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A Computational Approach to the Fredholm Integral Equation of the Second Kind

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Abstract—**The Fredholm integral equation of the second kind is of widespread use in many realms of engineering and applied mathematics. Among the variety of numerical solutions to this equation, the qudrature method and its modification are remarkable. The latter aims at reducing the computational complexity of the quadrature method. In this paper, we present Mathematica programs that utilize the modified quadrature method to solve the equation.**

Keywords: Fredholm integral equation, Mathematica, modified quadrature method

1 Introduction

Integral equations are of high applicability in different areas of applied mathematics, physics, and engineering. In particular, they are widely used in mechanics, geophysics, electricity and magnetism, kinetic theory of gases, hereditary phenomena in biology, quantum mechanics, mathematical economics, and queuing theory.

As witnessed by the literature, the Fredholm integral equation of the second kind is one of the most practical ones. A number of numerical solutions, such as quadrature collocation, Galerkin expansion, product integration, deferred correction, graded mesh, sinc collocation, Trefftz's method, Taylor's series, tau interpolation, and decomposition method, have already been proposed to this equation. Nevertheless, an efficient low-cost solution to this equation has remained a scientific inquiry. In particular, the modification made to the quadrature method is still of high complexity [1-11].

The main contribution of this paper is to propose an algorithm for solving the second kind of the Fredholm integral equation so as to be easily implemented in Mathematica. This paper goes on as follows: Section 2 provides a brief outline of the quadrature method as it is used in solving the Fredholm integral equation of the second kind. Section 3 explains a modification made to the quadrature method. Our proposal for a mechanized solving process is given in Section 4. Section 5 illustrates this process and, finally, Section 6 concludes the paper.

2 The Quadrature Method

The Fredholm integral equation of the second kind (FK2) is given by

$$f(x) - \lambda \int_a^b k(x,y) f(y) dy = g(x), \tag{1}$$

where it is assumed that λ is a regular value of the kernel and that $k(x,y)$ and $g(x)$ are piece-wise continuous.

Assume that $\int_a^b \phi(y)dy$ is approximated by the quadrature rule $J(\phi) = \sum_{j=0}^{n} w_j \phi(y_j)$. By such an approximation, for $a \le x \le b$, (1) is reduced to

$$\tilde{f}(x) - \lambda \sum_{j=0}^{n} w_j k(x, y_j) \tilde{f}(y_j) = g(x), \tag{2}$$

where its solution $\tilde{f}(x)$ is an approximation of the exact solution $f(x)$ to (1). A solution to the functional equation (2) may be obtained if we assign y_i's to x in which $i = 0, 1, ..., n$ and $a \le y_i \le b$. In this way, (2) is reduced to the system of equations

$$\tilde{f}(y_i) - \lambda \sum_{j=0}^{n} w_j k(y_i, y_j) \tilde{f}(y_j) = g(y_i), \tag{3}$$

where $i = 0, 1, ..., n$. Now, assume that $\tilde{f}(y_0)$, $\tilde{f}(y_1)$,..., and $\tilde{f}(y_n)$ make a solution to (3). For any $x \in [a, b]$, a solution to (2) can then be obtained by

$$\tilde{f}(x) = \lambda \sum_{j=0}^{n} w_j k(x, y_j) \tilde{f}(y_j) + g(x). \tag{4}$$

Moreover, (2) can be represented by

$$(\mathbf{I} - \lambda \mathbf{KD})\tilde{\mathbf{f}} = \mathbf{g}, \tag{5}$$

where $\tilde{\mathbf{f}} = [\tilde{f}(y_i)]^T$, $\mathbf{g} = [g(y_i)]^T$, $\mathbf{K} = [k(y_i, y_j)]$, and $\mathbf{D} = \mathbf{diag}(w_0, w_1, ..., w_n)$. It is worth noting that $\mathbf{I} - \lambda\mathbf{KD}$ may be singular for a chosen quadrature rule $J(\phi)$. However, under mild restrictions, one can preserve the non-singularity of $\mathbf{I} - \lambda\mathbf{KD}$ if he decides on a sufficiently accurate $J(\phi)$. In addition, Whether a quadrature rule is sufficiently accurate or not itself depends on λ, $k(x,y)$, and $g(x)$.

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3 The Modified Quadrature Method

We can reduce the error in the quadrature method, even if $k(x, y)$ is of bad behavior. To do so, (1) can be rewritten as follows:

$$f(x) - \lambda f(x) \int_a^b k(x,y)dy - \lambda \int_a^b k(x,y)(f(y) - f(x))dy = g(x). \tag{6}$$

Then, by using the quadrature rule $J(\phi)$, we have

$$\hat{f}(x)(1 - \lambda A(x)) - \lambda \sum_{j=0}^{n} w_j k(x, y_j)(\hat{f}(y_j) - \hat{f}(x)) = g(x), \tag{7}$$

where

$$A(x) = \int_a^b k(x,y)dy.$$

Thus,

$$\hat{f}(x)(1 - \lambda\Delta(x)) - \lambda \sum_{j=0}^{n} w_j k(x, y_j)\hat{f}(y_j) = g(x) \tag{8}$$

with

$$\Delta(x) = \sum_{j=0}^{n} w_j k(x, y_j) - A(x).$$

By setting $x = y_i$, (8) is reduced to the system of equations

$$\hat{f}(x)(1 - \lambda\Delta(y_i)) - \lambda \sum_{j=0}^{n} w_j k(y_i, y_j)\hat{f}(y_j) = g(y_i), \tag{9}$$

or in matrix notation,

$$(\mathbf{I} + \lambda(\mathbf{\Delta} - \mathbf{KD}))\hat{\mathbf{f}} = \mathbf{g}, \tag{10}$$

where $\mathbf{\Delta} = \mathbf{diag}(\Delta(y_0), \Delta(y_1), ..., \Delta(y_n))$. A method based on (8) or (10) is called a *modified* quadrature method.

The approximate solution to (1) obtained from computing $\hat{\mathbf{f}}$ assumes that $k(x, y)$ is weakly singular and has a discontinuous derivative at $x = y$. In fact, $\hat{f}$ may yield more accurate solutions than $\tilde{f}$, even if $k(x, y)$ is of good behavior. In particular, if $k(x, y)$ has a *hump* at $x = y$, we prefer the method of this section to that of Section 2.

4 Algorithms for the Fredholm Integral Equation

The mathematical mechanization is the deployment of mathematics in a constructive and algorithmic manner so that the reasoning about systems can be made automated [12]. The underlying notion of mathematical mechanization is to design mathematical algorithms and, then, convert them into the code [13-19]. The *Mathematica* is a tool for such mechanization that enjoys form high capability in symbolic operations and numerical calculations. In this section, we first propose the algorithms that can be used in solving the Fredholm integral equation of the second kind. Then, we code the algorithms using Mathematica. By applying them to a variety of equations, it is also shown that the proposed algorithms are effective in the sense that they yield very accurate approximate solutions.

4.1 The Quadrature Method

The following algorithm implements the method of Section 2. This algorithm yields good results for a normal $k(x, y)$.

Algorithm 1.

input a, b, n, λ, $g(x)$, $k(x,y)$
$h \leftarrow (b-a)/n$
for $i = a$ *to* b *step* h *do*
  $\mathbf{G}_i = g(i)$
  $\mathbf{S}_i = i$
  for $j = a$ *to* b *step* h *do*
    $\mathbf{K}_{ij} = k(i,j)$
  end do
end do
$W_0 = W_n \leftarrow h/2$
for $i = 1$ *to* $n-1$ *do*
  $W_i \leftarrow h$
end do
$\mathbf{D} \leftarrow \mathbf{diag}(W_0, ..., W_n)$
$\mathbf{lhs} \leftarrow \mathbf{I} - \lambda\mathbf{KD}$
$q \leftarrow$ the answer of $\mathbf{lhs}\,\mathbf{z} = \mathbf{G}$
$p(x) \leftarrow$ the interpolating polynomial at $[s_i, q_i]$ for $0 \le i \le n$
output $p(x)$

For example, consider the following equation.

$$f(x) = -\frac{2}{\pi}\cos(x) + \frac{4}{\pi}\int_0^{\frac{\pi}{2}} \cos(x-y)f(y)dy. \tag{11}$$

The exact solution to this equation is $f(x) = \sin(x)$. To solve this equation numerically, we make use of a program that implements the quadrature method with the trapezoid rule. Figure 1 compares the exact solution with the approximate one when $n = 60$.

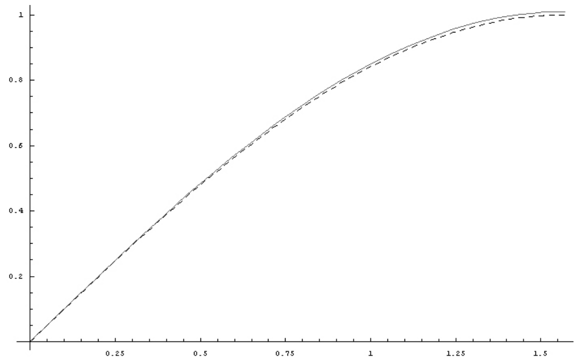

Figure 1: The solid line shows the exact solution, while the dashed line is the approximate one using the trapezoid rule.

4.2 The Modified Quadrature Method

The following algorithm modifies the Algorithm 1.

Algorithm 2.

input a, b, n, λ, $g(x)$, $k(x,y)$

$h \leftarrow (b-a)/n$

for $i = a$ *to* b *step* h *do*

  $\mathbf{G}_i = g(i)$

  $\mathbf{S}_i = i$

  for $j = a$ *to* b *step* h *do*

    $\mathbf{K}_{ij} = k(i,j)$

  end do

end do

$W_0 = W_n \leftarrow h/2$

for $i = 1$ *to* $n-1$ *do*

  $W_i \leftarrow h$

end do

$A(x) \leftarrow \int_a^b k(x,y)dy$

$delta(x) \leftarrow \sum_{j=0}^{n} W_j \mathbf{K}_{x,S_j} - A(x)$

$\omega \leftarrow delta(S_j)$ for $0 \le j \le n$

$\boldsymbol{\Delta} \leftarrow \mathbf{diag}(\omega_0, ..., \omega_n)$

$\mathbf{D} \leftarrow \mathbf{diag}(W_0, ..., W_n)$

$\mathbf{lhs} \leftarrow \mathbf{I} + \lambda(\boldsymbol{\Delta} - \mathbf{KD})$

$q \leftarrow$ the answer of $\mathbf{lhs}\,\mathbf{z} = \mathbf{G}$

$p(x) \leftarrow$ the interpolating polynomial at $[s_i, q_i]$ for $0 \le i \le n$

output $p(x)$

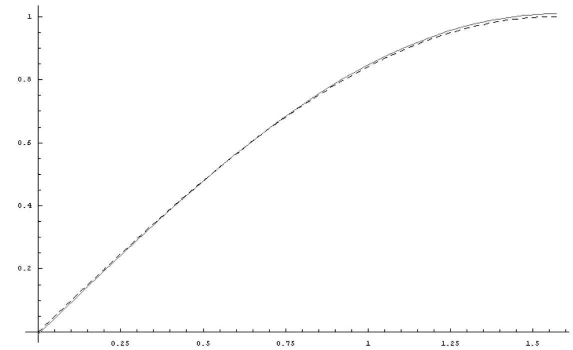

Figure 2: The results of applying the Algorithm 2 to (11). The solid and dashed lines are the exact and approximate solutions, respectively.

We apply the above algorithm to (11) where $n = 60$. The results are shown in Figure 2.

Now, consider the following equation with the exact solution $f(x) = \frac{2}{3}\sqrt{x}$.

$$f(x) + \int_0^1 \sqrt{xy} f(y)dy = \sqrt{x}. \tag{12}$$

Figure 3 shows the results of applying the Algorithm 2 to this equation with $n = 60$.

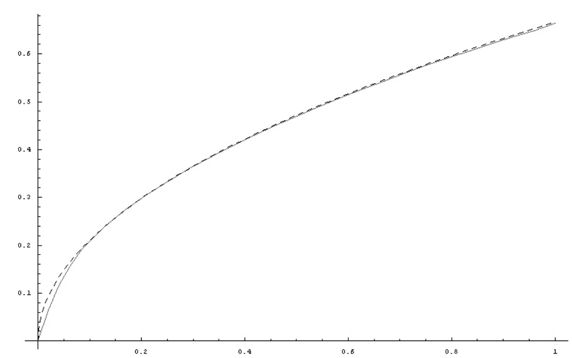

Figure 3: The results of applying the Algorithm 2 to (12). The solid and dashed lines are the exact and approximate solutions, respectively.

5 Examples

In this section, we give examples of the Fredholm integral equations of the second kind. These examples show that the Algorithm 2 yields good results for a variety of kernels.

Example 1. Consider the following equation.

$$f(x) = x + \int_{-1}^{1} xyf(y)dy. \tag{13}$$

The exact solution to this equation is $f(x) = 3x$. By applying the Algorithm 2 to (13) with $n = 40$, the maximum error is 0.0075. Figure 4 compares the exact solution with the approximate one.

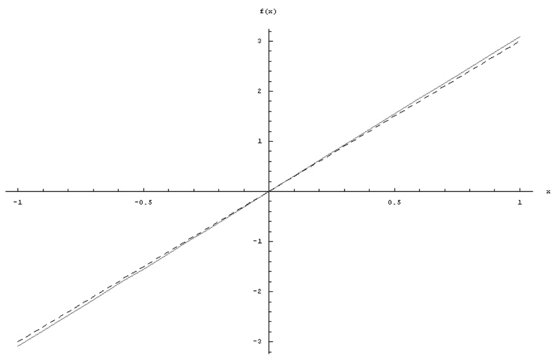

Figure 4: The results of applying the Algorithm 2 to (13). The solid and dashed lines are the exact and approximate solutions, respectively.

Example 2. Consider the following equation for $0 \leq x \leq 1$.

$$f(x) = x + \int_0^1 k(x,y)f(y)dy, \tag{14}$$

where

$$k(x,y) = \begin{cases} x, & x < y \\ y, & x \geq y \end{cases} . \tag{15}$$

The exact solution to this equation is $f(x) = \sec 1 \sin x$. By applying the Algorithm 2 to (15) with $n = 40$, the maximum error is 0.0002. Figure 5 compares the exact solution with the approximate one.

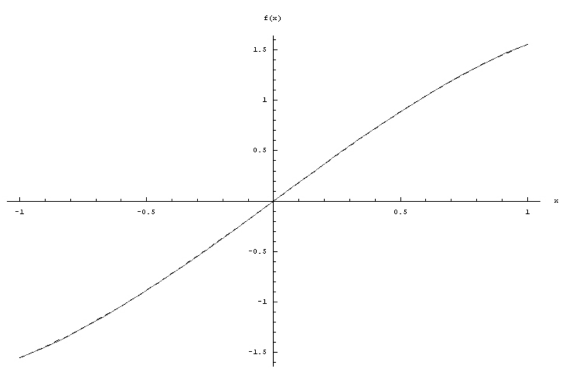

Figure 5: The results of applying the Algorithm 2 to (15). The solid and dashed lines are the exact and approximate solutions, respectively.

Example 3. Consider the equation

$$f(x) + \int_0^1 k(x,y)f(y)dy = xe + 1, \tag{16}$$

where

$$k(x,y) = \min(x,y).$$

The exact solution to this equation is $f(x) = \exp(x)$. By applying the Algorithm 2 to (16) with $n = 30$, the maximum error is 0.00034. Figure 6 compares the exact solution with the approximate one.

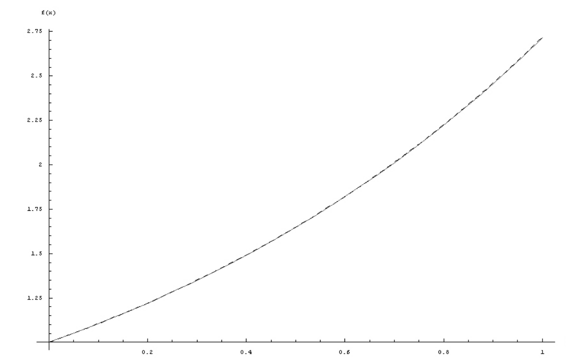

Figure 6: The results of applying the Algorithm 2 to (16). The solid and dashed lines are the exact and approximate solutions, respectively.

Example 4. Consider the equation

$$f(x) - \frac{1}{2}\int_{-1}^1 |x-y|f(y)dy = e^x. \tag{17}$$

The exact solution to this equation is $f(x) = \frac{1}{2}xe^x + c_1e^x + c_2e^{-x}$, where $c_1 = c_2 + (e^2+1)^{-1}$ and $c_2 = (e^4 + 6e^2 + 1)/8(e^2+1)$. By applying the Algorithm 2 to (17) with $n = 50$, the maximum error is 0.0037. Figure 7 compares the exact solution with the approximate one.

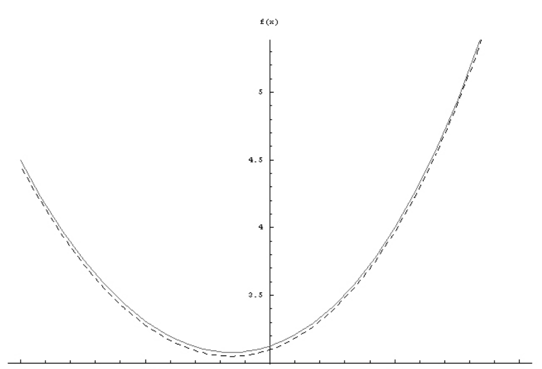

Figure 7: The results of applying the Algorithm 2 to (17). The solid and dashed lines are the exact and approximate solutions, respectively.

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Example 5. Consider the equation

$$f(x) = \sin x - \frac{x}{2} + \frac{1}{2}\int_0^{\frac{\pi}{2}} xyf(y)dy. \qquad (18)$$

The exact solution to this equation is $f(x) = \sin x$. By applying the Algorithm 2 to (18) with $n = 30$, the maximum error is 0.0005. Figure 8 compares the exact solution with the approximate one.

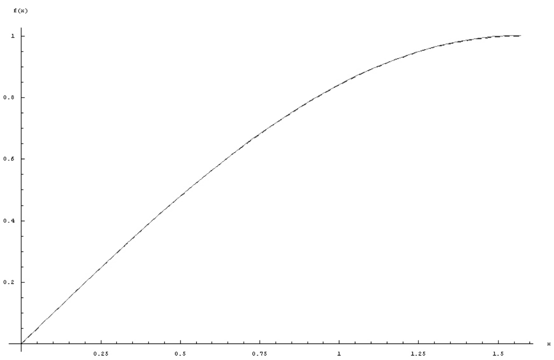

Figure 8: The results of applying the Algorithm 2 to (18). The solid and dashed lines are the exact and approximate solutions, respectively.

6 Conclusion

This paper deals with the effective algorithms for solving the Fredholm integral equation of the second kind. In fact, it provides the algorithms that implement the quadrature and its modification using Mathematica. It draws various examples of the Fredholm equations and shows that the algorithms yield acceptable results.

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