Functional Equations and Integral Equations in Spectral Domain for Scattering by Impedance Polygons

J.M.L. Bernard

***Abstract*—Some features of functional and integral equations involved in the spectral approach developed by the author (in Qu. J. of Mech. and Appl. Math., 59, 4, pp.517-550, 2006) for scattering by two-dimensional polygonal objects with arbitrary surface impedance conditions are presented. In this problem, the Wiener-Hopf method cannot be applied, while asymptotic methods can only be used if corners are widely spaced compared to wavelength, and the presence of imperfectly reflective surfaces particularly complicates the problem. After presenting our method to handle in a global manner the problem of n-part polygonal objects using the Sommerfeld-Maliuzhinets representation of the field, we detail the functional equations for the spectral functions, and the way to reduce them to a system of integral equations of the second kind with non-singular kernels, allowing approximations. We apply in particular this approach to the important class of three-part impedance polygons composed of a finite segment attached to two semi-infinite planes.**

***Index Terms*—spectral method, integral equations, functional equations, helmholtz equation, polygonal surface.**

I. INTRODUCTION

SOME features of functional and integral equations involved in the spectral approach developed by the author in [5] for scattering by two-dimensional polygonal objects with impedance boundary conditions are presented. In this delicate exterior problem, the Wiener-Hopf method cannot be applied [1-2], while asymptotic methods can only be used if corners are widely spaced compared to wavelength [3], and the presence of imperfectly reflective surfaces particularly complicates the problem.

To handle the problem, we consider the Sommerfeld-Maliuzhinets representation of the field,

$$u(\rho,\varphi)=\frac{1}{2\pi i}\int_{\gamma} f(\alpha+\varphi)e^{ik\rho\cos\alpha}d\alpha, \qquad (1)$$

which satisfies the Helmholtz equation $(\Delta+k^2)u(\rho,\varphi)=0$, in free space sector $-\Phi\leq\varphi\leq\Phi$ which contains the scatterer.

In this representation, f is an analytic function and the path γ consists of two branches: one, named γ_+, going from $(i\infty+\arg(ik)+(a_1+\frac{\pi}{2}))$ to $(i\infty+\arg(ik)-(a_2+\frac{\pi}{2}))$ with $0<a_{1,2}<\pi$, as $\mathrm{Im}\alpha\geq d$, above all the singularities of the integrand, and the other, named γ_-, obtained by inversion of γ_+ with respect to $\alpha=0$.

J.M.L. Bernard is with Département de Physique Théorique et Appliquée, CEA/DIF-Bruyères le Châtel, 91297 Arpajon cedex, and, with LRC MESO, CMLA-CEA, 61 av. du Pres. Wilson, 94235 Cachan, France e-mail: jean-michel.bernard@cea.fr

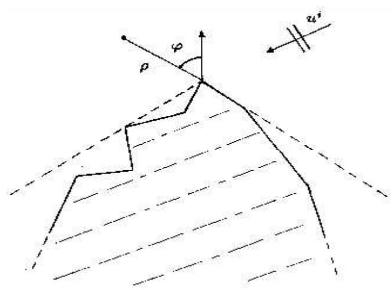

Fig. 1. geometry

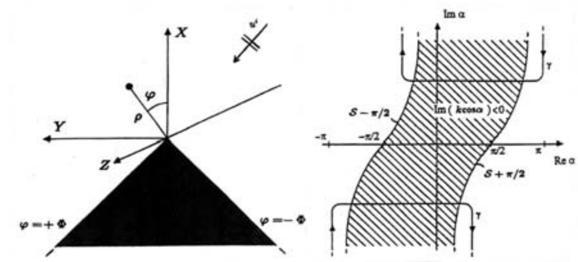

Fig. 2. coordinates and complex path

This representation has long been devoted to the rigorous analysis of isolated wedges. However, some of our recent developments permit us to consider a new integral expression of the spectral function, in some domain of complex angles, where it becomes possible to take globally account of boundary conditions on a complex geometry [4],[5]. The problem can be then reduced to original difference and integral equations that are studied.

II. SINGLE-FACE EXPRESSION OF f AND ITS USE FOR POLYGONAL OBJECTS

An expression allows to consider arbitrary shapes. For this, we show first [4-5] that

$$f(\pm\pi+\varphi)=\frac{1}{2}\int_0^{\infty}(iku(\rho',\pm\Phi)\sin(\varphi\mp\Phi)\pm\frac{\partial u}{\partial n}(\rho',\pm\Phi))e^{ik\rho'\cos(\varphi\mp\Phi)}d\rho', \qquad (2)$$

as $\frac{\pi}{2}<\Phi\mp\varphi_\circ<\frac{3\pi}{2}$ and $\frac{\pi}{2}<\Phi\mp\varphi<\frac{3\pi}{2}$, $|\arg(ik)|<\frac{\pi}{2}$, where $\varphi_\circ$ is the incident plane wave direction, with some general properties of the field permitting the convergence.

Using Green's theorem, we then note that the contour of integration along $\varphi = \pm\Phi$ can be deformed into any path $L^{\pm}_{0,\infty}$, provided that the integral remains bounded and no source passes through the path during the deformation. So, if we divide the semi-infinite paths $L^{\pm}_{0,\infty}$ (deriving from a deformation of the faces $\varphi = \pm\Phi$ enclosing the scatterer, described above) into $L^{\pm}_{0,\Delta^{\pm}}$ (i.e. $0 < l' < \Delta^{\pm}$) and $L^{\pm}_{\Delta,\infty}$ (i.e. $l' > \Delta^{\pm}$), we have

$$f(\pm\pi + \varphi) = \frac{1}{2}\int_{L^{\pm}_{0,\Delta^{\pm}}} (iku\sin(\varphi - \varphi'_t) \pm \frac{\partial u}{\partial n}) \times e^{ik\rho'\cos(\varphi-\varphi')} dl'(\rho', \varphi') + f_{L^{\pm}_{\Delta^{\pm},\infty}}(\pm\pi + \varphi), \qquad (3)$$

where $f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha) = e^{-ik\rho_{\Delta^{\pm}}\cos(\alpha - \varphi_{\Delta^{\pm}})} f^{\pm}_e(\alpha)$, $f^{\pm}_e(\alpha)$ is the spectral function related to the Sommerfeld-Maliuzhinets representation of the field in coordinates with origin at $l' = \Delta^{\pm}$. We can then write, by analytic continuation,

$$f(\alpha) = \frac{1}{2}\int_{L^{\pm}_{0,\Delta^{\pm}}} (-iku\sin(\alpha - \varphi'_t) \pm \frac{\partial u}{\partial n}) \times e^{-ik\rho'\cos(\alpha-\varphi')} dl'(\rho', \varphi') + f_{L^{\pm}_{\Delta^{\pm},\infty}}(\alpha), \qquad (4)$$

that is called henceforth the single-face expressions of f.

Let us consider a polygonal surface located inside the domain $|\varphi| > \Phi$ enclosing a scatterer. This surface is composed of two joined semi-infinite polygonal faces, denoted $+$ and $-$, respectively with $m^{\pm}$ segments of lengths $d^{\pm}_j$ with tangent angles $\pm\Phi^{\pm}_j$, $j = 1, 2, ..., m^{\pm}$ and a semi-infinite plane with tangent angles $\pm\Phi^{\pm}_e$. Then, the single face expression of the spectral function f becomes [5]

$$\begin{aligned} f(\alpha) \;=\; & \frac{1}{2}\sum_{1\leq j\leq m^{\pm}} e^{-ik\sum_{1\leq i<j} d^{\pm}_i \cos(\alpha\mp\Phi^{\pm}_i)} \\ & \int_0^{d^{\pm}_j} (-iku(\rho'_j, \pm\Phi^{\pm}_j)\sin(\alpha \mp \Phi^{\pm}_j) \\ & \pm\frac{\partial u}{\partial n}(\rho'_j, \pm\Phi^{\pm}_j))e^{-ik\rho'_j\cos(\alpha\mp\Phi^{\pm}_j)})d\rho'_j \\ & + e^{-ik\sum_{1\leq i\leq m^{\pm}} d^{\pm}_i\cos(\alpha\mp\Phi^{\pm}_i)} f^{\pm}_{e,m^{\pm}}(\alpha), \qquad (5) \end{aligned}$$

where $f^{\pm}_{e,m^{\pm}}(\alpha)$ is the analytic continuation of the integral expression

$$f^{\pm}_{e,m^{\pm}}(\alpha' \pm \Phi^{\pm}_e) = \frac{1}{2}\int_0^{\infty} (-iku(\rho'_e, \pm\Phi^{\pm}_e)\sin\alpha' \pm\frac{\partial u}{\partial n}(\rho'_e, \pm\Phi^{\pm}_e))e^{-ik\rho'_e\cos\alpha'} d\rho'_e, \qquad (6)$$

valid as $\mathrm{Re}(ik(\cos\alpha' - \cos(\Phi^{\pm}_e \mp \varphi_\circ))) > 0$, $|\mathrm{Re}\alpha'| < \pi, |\arg(ik)| < \frac{\pi}{2}$.

This original expression of the spectral function and its properties enable us to derive, for the first time, the functional equations for the spectral functions for scattering by a general impedance polygon with finite or infinite surface [5], and to reduce generally the problem to a system of integral equations of the second kind with non-singular kernels.

We apply in particular this approach to the important class of three-part impedance polygons composed of a finite segment attached to two semi-infinite planes.

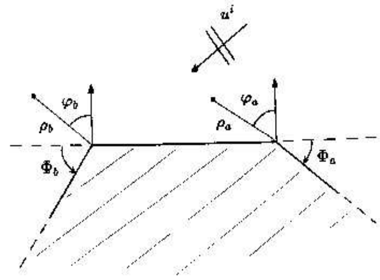

Fig. 3. geometry of three-part impedance polygons

III. Functional and Integral equations for a semi-infinite three-part impedance polygon

The functions f_a and f_b are the spectral functions associated with the Sommerfeld-Maliuzhinets representation of the field, in cylindrical coordinate systems (ρ_a, φ_a) and (ρ_b, φ_b), with origins at opposite ends of the finite segment. We have, in (ρ_a, φ_a) coordinates,

$$\begin{aligned} (\rho_a \in]0, \infty[, \varphi_a = -\tfrac{\pi}{2} - \Phi_a) \text{ with } \tfrac{\partial u}{\partial n} - ik\sin\theta_- u = 0, \\ (\rho_a \in [0, \Delta], \varphi_a = \tfrac{\pi}{2}) \text{ with } \tfrac{\partial u}{\partial n} - ik\sin\theta_1 u = 0, \qquad (7) \end{aligned}$$

with the incident field $u^i = e^{ik\rho_a\cos(\varphi_a - \varphi_\circ)}$, and, in coordinates (ρ_b, φ_b),

$$\begin{aligned} (\rho_b \in [0, \Delta], \varphi_b = -\tfrac{\pi}{2}) \text{ with } \tfrac{\partial u}{\partial n} - ik\sin\theta_1 u = 0, \\ (\rho_b \in]0, \infty[, \varphi_b = \tfrac{\pi}{2} + \Phi_b) \text{ with } \tfrac{\partial u}{\partial n} - ik\sin\theta_+ u = 0, \quad (8) \end{aligned}$$

with the incident field $u^i = e^{ik(\rho_b\cos(\varphi_b - \varphi_\circ) + \Delta\sin\varphi_\circ)}$.

IV. Functional equations for the spectral functions

Functional difference equations for f_a and f_b are obtained from (5)-(6) with (7)-(8), using parity of some expressions [5]. So, denoting $f_{br}(\alpha - \frac{\Phi_b}{2}) = f_b(\alpha)$ et $\Phi_+ = \frac{\pi}{2} + \frac{\Phi_b}{2}$, we have

$$\begin{aligned} & (\sin\alpha + \sin\theta_+)f_{br}(\alpha + \Phi_+) - \\ & \quad -(-\sin\alpha + \sin\theta_+)f_{br}(-\alpha + \Phi_+) = 0, \\ & (\sin\alpha - \sin\theta_1)f_{br}(\alpha - \Phi_+) - \\ & \quad -(-\sin\alpha - \sin\theta_1)f_{br}(-\alpha - \Phi_+) = \\ & = e^{-ik\Delta\cos\alpha}((\sin\alpha - \sin\theta_1)f_a(\alpha - \frac{\pi}{2}) - \\ & \quad -(-\sin\alpha - \sin\theta_1)f_a(-\alpha - \frac{\pi}{2})) = R^-_b(\alpha), \qquad (9) \end{aligned}$$

while, denoting $f_{ar}(\alpha + \frac{\Phi_a}{2}) = f_a(\alpha)$ and $\Phi_- = \frac{\pi}{2} + \frac{\Phi_a}{2}$, we obtain

$$\begin{aligned} & (\sin\alpha + \sin\theta_1)f_{ar}(\alpha + \Phi_-) - \\ & \quad -(-\sin\alpha + \sin\theta_1)f_{ar}(-\alpha + \Phi_-) = \\ & = e^{-ik\Delta\cos\alpha}((\sin\alpha + \sin\theta_1)f_b(\alpha + \frac{\pi}{2}) - \\ & \quad -(-\sin\alpha + \sin\theta_1)f_b(-\alpha + \frac{\pi}{2})) = R^+_a(\alpha), \\ & (\sin\alpha - \sin\theta_-)f_{ar}(\alpha - \Phi_-) \\ & \quad -(-\sin\alpha - \sin\theta_-)f_{ar}(-\alpha - \Phi_-) = 0. \qquad (10) \end{aligned}$$

ISBN:978-988-17012-3-7

A. Derivation of integral expressions and equations

From functional theory equations, the analytic function $\chi(\alpha)$ verifying

$$\chi(\alpha \pm \Phi) - \chi(-\alpha \pm \Phi) = \vartheta^{\pm}(\alpha), \tag{11}$$

and regular in the band $|\mathrm{Re}\alpha| \leq \Phi$ (even at infinity), is given as $|\mathrm{Re}\alpha| < \Phi$, by

$$\chi(\alpha) = \frac{\chi(i\infty) + \chi(-i\infty)}{2} + \frac{-i}{8\Phi}\int_{-i\infty}^{+i\infty} (\vartheta^{+}(\alpha') \times \times \tan(\nu(\alpha + \Phi - \alpha')) - \vartheta^{-}(\alpha')\tan(\nu(\alpha - \Phi - \alpha')))d\alpha' \tag{12}$$

with $\nu = \frac{\pi}{4\Phi}$, when the $\vartheta^{\pm}(\alpha)$ are regular and summable on imaginary axis. We then use the solutions $\Psi_{+1}(\alpha)$ (resp. $\Psi_{1-}(\alpha)$), without pole or zero in the band $|\mathrm{Re}\alpha| \leq \Phi_+$ (resp. $|\mathrm{Re}\alpha| \leq \Phi_-$) and $O(\cos(\pi\alpha/2\Phi_+))$ (resp. $O(\cos(\pi\alpha/2\Phi_-))$) in this domain, of previous functional equations taken without second member.

We then obtain for $-\pi/2 < \varphi_\circ < \pi/2 + \Phi_b$ and $-\Phi_+ <\mathrm{Re}\alpha < 3\Phi_+$,

$$\frac{f_{br}(\alpha)}{\Psi_{+1}(\alpha)} - \chi_b^i(\alpha) = = \frac{i}{8\Phi_+}\int_{-i\infty}^{+i\infty} \frac{R_b^-(\alpha')\tan(\frac{\pi}{4\Phi_+}(\alpha - \Phi_+ - \alpha'))}{(\sin\alpha' - \sin\theta_1)\Psi_{+1}(\alpha' - \Phi_+)}d\alpha' = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty} \frac{f_{ar}(\alpha' - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha' - \Phi_+)} \times \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_+)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha', \tag{13}$$

where the source term $\chi_b^i(\alpha)$ is given by $e^{ik\Delta\sin\varphi_\circ}\chi_{+1}^i(\alpha) = \frac{\pi}{2\Phi_+}(\frac{e^{ik\Delta\sin\varphi_\circ}\cos\frac{\pi\varphi_{\circ,b}}{2\Phi_+}}{\Psi_{+1}(\varphi_{\circ,b})(\sin\frac{\pi\alpha}{2\Phi_+} - \sin\frac{\pi\varphi_{\circ,b}}{2\Phi_+})})$, and, for $-\pi/2 - \Phi_b < \varphi_\circ < \pi/2$ et $-3\Phi_- <\mathrm{Re}\alpha < \Phi_-$,

$$\frac{f_{ar}(\alpha)}{\Psi_{1-}(\alpha)} - \chi_a^i(\alpha) = = \frac{-i}{8\Phi_-}\int_{-i\infty}^{+i\infty} \frac{R_a^+(\alpha')\tan(\frac{\pi}{4\Phi_-}(\alpha + \Phi_- - \alpha'))}{(\sin\alpha' + \sin\theta_1)\Psi_{1-}(\alpha' + \Phi_-)}d\alpha' = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty} \frac{f_{br}(\alpha' + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha' + \Phi_-)} \times \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_-)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha', \tag{14}$$

where $\chi_a^i(\alpha) = \frac{\pi}{2\Phi_-}(\frac{\cos\frac{\pi\varphi_{\circ,a}}{2\Phi_+}}{\Psi_{1-}(\varphi_{\circ,a})(\sin\frac{\pi\alpha}{2\Phi_-} - \sin\frac{\pi\varphi_{\circ,a}}{2\Phi_-})})$, with $\varphi_{\circ,a} = \varphi_\circ + \Phi_a/2$ et $\varphi_{\circ,b} = \varphi_\circ - \Phi_b/2$.

Integral equations for $f_{br}(\alpha + \pi/2 - \Phi_b/2)$ and $f_{ar}(\alpha - \pi/2 + \Phi_a/2)$ can be then derived.

B. non singular integral equations when $\Phi_{a,b} > -\pi/2$

When $\Phi_{a,b} > -\pi/2$, we have, from previous expressions,

$$\frac{f_{br}(\alpha + \pi/2 - \Phi_b/2)}{\Psi_{+1}(\alpha + \pi/2 - \Phi_b/2)} - \chi_{br}^i(\alpha + \pi/2 - \Phi_b/2) = = \frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty}(\frac{f_{ar}(\alpha' - \pi/2 + \Phi_a/2)}{\Psi_{+1}(\alpha' - \Phi_+)}) \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha', \tag{15}$$

as $-\Phi_+ <\mathrm{Re}(\alpha + \pi/2 - \Phi_b/2) < 3\Phi_+$, and,

$$\frac{f_{ar}(\alpha - \pi/2 + \Phi_a/2)}{\Psi_{1-}(\alpha - \pi/2 + \Phi_a/2)} - \chi_{ar}^i(\alpha - \pi/2 + \Phi_a/2) = = \frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty}(\frac{f_{br}(\alpha' + \pi/2 - \Phi_b/2)}{\Psi_{1-}(\alpha' + \Phi_-)}) \times \frac{e^{-ik\Delta\cos\alpha'}\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha', \tag{16}$$

as $-3\Phi_- <\mathrm{Re}(\alpha - \pi/2 + \Phi_a/2) < \Phi_-$, for $-\pi/2 < \varphi_\circ < \pi/2$. The equations can be solved numerically, or analytically by approximations, since the term depending on $k\Delta$ is simple and the source terms are not oscillating. Suitable for approximations when $k\Delta$ is large, we can also transform them by semi-inversion for approximations when $k\Delta$ is small.

V. A semi-inversion to obtain integral equations with kernels vanishing as $k\Delta \to 0$

We can modify equations and derive integral equations with kernels vanishing as $k\Delta \to 0$ for the three-part semi-infinite impedance polygon, for approximations when $k\Delta$ is small. For this, we begin with changing the unknowns in the equations (15)-(16), considering $f_{ar0}(\alpha) = f_{ar}(\alpha) - f_0(\alpha - (\Phi_+ - \frac{\pi}{2}), \varphi_\circ)$, $f_{br0}(\alpha) = f_{br}(\alpha) - e^{ik\Delta\sin\varphi_\circ}f_0(\alpha + (\Phi_- - \frac{\pi}{2}), \varphi_\circ)$ where $f_0(\alpha, \varphi_\circ)$, corresponding to the solution for $\Delta = 0$, is known [5]. These functions vanish as $\Delta = 0$, and satisfy, from (15)-(16),

$$\frac{f_{br0}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{+1}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2})} = \frac{-i}{4\Phi_+}(\int_{-i\infty}^{+i\infty}(\frac{f_{ar0}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)}) \times \frac{\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha' + + \int_{-i\infty}^{+i\infty}\frac{B_{a0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_+})}{\cos(\frac{\pi}{2\Phi_+}(\alpha - \Phi_b)) + \cos(\frac{\pi\alpha'}{2\Phi_+})}d\alpha') \tag{17}$$

where

$$B_{a0}(\alpha') = (\frac{f_{ar0}(\alpha' - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta\cos\alpha'} - 1) + +(\frac{e^{ik\Delta\sin\varphi_\circ}f_0(\alpha' - \frac{\pi}{2} + (\Phi_a - \Phi_b)/2)}{\Psi_{+1}(\alpha' - \Phi_+)})(e^{-ik\Delta(\cos\alpha' + \sin\varphi_\circ)} - 1) \tag{18}$$

as $-\Phi_+ <\mathrm{Re}(\alpha + \frac{\pi}{2} - \frac{\Phi_b}{2}) < 3\Phi_+$, and

$$\frac{f_{ar0}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})}{\Psi_{1-}(\alpha - \frac{\pi}{2} + \frac{\Phi_a}{2})} = \frac{i}{4\Phi_-}(\int_{-i\infty}^{+i\infty}(\frac{f_{br0}(\alpha' + \frac{\pi}{2} - \frac{\Phi_b}{2})}{\Psi_{1-}(\alpha' + \Phi_-)}) \times \frac{\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha' + + \int_{-i\infty}^{+i\infty}\frac{B_{b0}(\alpha')\sin(\frac{\pi\alpha'}{2\Phi_-})}{\cos(\frac{\pi}{2\Phi_-}(\alpha + \Phi_a)) + \cos(\frac{\pi\alpha'}{2\Phi_-})}d\alpha') \tag{19}$$

where

$$B_{b0}(\alpha') = (\tfrac{f_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta\cos\alpha'}-1)+ \\ +(\tfrac{f_0(\alpha'+\frac{\pi}{2}-(\Phi_b-\Phi_a)/2)}{\Psi_{1-}(\alpha'+\Phi_-)})(e^{-ik\Delta(\cos\alpha'-\sin\varphi_\circ)}-1) \quad (20)$$

as $-3\Phi_- <\mathrm{Re}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})<\Phi_-$, for $-\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_a)<\varphi_\circ<\min(\frac{\pi}{2},\frac{\pi}{2}+\Phi_b)$.

We then notice some similarity with the equations satisfied by f_0 when $\Delta=0$. Thus, we let

$$f'_{br0}(\alpha+\tfrac{\pi}{2}-\tfrac{\Phi_b}{2}) = \\ =\int_{-i\infty}^{i\infty} G(\varphi')f_0(\alpha+\tfrac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')d\varphi', \\ f'_{ar0}(\alpha-\tfrac{\pi}{2}+\tfrac{\Phi_a}{2}) = \\ =\int_{-i\infty}^{i\infty} G(\varphi')f_0(\alpha-\tfrac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')d\varphi' \quad (21)$$

as $|\mathrm{Re}(\alpha)|<\frac{\pi}{2}$, and search to define $G(\varphi')$ so that f'_{br0} and f'_{ar0} verifies (17)-(19). The functions $f_0(\alpha\pm\frac{\pi}{2}+(\Phi_a-\Phi_b)/2,\varphi')$ are regular and $O(1/\cos(\pi\varphi'/2\Phi_d))$ on the imaginary axis, and a pole at $\varphi'=\alpha\pm\frac{\pi}{2}$ ensures that, even if $\Phi_a=\Phi_b=0$, generally $f'_{br0}\neq f'_{ar0}$.

Using the equations (15)-(16) when $\Delta=0$ satisfied by f_0, we remark that we can write

$$\frac{f'_{br0}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})}{\Psi_{+1}(\alpha+\frac{\pi}{2}-\frac{\Phi_b}{2})} = \\ =\frac{-i}{4\Phi_+}\int_{-i\infty}^{+i\infty}\frac{f'_{ar0}(\alpha'-\frac{\pi}{2}+\frac{\Phi_a}{2})\sin(\frac{\pi\alpha'}{2\Phi_+})}{\Psi_{+1}(\alpha'-\Phi_+)(\cos(\frac{\pi}{2\Phi_+}(\alpha-\Phi_b))+\cos(\frac{\pi\alpha'}{2\Phi_+}))}d\alpha' \\ +\frac{\pi}{2\Phi_+}\int_{-i\infty}^{i\infty}\frac{G(\varphi')}{\Psi_{+1}(\varphi'-\frac{\Phi_b}{2})}\frac{\sin\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+}}{(\cos\frac{\pi(\alpha-\Phi_b)}{2\Phi_+}+\cos\frac{\pi(\varphi'+\frac{\pi}{2})}{2\Phi_+})}d\varphi', (22)$$

$$\frac{f'_{ar0}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})}{\Psi_{1-}(\alpha-\frac{\pi}{2}+\frac{\Phi_a}{2})} \\ =\frac{i}{4\Phi_-}\int_{-i\infty}^{+i\infty}\frac{f'_{br0}(\alpha'+\frac{\pi}{2}-\frac{\Phi_b}{2})\sin(\frac{\pi\alpha'}{2\Phi_-})}{\Psi_{1-}(\alpha'+\Phi_-)(\cos(\frac{\pi}{2\Phi_-}(\alpha+\Phi_a))+\cos(\frac{\pi\alpha'}{2\Phi_-}))}d\alpha' \\ +\frac{\pi}{2\Phi_-}\int_{-i\infty}^{i\infty}\frac{G(\varphi')}{\Psi_{1-}(\varphi'+\frac{\Phi_a}{2})}\frac{\sin\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-}}{(\cos\frac{\pi(\alpha+\Phi_a)}{2\Phi_-}+\cos\frac{\pi(\varphi'-\frac{\pi}{2})}{2\Phi_-})}d\varphi'. (23)$$

In the case where $G(\varphi')$ is regular in the band $|\mathrm{Re}(\varphi')|\leq\frac{\pi}{2}$, we can shift the integral paths in the integrals containing $G(\varphi')$. Comparing (17)-(19) with (22)-(23), we notice that (f'_{br0},f'_{ar0}) is a solution of the system of equations (17)-(19) if G satisfies the conditions

$$\frac{G(\alpha'+\frac{\pi}{2})}{\Psi_{1-}(\alpha'+\Phi_-)}-\frac{G(-\alpha'+\frac{\pi}{2})}{\Psi_{1-}(-\alpha'+\Phi_-)} = \\ =\frac{i}{2\pi}(B_{b0}(\alpha')-B_{b0}(-\alpha')), \\ \frac{G(\alpha'-\frac{\pi}{2})}{\Psi_{+1}(\alpha'-\Phi_+)}-\frac{G(-\alpha'-\frac{\pi}{2})}{\Psi_{+1}(-\alpha'-\Phi_+)} = \\ =\frac{-i}{2\pi}(B_{a0}(\alpha')-B_{a0}(-\alpha')), \quad (24)$$

where $\Phi_+=\frac{\pi}{2}+\frac{\Phi_b}{2}$ and $\Phi_-=\frac{\pi}{2}+\frac{\Phi_a}{2}$. Taking account of the properites of Ψ_{+1} and Ψ_{1-} , and letting $G(\alpha')=(\cos\alpha'+\sin\theta_1)g(\alpha')$, (24) can be written

$$g(\alpha'+\frac{\pi}{2})-g(-\alpha'+\frac{\pi}{2}) = \\ =\frac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(-\sin\alpha'+\sin\theta_1)}, \\ g(\alpha'-\frac{\pi}{2})-g(-\alpha'-\frac{\pi}{2}) = \\ =\frac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(-\sin\alpha'-\sin\theta_1)} (25)$$

Since $G(\alpha')$ is regular in the band $|\mathrm{Re}\alpha'|\leq\frac{\pi}{2}$ and $\mathrm{Re}(\sin\theta_1)>0$, $g(\alpha')$ is regular in this band. We can then use (11)-(12), and write, as $|\mathrm{Re}\alpha|<\frac{\pi}{2}$,

$$g(\alpha)=\frac{i}{4\pi}\int_{-i\infty}^{+i\infty}(\tfrac{i\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{2\pi(\sin\alpha'-\sin\theta_1)}\times \\ \times\tan(\tfrac{1}{2}(\alpha+\tfrac{\pi}{2}-\alpha'))-\tfrac{i\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{2\pi(\sin\alpha'+\sin\theta_1)}\times \\ \times\tan(\tfrac{1}{2}(\alpha-\tfrac{\pi}{2}-\alpha')))d\alpha', \quad (26)$$

Using (21) and (26), we obtain the equations with kernels vanishing as $k\Delta\to 0$:

$$f_{br0}(\alpha+\tfrac{\pi}{2}-\tfrac{\Phi_b}{2}) = \\ =\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}d\alpha'(\tfrac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}M_+(\alpha,\alpha') \\ -\tfrac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}M_-(\alpha,\alpha')), \quad (27)$$

$$f_{ar0}(\alpha-\tfrac{\pi}{2}+\tfrac{\Phi_a}{2}) = \\ =\frac{1}{8\pi^2}\int_{-i\infty}^{+i\infty}(\tfrac{\Psi_{1-}(\alpha'+\Phi_-)(B_{b0}(\alpha')-B_{b0}(-\alpha'))}{\sin\alpha'-\sin\theta_1}N_+(\alpha,\alpha') \\ -\tfrac{\Psi_{+1}(\alpha'-\Phi_+)(B_{a0}(\alpha')-B_{a0}(-\alpha'))}{\sin\alpha'+\sin\theta_1}N_-(\alpha,\alpha'))d\alpha', \quad (28)$$

where $M_\pm(\alpha,\alpha')=L_\pm(\alpha+\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$, $N_\pm(\alpha,\alpha')=L_\pm(\alpha-\frac{\pi}{2}+\frac{(\Phi_a-\Phi_b)}{2},\alpha')$,

$$L_\pm(\alpha,\alpha')=\frac{\pi\sin\alpha'\Psi_{+-}(\alpha)}{2\Phi_d}\int_{-i\infty}^{i\infty}\frac{\cos(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d})}{\Psi_{+-}(\varphi'+(\Phi_a-\Phi_b)/2)} \\ \frac{(\cos\varphi'+\sin\theta_1)}{\cos(\varphi'\pm\frac{\pi}{2})+\cos\alpha'}\frac{1}{(\sin(\frac{\pi\alpha}{2\Phi_d})-\sin(\frac{\pi(\varphi'+(\Phi_a-\Phi_b)/2)}{2\Phi_d}))}d\varphi', \quad (29)$$

In the particular case $\Phi_a=\Phi_b=0$, $\Phi_d=\frac{\pi}{2}$, the functions $L_\pm$ can be simplified so that we recover the expressions found in [4] for the three-part impedance plane.

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