Proceedings of the World Congress on Engineering 2008 Vol II
WCE 2008, July 2 - 4, 2008, London, U.K.

Feller Properties on the Torus

Emil Popescu *

Abstract—**A representation of the operators which form a Feller semigroup defined on a compact abelian group is given. Thus, are obtained the symbols which characterize the semigroup and its infinitesimal generator. The conditions on the symbols determine properties of the semigroup and conversely. A characterization for the conservativeness of the semigroup and applications on a torus are given.**

Keywords: compact abelian group, Feller semigroup, Fourier transform

1 Introduction

Let $A : D(A) \to C_\infty(\mathbf{R}^n)$ be a linear operator, where $D(A)$ is a linear dense subspace of $C_\infty(\mathbf{R}^n)$, the Banach space of all continuous functions on $\mathbf{R}^n$ vanishing at infinity with the supremum norm. A satisfies *the positive maximum principle* on $D(A)$ if for all $u \in D(A)$ and $x_0 \in \mathbf{R}^n$ such that

$$\sup_{x\in\mathbf{R}^n} u(x) = u(x_0) \geq 0$$

it follows that $Au(x_0) \leq 0$. It is not difficult to verify that infinitesimal generators of Feller semigroups on $\mathbf{R}^n$ satisfy this principle. In this case, P. Courrège showed in [4] that for all $u \in C_0^\infty(\mathbf{R}^n) \subset D(A)$,

$$Au(x) = \sum_{i,j=1}^{n} a_{ij}(x)\,\partial_i\partial_j u(x) + \sum_{i=1}^{n} b_i(x)\,\partial_i u(x) + k(x)u(x) + \int_{\mathbf{R}^n} [u(y) - \sigma(x,y)(u(x) + \sum_{i=1}^{n} \partial_i u(x)(y_i - x_i))]\nu(x,dy),$$

where each b_i, a_{ij} and k are measurable functions with each $(a_{ij}(x))$ a symmetric pozitive definite matrix and ν is a Lévy kernel. Courrège also showed that A can be represented as a pseudo differential operator,

$$Au(x) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle} a(x,\xi)\,\widehat{u}(\xi)\,d\xi,$$

where a is the *symbol* of A ($\widehat{u}$ is the Fourier transform of u). This fact has been exploited in recent investigations of Feller processes by N. Jacob ([7]). Moreover, in [6] and [9] has been proved that the operators which form a Feller semigroup on $\mathbf{R}^n$ can be represented as pseudo differential operators with positive definite symbols. All these results use the fact that $\mathbf{R}^n$ is a commutative group and the Feller semigroup is compatible with the group structure. The result of Courrège on operators satisfying the positive maximum principle can be interpreted, in certain sense, as a perturbation of the translation invariant case, the coefficients becoming real-valued functions. It is natural to search for extensions to other structures than $\mathbf{R}^n$. It turns out that there are several extensions for the translation invariant case. The representation of the generator of a convolution semigroup as pseudo differential operator is given in the book of Berg and Forst ([3]), where $\mathbf{R}^n$ is replaced by a locally compact abelian group. For the compact abelian group $\mathbf{T}^\infty$, the infinite dimensional torus, important results are due to Bendikov ([2]). In [1] are considered Feller semigroups on Lie groups. In the framework of locally compact abelian group a class of symbols has been studied in [8].

Despite of all the above papers, we have not found results for the non-translation invariant case in other structures than $\mathbf{R}^n$. This is why we present in this paper a theorem of representation for a semigroup of operators defined on a compact abelian group G, in the non-translation invariant case. For the Feller semigroup $(P_t)_{t>0}$ on G we find

$$P_t u(x) = \sum_{\gamma\in\Gamma} \gamma(x)\; p_t(x,\gamma)\; \widehat{u}(\gamma),$$

where Γ is the dual group of G. Thus, some functions named symbols, which characterize the semigroup and its infinitesimal generator, exist. Imposing conditions on the symbols we obtain properties of the semigroup and conversely. A condition for the conservativeness of the semigroup and applications on a torus are given.

The interaction between probability theory and groups is an active area of research, which contains random matrix theory, random walks and invariant diffusions in groups, Lévy processes on a topological group etc. The compact groups can be seen as a generalization of finite groups and the Pontryagin duality describes the theory for commutative groups, as a generalised Fourier transform. In this framework, we obtain new extensions of the results concerning the physical processes of diffusion, given by the Laplacian and its Brownian semigroup.

*Manuscript received March 22, 2008. Emil Popescu is with the Technical University of Civil Engineering Bucharest, Romania Department of Mathematics and Informatics, Tel: +(40) 21 242 12 08/208, Fax: +(40) 21 242 07 81, Email: epopescu@utcb.ro

2 Framework and auxiliary results

This section is introductory. It lists basic definitions and some theorems which are used in the present paper. In the beginning, we consider the context of the locally compact abelian groups ([3],[10]).

Let $(G,+)$ be a locally compact abelian group with the neutral element denoted 0. A function $\gamma : G \to \mathbf{C}$ is called a *character* of G if $|\gamma(x)| = 1$ for all $x \in G$ and if the functional equation

$$\gamma(x+y) = \gamma(x) + \gamma(y), \quad x, y \in G,$$

is satisfied. The set of all continuous characters on G forms a group Γ, the dual group of G, where the group operation is defined by

$$(\gamma_1 + \gamma_2)(x) = \gamma_1(x) \cdot \gamma_2(x)$$

for all γ_1, $\gamma_2 \in \Gamma$ and $x \in G$. The neutral element in Γ is the character $x \longmapsto 1$, denoted $\mathbf{0}$. The inverse element of $\gamma \in \Gamma$ is the character

$$-\gamma : x \longmapsto \overline{\gamma(x)}.$$

We denote by $C(G)$ the set of continuous complex functions on G. $C(G)$ is endowed with the topology of compact convergence. Equipped with the subspace topology from $C(G)$, Γ is a locally compact abelian group.

Once and for all we choose and fix a Haar measure on G, denoted ω_G or dx. The L^p- spaces with respect to ω_G are denoted by $L^p(G)$, $p \in [1, \infty)$. For $f \in L^1(G)$ and $\gamma \in \Gamma$ we denote by $\widehat{f}$ the Fourier transform of f,

$$\widehat{f}(\gamma) = \int_G \overline{\gamma(x)} f(x) dx.$$

By $C_\infty(G)$, we denote the set of continuous complex functions on G which tend to zero at infinity and this space will be endowed with the topology of uniform convergence. $C_0(G)$ is the set of continuous complex functions on G which have compact support. $M(G)$ is the space of Radon measures on G. By $M_b(G)$ we denote the set of the bounded Radon measures on G. The Dirac measure at $a \in G$ is denoted ε_a. We shall not distinguish between the locally integrable Borel function f on G and the measure $f\omega_G$, defined by

$$< f\omega_G, g >= \int_G f(x) g(x) dx,$$

for $g \in C_0(G)$.

The Haar measure ω_Γ (or just $d\gamma$) on Γ is the dual Haar measure to ω_G on G (see [3], [10]). If $\mu \in M_b(G)$ such that the Fourier transform $\widehat{\mu}$ belongs to $L^1(\Gamma)$, then μ has a continuous density φ with respect to ω_G given by

$$\varphi(x) = \int_\Gamma \gamma(x) \widehat{\mu}(x) d\omega_\Gamma(\gamma),$$

for all $x \in G$.

For $f \in L^1(G)$ the Fourier transform $\widehat{f}$ tends to zero at infinity. The set $\left\{ \widehat{f} \mid f \in L^1(G) \right\}$ is a dense subalgebra of $C_0(\Gamma)$. The Fourier transform, considered as a map of $L^1(G)$ into $C_\infty(\Gamma)$ is continuous and

$$\left\| \widehat{f} \right\|_\infty \leq \|f\|_1.$$

Definition. A function $\varphi : G \to \mathbf{C}$ is called *positive definite*, if for all $n \in \mathbf{N}$, for all n-tuples $(x_1, x_2, ...x_n)$ of elements from G and for every choise of complex numbers $c_1, c_2, ..., c_n$ we have

$$\sum_{i=1}^n \sum_{j=1}^n c_i \overline{c_j} \varphi(x_i - x_j) \geq 0.$$

If φ is positive definite, then

$$\varphi(-x) = \overline{\varphi(x)}, \qquad |\varphi(x)| \leq \varphi(0).$$

Every character is positive definite, hence so is every finite linear combination of characters, if the coefficients are positive. From the theorem of Bochner, a continuous function φ on G is positive definite if and only if there exists a positive bounded measure σ on the dual group Γ such that

$$\varphi(x) = \int_\Gamma \gamma(x) d\sigma(\gamma), \quad x \in G.$$

Definition. A strongly continuous contraction semigroup $(P_t)_{t \geq 0}$ on $C_\infty(G)$ for which all the operators P_t are positive, i.e. for all $t \geq 0$ and $f \in C_\infty^+(G)$ we have $P_t f \in C_\infty^+(G)$, is called a *Feller semigroup* on G.

A Feller semigroup $(P_t)_{t \geq 0}$ on G is said to be *conservative* if $P_t \mathbf{0} = \mathbf{0}$, for all $t \geq 0$.

If G is compact, then Γ is countable; of course, G is metrisable and $C(G)$ is separable. An *approximate unit* on G is a sequence $(\varphi_k)_{k \geq 1}$ of functions $\varphi_k \in C^+(G)$ such that for every k,

$$\int_G \varphi_k(x) d\omega_G(x) = 1.$$

For an approximate unit $(\varphi_k)_{k \geq 1}$, the sequence $(\varphi_k \omega_G)_{k \geq 1}$ of probability measures converges to ε_0 in $M(G)$ with the vague topology. Let E be one of the following spaces: $C(G)$ with the topology of uniform convergence or $M(G)$ with the vague topology. For an approximate unit $(\varphi_k)_{k \geq 1}$ on G and an element $f \in E$ we have $\varphi_k * f \in E$, for all $k \geq 1$ and $\varphi_k * f$ converges to f in the topology of E ($*$ is the convolution).

Definition. Any function of the form

$$f(x) = \sum_{j=1}^{n} a_j \gamma_j(x),$$

where $a_j \in \mathbf{C}$, $j = 1, 2, ..., n$ and $n \in \mathbf{N}$, is called a *generalized trigonometrical polinomial* on G.

The set of all generalized trigonometrical polynomials is denoted by $S(G)$. Since G is compact then $S(G)$ is dense in the spaces $C(G)$ and $L^p(G)$, $1 \leq p < \infty$, respectively. Every $f \in L^2(G)$ can represented as

$$f(x) = \sum_{\gamma \in \Gamma} \widehat{f}(\gamma)\gamma(x),$$

where the series converges in $L^2(G)$, hence the limit does not depend on the order of summation.

3 The main result

In the sequel, we consider that G is a compact abelian group, $C(G)$ represents the Banach space of continuous complex functions on G and $S(G)$ is the set of all generalized trigonometrical polynomials on G. Starting from [6] and [9] we have the following result.

Theorem 1. *Let $P : C(G) \to C(G)$ be a linear and continuous operator such that for every $f \geq 0$ we have $Pf \geq 0$. Then there exists a function*

$$p : G \times \Gamma \to \mathbf{C}$$

continuous, bounded and such that for every $x \in G$ fixed,

$$\gamma \to p(x, \gamma)$$

is positive definite with the property that for all $u \in S(G)$ we have

$$Pu(x) = \sum_{\gamma \in \Gamma} \gamma(x)p(x,\gamma)\widehat{u}(\gamma).$$

Proof. Let $p : G \times \Gamma \to \mathbf{C}$,

$$p(x,\gamma) := \overline{\gamma(x)}(P\gamma)(x).$$

p is continuous and bounded. Let $x \in G$ be fixed. The function

$$p_x : \Gamma \to \mathbf{C}, \quad p_x(\gamma) := p(x, \gamma)$$

is positive definite since for all $m \in \mathbf{N}$, $\gamma_1, \gamma_2, ..., \gamma_m \in \Gamma$ and $z_1, z_{2,...,} z_m \in \mathbf{C}$ we have $\sum\limits_{k,l=1}^{m} p_x(\gamma_k - \gamma_l) z_k \overline{z_l} = \sum\limits_{k,l=1}^{m} [\overline{(\gamma_k - \gamma_l)(x)} \cdot P(\gamma_k - \gamma_l)(x)] z_k \ \overline{z_l} = \sum\limits_{k,l=1}^{m} [\overline{(\gamma_k - \gamma_l)(x)} \int\limits_G (\gamma_k - \gamma_l)(y) P(x, dy)] z_k \ \overline{z_l} = \sum\limits_{k,l=1}^{m} [\int\limits_G \overline{(\gamma_k - \gamma_l)(y-x)} P(x,dy)] z_k \ \overline{z_l} = \int\limits_G (\sum\limits_{l=1}^{m} z_l \gamma_l (y - x)) \cdot \overline{(\sum\limits_{k=1}^{m} z_k \gamma_k(y-x))} \ P(x,dy) = \int\limits_G \mid (\sum\limits_{l=1}^{m} z_l \gamma_l(y-x)) \mid^2 \cdot P(x, dy) \geq 0.$

Let $(\varphi_k)_{k \geq 1}$ be an approximate unit on G such that when $k \to \infty$, $\varphi_k \to \varepsilon_0$ in $M(G)$ with the vague topology and

$$\widehat{\varphi_k}(\gamma) \to 1,$$

for every $\gamma \in \Gamma$. Indeed,

$$\widehat{\varphi_k}(\gamma) = \widehat{\varphi_k}(\gamma) \cdot \widehat{\varepsilon_0}(\gamma) = \widehat{\varphi_k * \varepsilon_0}(\gamma) \to \widehat{\varepsilon_0}(\gamma) = 1,$$

for every $\gamma \in \Gamma$.

For $u \in S(G)$, we have: $\sum\limits_{\gamma \in \Gamma} \gamma(x) \ p(x,\gamma) \widehat{\varphi_k}(\gamma) \ \widehat{u}(\gamma) = \sum\limits_{\gamma \in \Gamma} \gamma(x) \ \overline{\gamma(x)} \ (P\gamma)(x) \widehat{\varphi_k}(\gamma) \ \widehat{u}(\gamma) = \sum\limits_{\gamma \in \Gamma} \int\limits_G (P\gamma)(x) \ \widehat{\varphi_k}(\gamma) \ \overline{\gamma(y)} \ u(y) \ dy = \sum\limits_{\gamma \in \Gamma} \int\limits_G \overline{\gamma(y)} \int\limits_G \gamma(z) P(x,dz) \ \widehat{\varphi_k}(\gamma) \ u(y) \ dy = \int\limits_G \int\limits_G (\sum\limits_{\gamma \in \Gamma} \overline{\gamma(y)} \ \gamma(z) \ \widehat{\varphi_k}(\gamma)) \ P(x,dz) \ u(y) \ dy = \int\limits_G \int\limits_G (\sum\limits_{\gamma \in \Gamma} \overline{\gamma(y-z)} \widehat{\varphi_k}(\gamma) \ ) \ P(x,dz) \ u(y) \ dy = \int\limits_G \int\limits_G (\sum\limits_{\gamma \in \Gamma} \gamma(z-y) \widehat{\varphi_k}(\gamma)) \ P(x,dz) \ u(y) \ dy = \int\limits_G \int\limits_G \varphi_k(z-y) P(x,dz) \ u(y) dy = \int\limits_G (\int\limits_G \varphi_k(z-y) \ u(y) dy) P(x,dz) = \int\limits_G (\varphi_k * u)(z) P(x,dz).$

When $k \to \infty$, we have $\varphi_k * u \to u$ in $C(G)$. Then

$$\lim_{k\to\infty} \int_G (\varphi_k * u)(z) P(x,dz) = \int_G u(z)P(x,dz) = (Pu)(x).$$

On the other hand, we observe that

$$\lim_{k \to \infty} \sum_{\gamma \in \Gamma} \gamma(x) \ p(x,\gamma) \widehat{\varphi_k}(\gamma) \ \widehat{u}(\gamma) = \sum_{\gamma \in \Gamma} \gamma(x) \ p(x,\gamma) \ \widehat{u}(\gamma).$$

Corollary 2. *Let $(P_t)_{t \geq 0}$ be a Feller semigroup on G. For every $t \geq 0$ there exists a function*

$$p_t : G \times \Gamma \to \mathbf{C}$$

continuous, bounded and such that for any $x \in G$,

$$\gamma \to p_t(x,\gamma)$$

is positive definite with the property that for all $u \in S(G)$ we have

$$P_t u(x) = \sum_{\gamma \in \Gamma} \gamma(x) \ p_t(x,\gamma) \ \widehat{u}(\gamma).$$

Theorem 3. *Let $(P_t)_{t\geq 0}$ be a Feller semigroup on G with the infinitesimal generator $(A, D(A))$. Suppose that $S(G) \subset D(A)$ and the functions*

$$\gamma \to (A\gamma)(x) \cdot \widehat{u}(\gamma)$$

belong to $L^1(\Gamma)$ for every $x \in G$ and $u \in S(G)$. Then for every $u \in S(G)$,

$$Au(x) = \sum_{\gamma\in\Gamma} \gamma(x)\ a(x,\gamma)\ \widehat{u}(\gamma),$$

where $a : G \times \Gamma \to \mathbf{C}$,

$$a(x,\gamma) = \frac{d}{dt}\ p_t\,(x,\gamma)\ |_{t=0}\ .$$

Proof. Let $u \in S(G)$. From Corollary 2, it follows that

$$\frac{P_t u(x) - u(x)}{t} = \sum_{\gamma\in\Gamma} \gamma(x)\ \left[\frac{p_t(x,\gamma) - 1}{t}\right]\ \widehat{u}(\gamma) =$$

$$\sum_{\gamma\in\Gamma} \gamma(x)\ \left[\frac{\overline{\gamma(x)}(P_t\gamma)(x) - 1}{t}\right]\ \widehat{u}(\gamma).$$

Since $S(G) \subset D(A)$ there exists

$$\lim_{t\to 0} \frac{P_t u - u}{t} = Au,$$

for every $u \in S(G)$. For fixed $x \in G$ and $\gamma \in \Gamma$, the following limit exists:

$$\lim_{t\to 0} \frac{\overline{\gamma(x)}(P_t\gamma)(x) - 1}{t} = \overline{\gamma(x)} \lim_{t\to 0} \frac{(P_t\gamma)(x) - \gamma(x)}{t} =$$

$$\overline{\gamma(x)}(A\gamma)(x).$$

We define

$$a(x,\gamma) := \lim_{t\to 0} \frac{p_t(x,\gamma) - 1}{t} =$$

$$\frac{d}{dt}\ p_t\,(x,\gamma)\ |_{t=0} = \overline{\gamma(x)}(A\gamma)(x).$$

Since, by ypothesis,

$$\gamma \to (A\gamma)(x) \cdot \widehat{u}(\gamma)$$

belong to $L^1(\Gamma)$ for every $x \in G$ and $u \in S(G)$, we obtain the wanted formula.

Remark. The functions $p_t\,(x,\gamma)$ and $a(x,\gamma)$, which characterize the semigroup $(P_t)_{t>0}$ and its infinitesimal generator, are named *symbols.* Imposing conditions on the symbols we obtain properties of the semigroup and conversely.

4 Conservativeness

The intuitive meaning of conservativeness of the Feller semigroup $(P_t)_{t\geq 0}$ on G can be given with the help of the associated stochastic processes $\{X_t\}_{t\geq 0}$. The relation between them is given by

$$P_t\chi_B\,(x) = E^x\,(X_t \in B)\,,$$

for all $x \in G$ and all Borel sets $B \subset G$, where χ_B is the indicator function of B. Therefore,

$$\mathbf{0} = P_t\mathbf{0}\,(x) = E^x 1 = E^x\,(X_t \in G) \qquad a.s.$$

and the conservative process $\{X_t\}_{t\geq 0}$ has *a.s. infinite lifetime.*

In the following we consider that the assumptions of Theorem 3 are true. $p_t\,(x,\gamma)$ and $a(x,\gamma)$ are the symbols defined above.

Theorem 4. *The Feller semigroup $(P_t)_{t\geq 0}$ is conservative if and only if $a\,(x,\mathbf{0}) = 0$ holds for all $x \in G$, where*

$$a\,(x,\gamma) = \frac{d}{dt}\ p_t\,(x,\gamma)\ |_{t=0}\ .$$

Proof. We suppose $P_t\mathbf{0} = \mathbf{0}$ for every $t \geq 0$. Then

$$A\mathbf{0} = \lim_{t\to 0} \frac{P_t\mathbf{0} - \mathbf{0}}{t} = 0.$$

Since

$$a : G \times \Gamma \to \mathbf{C},\ \ a(x,\gamma) = \overline{\gamma\,(x)}\,(A\gamma)\,(x)$$

it follows that

$$a\,(x,\mathbf{0}) = A\mathbf{0}\,(x) = 0,$$

for all $x \in G$.

Conversely, if $a\,(x,\mathbf{0}) = 0$ for every $x \in G$ then we have

$$\frac{d}{dt}\ p_t\,(x,\mathbf{0})\ |_{t=0} = 0,$$

for every $x \in G$. By Taylor's formula,

$$p_t\,(x,\gamma) - p_0\,(x,\gamma) = \int_0^t \frac{d}{ds}\ p_s\,(x,\gamma)\,ds.$$

We obtain

$$p_t\,(x,\gamma) - \mathbf{0} = t\int_0^1 \frac{d}{ds}\ p_s\,(x,\gamma)\ |_{s=\rho t}\ d\rho$$

$$= t\int_0^1 \frac{d}{ds}\ \left[\overline{\gamma\,(x)}\,(P_s\gamma)\,(x)\right]\ |_{s=\rho t}\ d\rho.$$

ISBN:978-988-17012-3-7

Since

$$\frac{d}{ds}P_s u = P_s A u,$$

it follows that

$$p_t(x,\gamma) - \mathbf{0} = t\int_0^1 \overline{\gamma(x)}\,(P_{\rho t}A\gamma)(x)\,d\rho$$

$$= -t\int_0^1 \overline{\gamma(x)}P_{\rho t}\,(a(\cdot,\gamma)\,\gamma(\cdot))\,(x)\,d\rho.$$

Thus we find for $\gamma = \mathbf{0}$, $p_t(x,\mathbf{0}) = \mathbf{0}$, for every $x \in G$. Since

$$p_t(x,\gamma) = \overline{\gamma(x)}\,(P_t\gamma)(x),$$

we obtain $\mathbf{0} = P_t\mathbf{0}(x)$, for every $x \in G$, i.e. $P_t\mathbf{0} = \mathbf{0}$ and $(P_t)_{t\geq 0}$ is conservative.

We present two examples of conservative semigroups.

The brownian semigroup $(\Lambda_t)_{t>0}$ on the compact abelian group given by the torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$ ([5]) is

$$\Lambda_t f(\theta) = \sum_{n=-\infty}^{\infty} e^{in\theta}e^{-tn^2/2}\widehat{f}(n).$$

If

$$f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta} \qquad (f \in S(\mathbf{T})),$$

then

$$\Lambda_t f(\theta) = \sum_{n=-\infty}^{\infty} a_n e^{-tn^2/2}e^{in\theta},$$

where

$$p_t(\theta,n) = e^{-tn^2/2}.$$

Thus,

$$a(\theta,n) = \frac{d}{dt}\,p_t(\theta,n)\,|_{t=0} = -\frac{n^2}{2}e^{-tn^2/2}\,|_{t=0} = -\frac{n^2}{2}$$

and the formula of the infinitesimal generator A of $(\Lambda_t)_{t>0}$ is

$$Af(\theta) = \sum_{n=-\infty}^{\infty} -\frac{n^2}{2}a_n e^{in\theta} = \frac{1}{2}f''(\theta).$$

We observe that $a(\theta,0) = 0$ and $(\Lambda_t)_{t>0}$ is conservative.

In the context of the infinite-dimensional compact group $G = \mathbf{T}^\infty$ with the dual group $\Gamma = \mathbf{Z}^{(\infty)}$, where $\mathbf{Z}^{(\infty)}$ is the subgroup of $\mathbf{Z}^\infty$ consisting of sequences $\gamma = (\gamma_k)_{k=1}^\infty$, which are eventually zero, let X be a space-homogeneous Markov process on G with almost surely continuous trajectories, the transition function being invariant with respect to the action of the group ([2]). The semigroup $(P_t)_{t>0}$ of the process X is a Feller semigroup and the formula of Corollary 2 is valid on a set $\mathcal{D}$ of cylindrical infinitely differentiable functions with compact supports with

$$p_t(x,\gamma) = \exp\{-t[\Psi(\gamma) - i\ell(\gamma)]\},\ \gamma \in \Gamma,$$

where $\Psi(\gamma) \geq 0$ is a quadratic form and $\ell(\gamma)$ is a linear form on Γ. In the natural basis of group Γ these forms can be written

$$\Psi(\gamma) = \sum_{i,j=1}^{\infty} a_{ij}\gamma_i\gamma_j,\ \gamma = (\gamma_i)_{i=1}^\infty \in \Gamma,$$

$$\ell(\gamma) = \sum_{k=1}^{\infty} b_k\gamma_k.$$

From Theorem 3, it follows that on $\mathcal{D}$ the infinitesimal generator of the semigroup $(P_t)_{t>0}$ is the infinite-dimensional differential operator

$$\mathcal{L} = \sum_{i,j=1}^{\infty} a_{ij}\partial_i\partial_j + \sum_{i=1}^{\infty} b_i\partial_i,$$

where ∂_i is the operator of differentiation respect to variable x_i $(x = (x_i)_{i=1}^\infty \in G)$. Here

$$a(x,\gamma) = -\Psi(\gamma) + i\ell(\gamma) = -\sum_{i,j=1}^{\infty} a_{ij}\gamma_i\gamma_j + i\sum_{k=1}^{\infty} b_k\gamma_k$$

and $a(x,0) = 0$, hence $(P_t)_{t>0}$ is conservative.

References

[1] Applebaum, D., "On a class of non-translation invariant Feller semigroups on Lie groups," *Potential Analysis,* V16, pp. 103-114, 2002.

[2] Bendikov, A., *Potential theory on infinite-dimensional abelian groups*, Walter de Gruyter, 1995.

[3] Berg, C., Forst, G., *Potential theory on locally compact abelian groups*, Springer, 1975.

[4] Courrège, P., "Sur la forme integro-differentielle des operateurs de dans C satisfaisant au principe du maximum", *Sem. Theorie du Potentiel*, pp.38, 1965/1966.

[5] Dellacherie, C., Meyer, P. A., *Probabilities and Potential*, North-Holland, 1988.

[6] Jacob, N., "Characteristic functions and symbols in the theory of Feller processes", *Potential Analysis*, V8, pp. 61-68, 1998.

[7] Jacob, N., *Pseudo-differential Operators and Markov Processes*, Vol.2: *Generators and their Potential Theory*, Imperial College Press, London, 2002; Vol.3: *Markov Processes and Applications*, Imperial College Press, London, 2005.

[8] Popescu, E., "PD-opérateurs k-elliptiques sur les groupes abéliens local compacts", *Rev.Roumaine Math. Pures Appl*, V39, pp. 487-499, 1994.

[9] Popescu, E., "A Note on Feller Semigroups", *Potential Analysis*, V14, pp. 207-209, 2001.

[10] Rudin, W., *Fourier analysis on groups,* Interscience, 1962.