{kk}(s)$) are calculated with the precision ϵ. Quantity ϵ is used to determine on the convergence of the Cauchy sequence $\{\pi_{kk}^n(s^*)\}_{n=0}^{\infty}$. Improvement of the algorithms can be achieved using the acceleration scheme elaborated in [8].

5 Examples of numerical solution

A. The numerical results for priority policy P11

Example 1. Consider the system $M_7|G_7|1$ with all interarrival times distributed Exponentially, $Exp(10)$ and all service times being distributed Erlang, $Er(3, 200)$. The switchover times C_k are all distributed as $Exp(100)$, $k = 1, \ldots, 7$. The quantity ϵ was taken to be 0.000001. The solution is evaluated at point $s^* = 0.1$.

The numerical results for Example 1 and priority policy P11 are presented in Table I.

Table I.

k	$h_k(s)$	$\nu_k(s)$	$\pi_{kk}(s)$	$\pi_k(s)$
1	0.998501	0.999001	0.998238	0.997065
2	0.997870	0.998708	0.997295	0.995768
3	0.996687	0.998157	0.995065	0.992693
4	0.993816	0.996818	0.984686	0.978427
5	0.980222	0.990463	0.834719	0.775830
6	0.802164	0.898404	0.651123	0.539368
7	0.618285	0.782853	0.533606	0.399182

B. The numerical results for priority policy P11

Example 2. Consider the system $M_7|G_7|1$ with all interarrival times distributed exponentially, $Exp(1)$ and all service times being distributed $Exp(200)$. The switchover times C_k are all distributed as $Exp(100)$, $k = 1, \ldots, 7$. The quantity ϵ was taken to be 0.000001.The solution is evaluated at point $s^* = 0.1$.

The numerical results for Example 2 and priority policy P12 are presented in Table II.

Table II.

k	$h_k(s)$	$\nu_k(s)$	$\pi_{kk}(s)$	$\pi_k(s)$
1	0.999500	0.999001	0.999498	0.998494
2	0.999488	0.998986	0.999485	0.998469
3	0.999475	0.998970	0.999472	0.998443
4	0.999461	0.998954	0.999458	0.998415
5	0.999447	0.998938	0.999444	0.998387
6	0.999433	0.998921	0.999430	0.998358
7	0.999418	0.998903	0.999415	0.998328

C. The numerical results for priority policy P11

Example 3. Consider the system $M_7|G_7|1$ with interarrival times all distributed as $Exp(1)$, and with the times of service of the requests from the priority classes $k = 1$, $k = 2$ being distributed exponentially, $Exp(200)$, from the priority classes $k = 3$, $k = 4$ being distributed uniformly, U[0,1] and from the priority classes $k = 5$, $k = 6$ and $k = 7$ being distributed as $Er(3, 200)$. The switchover times C_k are all distributed as $Exp(100)$, $k = 1, \ldots, 7$. The quantity ϵ was taken to be 0.000001.The solution is evaluated at point $s^* = 0.1$.

The numerical results for Example 3 and priority policy P21 are presented in Table III.

Table III.

k	$h_k(s)$	$\nu_k(s)$	$\pi_{kk}(s)$	$\pi_k(s)$
1	0.999500	0.999001	0.999498	0.998494
2	0.999488	0.998986	0.999485	0.998469
3	0.010457	0.998970	0.099735	0.689462
4	0.099921	0.989789	0.169353	0.569836
5	0.972591	0.982119	0.972228	0.647840
6	0.971745	0.981732	0.971360	0.699508
7	0.970859	0.981326	0.970450	0.736105

Remark 4. Note that numerical solution presented in Tables I-III is obtained in the terms of LST. Namely, LST characteristic is required for the calculation of some key characteristics of the generalized system, such as traffic intensity, probabilities of states, blocking probabilities etc. For obtaining the distribution functions $N_k(x)$,...,$\Pi_k(x)$ themselves it is necessary to LST invert numerically, i.e. to solve numerically the integral equations of the type $\pi_k(s) = \int\limits_0^\infty e^{-sx} d\Pi_k(x)$.

References

[1] Vishnevsky V.M. *Theoretical bases of design of the computer communication networks.* Moscow, 2003 (in Russian).

[2] Mishkoy Gh., Giordano S., Andronati N., Bejan A. *Priority Queueing Systems with Switchover Times: Generalized Models for QoS and CoS Network Technologies and Analysis.* Technical report, 2006. web:http: www.vitrum.md andrew PQSST.pdf

[3] Mishkoy Gh. *An analog of Kendall-Takacs equation for generalized queueing systems.* Proceedings of the 31th Annual Congress of the American Roumanian Academy of Arts and Sciences, Brasov, Romania, 2007, pp.483-486.

[4] Mishkoy Gh.K., Rykov V.V., Giordano S., Bejan A.Iu. *Multidimensional analogs of the Kendall equation for priority queueing systems: computational aspects.* Automatics and Remote Control N.5, Moscow, 2008 (to be published).

[5] Mishkoy Gh., *A virtual analog of Pollaczek-Khintchin transform equation.* Bul. Acad. of Science of Moldova Mat., N.2, 2008 (to be published).

[6] Gnedenko B.V. and Kovalenko I.N. *Introduction to Queueing Theory.* Moscow, 2005 (in Russian).

[7] Takagi H. *Queueing Analysis.* volume 1: Vacation and Priority Systems. North-Hodland, Elsevier Science Publ., 1991.

[8] Bejan A. *Numerical treatment of the Kendall equation in the analysis of priority queueing systems* Bul. Acad. of Sc. of Moldova, Mat., 51(2), 2006, pp.17-28.