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Some Properties of Sign Regular Matrix in Neville Elimination Method

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***Abstract*— In this paper we study some properties of sign regular matrices in Neville Elimination method and show that if we apply the second method of Neville Elimination brought in [1], then this method preserves both diagonally dominant and sign regular properties.**

***Index Terms*— Sign regular matrix, Neville algorithm.**

I. Introduction

Definitions. If all $k \times k$ minors of each k have the same sign or equal zero, the matrix is called sign regular. If none of them is zero, the matrix will be called strictly sign regular. It is not needed that all different k have the same sign, but if they are all non-negative, the matrix is called totally non-negative. The symbol ε_k is employed to display the sign of $k \times k$ minors. That is, if it is written $\varepsilon_k(A) = +1$ ($\varepsilon_k(A) = -1$), it means that all $k \times k$ submatrices of A have non-negative(resp, non-positive) determinants.

For a nonsingular matrix A of order n, the Neville elimination procedure consists of $n-1$ successive steps, resulting in sequence of matrices as follows:

$$A = A^{(1)} \rightarrow \tilde{A}^{(1)} \rightarrow A^{(2)} \rightarrow \tilde{A}^{(2)} \rightarrow \cdots \rightarrow A^{(n)} \rightarrow \tilde{A}^{(n)} = U$$

where U is an upper triangular matrix.

Remark : If for all $j = 1, \cdots, k$ the $j \times j$ submatrices have the same sign or have zero value and are denoted by SR_k, a matrix A is called sign regular of order k.

Lemma 1.1. *If $A = (a_{ij})_{1 \le i,j \le n}$ is nonsingular and SR_3 matrix with $a_{11} \neq 0$ and $a_{n1} \neq 0$ then at least one of the determinants below:*

$$d_1 := det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, d_2 := det\begin{pmatrix} a^{(t)}_{n-1,1} & a^{(t)}_{n-1,2} \\ a^{(t)}_{n1} & a^{(t)}_{n2} \end{pmatrix}$$

is not zero.

2. Properties Of Matrix In Neville

Lemma 2.1. *Neville elimination can not preserve row diagonally dominant in matrix.*

Lemma 2.2. *Let $A = (a_{ij})_{1 \le i,j \le n}$ be a nonsingular and SR_2 matrix.*

(i) if $\varepsilon_2(A) = +1$, then $a_{11} \neq 0$ and we perform the first step of Neville elimination without exchanging rows.

(ii) if $\varepsilon_2(A) = -1$, then $a_{n1} \neq 0$ and if we reverse the order of the rows of A then we perform the first step of Neville elimination without rows exchange.

Lemma 2.3. *Assume $A = (a_{ij})_{1 \le i,j \le n}$ is diagonally dominant and sign regular matrix, then the first step of the first method of Neville Elimination is preserving row diagonally dominant.*

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Proof. To start we provide a proof for 3×3 matrix and $\varepsilon_1 = \varepsilon_2 = +1$. In this case all the entries of matrix are equal to or greater than zero and all determinants of its 2×2 submatrices are similarly equal to or greater than zero. The case of $\varepsilon_2 = -1$ is not considered because it is in contradiction to diagonally dominant. The cases of $\varepsilon_1 = -1$ and $\varepsilon_2 = +1$ will be presented in the next theorem. At the first step of this algorithm we have

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \longrightarrow \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} - \frac{a_{21}}{a_{11}} \times a_{12} & a_{23} - \frac{a_{21}}{a_{11}} \times a_{13} \\ 0 & a_{32} - \frac{a_{31}}{a_{21}} \times a_{22} & a_{33} - \frac{a_{31}}{a_{21}} \times a_{23} \end{bmatrix},$$

diagonally dominant property of A gives:

$$a_{11} \ge a_{12} + a_{13} \quad (1)$$

$$a_{22} \ge a_{21} + a_{23} \quad (2)$$

$$a_{33} \ge a_{31} + a_{32} \quad (3)$$

and since A is SR_2 and $\varepsilon_2 = +1$, then

$$a_{11}a_{22} - a_{12}a_{21} \ge 0$$

$$a_{12}a_{23} - a_{13}a_{22} \ge 0, \cdots .$$

The number of above relations is 9, Since we have nine 2×2 matrices.

To prove the axiom it is necessary that:

$$a_{11}a_{23} - a_{21}a_{13} \le a_{22}a_{11} - a_{21}a_{12} \quad (4)$$

$$a_{32}a_{21} - a_{31}a_{22} \le a_{33}a_{21} - a_{31}a_{23}, \quad (5)$$

note that: $a_{12} \ge a_{13}$ since $a_{12}a_{23} - a_{13}a_{22} \ge 0$ and $a_{22} \ge a_{23}$. In the same way we have $a_{32} \ge a_{31}$. We claim that $a_{11} \neq 0$, otherwise diagonally dominant property causes that $a_{12} = a_{13} = 0$, which means that all of entries the first row of matrix are zero and this is in contradiction to nonsingularity property. If two entries a_{12} and a_{13} are zero, then there is not any dilemma for problem, since $a_{12} = a_{13} = 0$, by applying the first step of Neville algorithm we have diagonally dominant matrix.

Two other cases remain,in which one of two entries is zero and another entry is nonzero.

case 1: $a_{12} = 0$ and $a_{13} \neq 0$, this case is impossible, except $a_{22} = 0$ and this means a row of matrix equals zero and this is contradictory.

case 2: $a_{13} = 0$ and $a_{12} \neq 0$, then by Neville algorithm the 3th row does not change and diagonally dominant holds.

according to lemma (**2.2**), the first step of Neville algorithm in the first method is done without changing rows, Accordingly we consider three entries of the first column are nonzero. Now by reductio ad absurdum we show that if at least one of the above relations does not hold, contradiction occurs. Assuming that relation (4) does not hold, we have

$$a_{11}a_{23} - a_{21}a_{13} > a_{22}a_{11} - a_{21}a_{12} \Rightarrow a_{11}a_{22} - a_{11}a_{23} < a_{21}a_{12} - a_{21}a_{13}$$

$$\Rightarrow a_{11}(a_{22} - a_{23}) < a_{21}(a_{12} - a_{13}).$$

On the other hand $a_{22} \ge a_{21} + a_{23} \Rightarrow a_{22} - a_{23} \ge a_{21}$, then we have

$$a_{11}a_{21} \le a_{11}(a_{22} - a_{23}) < a_{21}(a_{12} - a_{13}).$$

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Now by dividing left and right sides of above relation by a_{21}:

$$a_{11} < (a_{12} - a_{13})$$

which contradicts diagonally dominant of the first row, then (4) holds. Now assume relation (5) doesn't hold, we have

$$a_{32}a_{21} - a_{31}a_{22} > a_{33}a_{21} - a_{31}a_{23}$$

and furthermore

$$a_{32} \leq a_{33} \Rightarrow -a_{21}a_{32} \geq -a_{21}a_{33}$$

by adding two above relations, we have:

$$-a_{31}a_{22} \geq -a_{31}a_{23} \Rightarrow a_{22} \leq a_{23}$$

which this contradicts diagonally dominant of second rows. This shows our axiom holds for 3×3 matrices, now we extend the topic for $n \times n$ matrix. For this purpose we consider an arbitrary row of matrix, applying Neville algorithm for it, and show that it preserves diagonally dominant. Since our considered matrix is SR_2 with $\varepsilon_2 = +1$, then $a_{11} \neq 0$ and the first step of Neville's algorithm (by lemma (**2.2**)) is done by changing rows, so for the ith row, we show that:

$$a_{ii} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,i} \geq \sum_{k\neq i} a_{ik} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,k}$$

If relation above does not hold, then

$$a_{ii} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,i} < \sum_{k\neq i} a_{ik} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,k}.$$

By diagonally dominant property we have

$$a_{ii} \geq \sum_{k=1,k\neq i}^{n} a_{ik} \Longrightarrow a_{ii} - \sum_{k\neq i,1} a_{ik} \geq a_{i1}$$

and

$$a_{ii} - \sum_{k\neq i} a_{ik} < \frac{a_{i1}}{a_{i-1,1}}\left(a_{i-1,1} - \sum_{k\neq i} a_{i-1,k}\right)$$

$$\Longrightarrow a_{i1} \leq a_{ii} - \sum_{k\neq i} a_{ik} < \frac{a_{i1}}{a_{i-1,1}}\left(a_{i-1,i} - \sum_{k\neq i} a_{i-1,k}\right).$$

Both sides of relation above are divided by a_{i1} and multiplied by $a_{i-1,1}$, therefore we have

$$a_{i-1,1} < \left(a_{i-1,i} - \sum_{k\neq i} a_{i-1,k}\right) = a_{i-1,i} - a_{i-1,i-1} - \sum_{k\neq i,i-1} a_{i-1,k} < 0$$

which contradicts $\varepsilon_1(A) = +1$

Theorem 2.4. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be nonsingular and sign regular matrix (strictly sign regular), and Neville Elimination with second method be applied, then for all $k \in \{1, 2, \cdots, n\}$, all submatrices $A^{(k)}[k, \cdots, n]$ are sign regular (strictly sign regular) matrix and $\varepsilon_1(A^{(k)}[k, \cdots, n]) = \varepsilon_1(A)$.*

Proof in [**1**].

Theorem 2.5. *Let $A = (a_{ij})_{1\leq i,j\leq n}$ be SR nonsingular and diagonally dominant matrix. By applying Neville Elimination with second method for all k, $A^{(k)}[k, \cdots, n]$ has properties of A.*

Proof. At first we consider a $k\times k$ nonsingular and sign regular matrix in which the following cases exist:

case 1: $\varepsilon_1 = \varepsilon_2 = +1$,

in this step $t = 1$, to reach $\tilde{A}^{(1)}$ from $A^{(1)} = A$ we apply the second method. Then we have:

$$a_{11} \neq 0 \longrightarrow \begin{cases} a_{n1} = 0 \Rightarrow \tilde{A}^{(1)} = A^{(1)}. \\ \\ a_{n1} \neq 0 \Rightarrow compute(d_1) \quad \rightarrow \begin{cases} d_1 > 0 \Rightarrow \tilde{A}^{(1)} = A^{(1)}. \\ \\ d_1 = 0 \rightarrow compute(d_2) \\ \rightarrow d_2 > 0 \Rightarrow \tilde{A}^{(1)} = A^{(1)}. \end{cases} \end{cases}$$

the case of $d_1 < 0$ does not exist because we assume $\varepsilon_2 = +1$.

In this step since $\tilde{A}^{(1)} = A^{(1)}$, so without any changing to reach $A^{(2)}$ from $\tilde{A}^{(1)}$ which in this case by proof of lemma (**2.3**) and theorem (**2.4**) two properties sign regular and diagonally dominant will be copied to $A^{(2)}$. By theorem (**2.4**) we have $\varepsilon_1 = +1$ for matrix $A^{(2)}[2, \cdots, n]$.

For ε_2 we have two cases:

- $\varepsilon_2 = +1$, then the process above is repeated and we can claim two aforementioned properties are transformable in n step.

- $\varepsilon_2 = -1$, this case is impossible, because it is contradictory to diagonally dominant property of $A^{(2)}[2, \cdots, n]$.

case 2: $\varepsilon_1 = \varepsilon_2 = -1$ or $\varepsilon_1 = +1$ & $\varepsilon_2 = -1$

This is impossible for cases above, because our matrix is diagonally dominant.

case 3: $\varepsilon_1 = -1$ & $\varepsilon_2 = +1$,

similarly, in this case we can show that matrix $\tilde{A}^{(1)}$ exactly equals $A^{(1)} = A$. But it is necessary that we show $A^{(2)}[2, \cdots, n]$ is diagonally dominant. Since all entries of matrix are negative when we remove absolute value, the direction of inequality will change, therefore for all i, $1 \leq i \leq n$ we have:

$$a_{ii} \leq \sum_{j\neq i} a_{ij}. \tag{6}$$

Now we consider an arbitrary row of matrix $A^{(2)}$ and prove that it satisfies diagonally dominant property. To prove we should show that:

$$| a_{ii} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,i} | \geq \sum_{j=2,j\neq i}^{n} | a_{ij} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,j} |$$

If the relation above does not hold, then

$$| a_{ii} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,i} | < \sum_{j=2,j\neq i}^{n} | a_{ij} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,j} |$$

since all entries of matrix $A^{(2)}[2, \cdots, n]$ are negative, then we can remove absolute value and we have

$$a_{ii} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,i} > \sum_{j=2,j\neq i}^{n} a_{ij} - \frac{a_{i1}}{a_{i-1,1}}a_{i-1,j}. \tag{7}$$

On the other hand by relations (6) and (7) we have:

$$a_{i1} \geq a_{ii} - \sum_{j\neq i,1} a_{ij}$$

$$\Rightarrow a_{ii} - \sum_{j\neq i,1} a_{ij} > \frac{a_{i1}}{a_{i-1,1}}\left(a_{i-1,i} - \sum_{j\neq i} a_{i-1,j}\right)$$

through dividing relation above by a_{i1} and multiplying by $a_{i-1,1}$ (These entries are nonzero, because the Neville Elimination is not applied for them) we have:

$$a_{i-1,1} > a_{i-1,i} - \sum_{j\neq i} a_{i-1,j} = a_{i-1,i} - a_{i-1,i-1} - \sum_{j\neq i,i-1} a_{i-1,j} > 0$$

and this is in contradiction to being negative of entries of $A^{(2)}[2, \cdots, n]$.

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