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Switchings Conditions Study for a Particular Class of Hybrid Dynamical Systems

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Abstract—**This paper aims to give general conditions on switchings thresholds to ensure the switchings existence for a particular class of hybrid dynamical systems. We develop here the study considering systems of this class in dimension one and, above all, in dimension two. We finally conclude that those conditions could be applied to any dimension systems of this hybrid systems class and we illustrate all results with a thermal application.**

Keywords: hybrid dynamical system, switchings conditions, thresholds, hysteresis phenomenon, thermal application

1 Introduction

A hybrid dynamical system is a dynamical system that exhibits both continuous and discrete behaviors [5], [6]. It is generally described by a differential equations system (which represents the continuous dynamics) and by a switching equation (which describes the discrete dynamics).

In this paper, we consider a particular class of hybrid dynamical systems (h.d.s.) with autonomous switchings, these switchings being generated by a hysteresis phenomenon. This class is very interesting to study. Indeed, first, it models an important number of industrial applications in many research areas (thermic, electronics, automotive...) and in any dimension. Moreover, it has a mathematical model which permits to develop a depth analysis. Thus, for example, we can solve optimization problems [2], we can study and determine equations for one-period cycles [3], for two-period cycles [4] and we can be confronted to chaos problems with period-doubling bifurcations and positive value for the largest Lyapunov exponent [1]. So, despite a relatively simple mathematical model, a very complex and important analysis can be made.

Nevertheless, all these problems imply that switchings exist and in applications, numerical values were chosen in order to have a cycle for solution. So, all this study makes sense only if switchings exist which is not always the case. That's why, in this paper, we give conditions on some system parameters (and particularly on switchings thresholds) to ensure switchings existence and we try to conclude that these conditions are the same for any dimension systems belonging to the studied h.d.s. class.

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2 Presentation of the studied h.d.s. class

In R^N, $N \geq 1$, we consider a basis which is generally the canonical basis or an eigenvectors basis. In relation to this basis, we consider the following hybrid dynamical system (h.d.s.) of order N :

$$\begin{cases} \dot{X}(t) = AX(t) + q(\xi(t))B + C, \\ \xi(t) = LX(t), \end{cases} \qquad (1)$$

where A is a square matrix of order N, B, C and X are columns matrices of order N, L is a row matrix of order N, all these matrices taking real values. Moreover, we suppose that matrix A only has eigenvalues with strictly negative real part and that X, and so ξ are continuous.

In this model, the discrete variable is q, which can take two values 0 and 1 according to ξ following a hysteresis phenomenon described on figure 1. If ξ reaches its lower

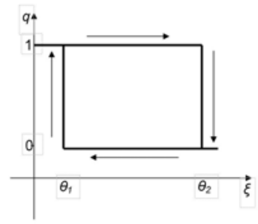

Figure 1: Hysteresis phenomenon

threshold θ_1 by decreasing value, then q goes from value 0 to value 1. Identically, if ξ reaches its upper threshold θ_2 by increasing value, then q goes from value 1 to value 0. In these conditions, the multifunction $q(\xi)$ is explicitly given by :

$$\begin{cases} q(\xi(t)) = 0 \text{ if } \xi(t^-) = \theta_2 \text{ and } q(\xi(t^-)) = 1, \\ q(\xi(t)) = 1 \text{ if } \xi(t^-) = \theta_1 \text{ and } q(\xi(t^-)) = 0, \\ q(\xi(t)) = q(\xi(t^-)) \text{ otherwise.} \end{cases} \qquad (2)$$

In the first two cases of (2), t is called switching time and θ_1 and θ_2 are respectively called lower and upper switchings thresholds.

3 Study of switchings conditions

To find switchings conditions, first, we need to have cycles equations that have been generally determined in [1]. Repeat here the same reasoning.

We consider t_0 a given initial time and $t_1 < t_2 < ... < t_n < t_{n+1} < ...$ an increasing suite of successive switchings times in $[t_0, +\infty[$, all necessarily distincts because definition of $q(\xi(t))$ implies $\xi(t_n) \neq \xi(t_{n-1})$. So, we suppose that instant t_n of the n-th switching exists and is finite.

To simplify notations, we set $q_n = q(\xi(t_n))$ and we have $q(\xi(t)) = q_n$ in $[t_n, t_{n+1}[$. Identically, we set $\xi_n = \xi(t_n)$ and $\Delta q_n = q_n - q_{n-1}$. A classical integration of general differential system (1) in $[t_n, t_{n+1}[$ gives:

$$X(t) = \mathrm{e}^{(t-t_n)A}\Gamma_n - A^{-1}(q_nB + C), \quad (3)$$

where Γ_n is a column matrix of order N corresponding to integration constant, function of n. Thus, introducing notation $\sigma_n = t_n - t_{n-1} > 0$, $\forall n \geq 1$, and considering the continuity assumption of the state at t_n, we obtain:

$$\forall n \geq 1,\ \Gamma_n = \mathrm{e}^{\sigma_n A}\Gamma_{n-1} + \Delta q_n A^{-1}B. \quad (4)$$

Moreover, constant Γ_0 is given by equation (3) when $n = 0$ and $t = t_0$ by the following expression:

$$\Gamma_0 = X(t_0) + A^{-1}(q_0B + C). \quad (5)$$

Then, we have $\forall n \geq 1$, $\xi_n = q_n\theta_1 + q_{n-1}\theta_2$ which traduces the fact that ξ_n takes value θ_1 or θ_2 according to hysteresis variable q. We also have by definition $\forall n \geq 1$, $\xi_n = LX(t_n) = L(\Gamma_n - A^{-1}(q_nB + C))$. Combining these two expressions, we finally obtain the following equation:

$$\forall n \geq 1,\ L(\Gamma_n - A^{-1}(q_nB+C)) - q_n\theta_1 - q_{n-1}\theta_2 = 0. \quad (6)$$

Resolution of system (1), (2) with unknowns $X(t)$, $(t_n)_{n\in N}$ is reduced to the one of system (4), (6) with unknowns $(\Gamma_n)_{n\geq 1}$, $(\sigma_n)_{n\geq 1}$. It is from these two last equations that we will study switchings conditions.

Moreover, such globally non linear systems can admit none, one or several solutions. Equation $\xi_n = \theta_1$ or θ_2 implies that search for cycles solution corresponds to search for periodic suites $(\sigma_{2n}, \Gamma_{2n}, \sigma_{2n+1}, \Gamma_{2n+1})_{n\in N}$ of period k, $k \geq 1$. For example, a cycle of period one is characterized by the existence of two different durations between two successive switchings times and corresponds to a constant suite $(\sigma_{2n}, \Gamma_{2n}, \sigma_{2n+1}, \Gamma_{2n+1})$.

So, like in [1], introducing notation $U_n^i = U_{2kn+i}$, $n \geq 0$, k being the period of the cycle, for any suite $(U_n)_{n\in N}$ and applying to our system given by (4), (6), we obtain $\forall i = 1, ..., 2k$:

$$\left\{\begin{array}{l} \Gamma_{n+1}^i - \mathrm{e}^{\sigma_{n+1}^i A}\Gamma_r^{i-1} - \Delta q_i A^{-1}B = 0, \\ L(\Gamma_{n+1}^i - A^{-1}(q_iB + C)) - q_i\theta_1 - q_{i-1}\theta_2 = 0, \end{array}\right. \quad (7)$$

with $r = n$ if $i = 1$, $r = n+1$ otherwise, $\Gamma_n^0 = \Gamma_n^{2k}$, $q_i = q_0$ if i is even and $q_i = q_1$ if i is odd.

It is from system (7) that we will study switchings conditions only considering period-one cycle ($k = 1$).

3.1 Dimension one

In dimension one, system (1) becomes:

$$\left\{\begin{array}{l} \dot{x}(t) = -ax(t) + q(x(t))b + c, \\ \xi(t) = x(t) \end{array}\right.$$

where $X(t) = x(t)$, $A = -a$, $B = b$, $C = c$, $L = 1$.

As mentioned in [3], system (7) becomes here $\forall n \geq 0$:

$$\left\{\begin{array}{l} \sigma_{n+1}^1 = -\frac{1}{a}\ln\left(\frac{\gamma_{n+1}^1 + \Delta q_1 a^{-1}b}{\gamma_n^2}\right) \\ \sigma_{n+1}^2 = -\frac{1}{a}\ln\left(\frac{\gamma_{n+1}^2 - \Delta q_1 a^{-1}b}{\gamma_{n+1}^1}\right) \\ \gamma_{n+1}^1 + a^{-1}(q_1b + c) - q_1\theta_1 - q_0\theta_2 = 0 \\ \gamma_{n+1}^2 + a^{-1}(q_0b + c) - q_0\theta_1 - q_1\theta_2 = 0, \end{array}\right.$$

where $\Gamma_{n+1}^i = \gamma_{n+1}^i$ for $i = 1, 2$.

Switchings exist if the first switching exists and if durations between different switching times σ_{n+1}^i, $i = 1, 2$, $n \geq 0$ also exist. Considering case $q_0 = 0$, $q_1 = 1$ or $q_0 = 1$, $q_1 = 0$, we finally obtain like in [3] following conditions for switchings existence.

If the necessary condition of the σ_{n+1}^i, $i = 1, 2$, $n \geq 0$ that is $a^{-1}c < \theta_1 < \theta_2 < a^{-1}(b + c)$ is satisfied and, moreover, if the condition of the first switching existence given by ($q_0 = 1$ and $x_0 < \theta_2 < a^{-1}(b + c)$) or ($q_0 = 1$ and $x_0 > \theta_1 > a^{-1}c$) is also satisfied, then, the hybrid system $\dot{x}(t) = -ax(t) + qb + c$ has for unique solution a cycle.

For the application, we consider a convector in direct exchange with the outside fluid. System is given here by:

$$Q_c\dot{x}(t) = -\frac{1}{R_c}x(t) + qP_c + \frac{\theta_e}{R_c},$$

where x (in K) is the convector temperature, R_c (in K.W^{-1}) is the convector resistance, $Q_c = m_cC_c$ with m_c (in kg) its mass, C_c (in J.kg^{-1}.W^{-1}) its heat capacity, P_c (in W) is its power and θ_e (in K) is the outside temperature. We choose the following numerical values

R_c	Q_c	P_c	θ_e	q_0
10	100	5	273	1

So, we have $a = \frac{1}{R_cQ_c} = 10^{-3}$, $b = \frac{P_c}{Q_c} = 0.05$, $c = \frac{\theta_e}{R_cQ_c} = 0.273$. To ensure switchings existence, we must choose $273 = a^{-1}c < \theta_1 < \theta_2 < a^{-1}(b + c) = 323$ and $x_0 < \theta_2 < a^{-1}(b + c)$.

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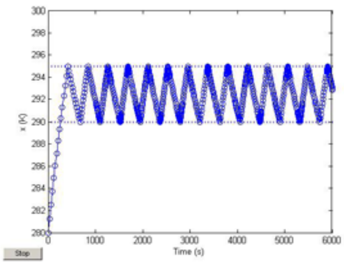


Figure 2: convector temperature simulation for $\theta_1 = 290$ K, $\theta_2 = 295$ K, $x_0 = 280$ K

• For example, if we set $\theta_1 = 290$ K, $\theta_2 = 295$ K, $x_0 = 280$ K, all conditions are satisfied then switchings exist as figure 2 shows.

• However, if we choose $\theta_1 = 290$ K, $\theta_2 = 295$ K, $x_0 = 300$ K, the first switching does not exist since $x_0 > \theta_2$ (see figure 3 on the left).

• Finally, if we set $\theta_1 = 272$ K, $\theta_2 = 295$ K, $x_0 = 280$ K, the first switching exists since the condition is satisfied but not the others since $\theta_1 < a^{-1}c = 273$ (see figure 3 on the right).

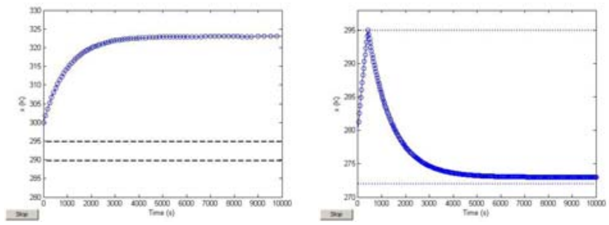


Figure 3: convector temperature simulation for $\theta_1 = 290$ K, $\theta_2 = 295$ K, $x_0 = 300$ K (on the left) and for $\theta_1 = 272$ K, $\theta_2 = 295$ K, $x_0 = 280$ K (on the right)

3.2 Dimension two

In dimension two, we consider a thermostat which controls a convector located in the same room. System (1) becomes:

$$\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = A \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + q(x) \begin{pmatrix} 0 \\ p_c \end{pmatrix} + \begin{pmatrix} d\theta_e \\ 0 \end{pmatrix} \quad (8)$$

where x is the room temperature and y is the convector temperature, $A = \begin{pmatrix} -(b+d) & b \\ c & -c \end{pmatrix}$, $b = \frac{1}{R_c m_p C_p}$, $c = \frac{1}{R_c m_c C_c}$, $d = \frac{1}{R_m m_p C_p}$, $p_c = \frac{P_c}{m_c C_c}$, $L = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Coefficients R_c, R_m (in K.W^{-1}) are thermal resistances, C_c, C_p (in J.kg^{-1}.K^{-1}) are heat capacities and m_c, m_p (in kg) are masses according to indices c, p, m which respectively represent the convector, the room and the house wall. Moreover, P_c (in W) is the convector power and θ_e (in K) is the outside temperature.

As in dimension one, for some values of the parameters, the system does not switch since the room temperature never reaches its upper or lower threshold. Moreover, for other values, the system can only switch once because the room temperature reaches a first time its lower (or upper) threshold but then, it never reaches its upper (or lower) threshold.

The reasoning to determine switchings conditions is the same than in dimension one. The only difference which introduces more difficulties, is that each component of Γ^i_{n+1}, $i = 1, 2$, $n \geq 0$ noted Γ^{ij}_{n+1}, $j = 1, 2$ can not only be expressed as a function of system parameters. To simplify calculuses, let us consider $\mathcal{B}$ a basis of eigenvectors of matrix A. According to this basis, system in dimension two is written $\dot{X} = AX + qB + C$ where:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} -\frac{p_c(c+A_2)}{A_1-A_2} \\ \frac{(c+A_1)(c+A_2)p_c}{c(A_1-A_2)} \end{pmatrix}$$

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{cd\theta_e}{A_1-A_2} \\ -\frac{(c+A_2)d\theta_e}{A_1-A_2} \end{pmatrix}, L = \begin{pmatrix} \frac{c+A_1}{c} \\ 1 \end{pmatrix}^t$$

where A_i, $i = 1, 2$ correspond to A eigenvalues, $A_1 = -\frac{1}{2}(b+c+d-\sqrt{\Delta})$, $A_2 = -\frac{1}{2}(b+c+d+\sqrt{\Delta}) < 0$, $\Delta = (b+c+d)^2 - 4cd$.

System (7) is explicitly given by:

$$\left\{ \begin{array}{l} \Gamma^{11} - e^{A_1\sigma^1}\Gamma^{21} - \Delta q_1 A_1^{-1} B_1 = 0 \\ \Gamma^{12} - e^{A_2\sigma^1}\Gamma^{22} - \Delta q_1 A_2^{-1} B_2 = 0 \\ \Gamma^{21} - e^{A_1\sigma^2}\Gamma^{11} + \Delta q_1 A_1^{-1} B_1 = 0 \\ \Gamma^{22} - e^{A_2\sigma^2}\Gamma^{12} + \Delta q_1 A_2^{-1} B_2 = 0 \\ L_1\Gamma^{11} + L_2\Gamma^{12} - A_1^{-1}L_1(q_1B_1 + C_1) - A_2^{-1}L_2(q_1B_2 + C_2) \\ -q_1\theta_1 - q_0\theta_2 = 0 \\ L_1\Gamma^{21} + L_2\Gamma^{22} - A_1^{-1}L_1(q_0B_1 + C_1) - A_2^{-1}L_2(q_0B_2 + C_2) \\ -q_0\theta_1 - q_1\theta_2 = 0 \end{array} \right. \quad (9)$$

where Γ^{ij} is the limit of Γ^{ij}_n, $i, j = 1, 2$ and σ^i is the limit of σ^i_n, $i = 1, 2$ when $n \to +\infty$.

From the first four equations of (9), we can write:

$$\left\{ \begin{array}{l} \sigma^1 = \frac{1}{A_1} \ln\left(\frac{\Gamma^{11} - \Delta q_1 A_1^{-1} B_1}{\Gamma^{21}}\right) = \frac{1}{A_2} \ln\left(\frac{\Gamma^{12} - \Delta q_1 A_2^{-1} B_2}{\Gamma^{22}}\right) \\ \sigma^2 = \frac{1}{A_1} \ln\left(\frac{\Gamma^{21} + \Delta q_1 A_1^{-1} B_1}{\Gamma^{11}}\right) = \frac{1}{A_2} \ln\left(\frac{\Gamma^{22} + \Delta q_1 A_2^{-1} B_2}{\Gamma^{12}}\right) \end{array} \right. . \quad (10)$$

We conclude that σ^i, $i = 1, 2$ exist if:

$$\left\{ \begin{array}{l} 0 < \frac{L_1\Gamma^{11} - \Delta q_1 L_1 A_1^{-1} B_1}{L_1\Gamma^{21}} < 1, \; 0 < \frac{L_1\Gamma^{21} + \Delta q_1 L_1 A_1^{-1} B_1}{L_1\Gamma^{11}} < 1 \\ 0 < \frac{L_2\Gamma^{12} - \Delta q_1 L_2 A_2^{-1} B_2}{L_2\Gamma^{22}} < 1, \; 0 < \frac{L_2\Gamma^{22} + \Delta q_1 L_2 A_2^{-1} B_2}{L_2\Gamma^{12}} < 1. \end{array} \right. \quad (11)$$

Sixteen possibilities come from system (11) combining all these following cases:

$$\mathbf{1a}\begin{cases} L_1\Gamma^{11}-\Delta q_1 L_1 A_1^{-1}B_1>0\\ L_1\Gamma^{21}>0\\ L_1\Gamma^{11}-\Delta q_1 L_1 A_1^{-1}B_1<L_1\Gamma^{21}\end{cases} \text{or } \mathbf{1b}\begin{cases} L_1\Gamma^{11}-\Delta q_1 L_1 A_1^{-1}B_1<0\\ L_1\Gamma^{21}<0\\ L_1\Gamma^{11}-\Delta q_1 L_1 A_1^{-1}B_1>L_1\Gamma^{21}\end{cases}$$

$$\mathbf{2a}\begin{cases} L_2\Gamma^{12}-\Delta q_1 L_2 A_2^{-1}B_2>0\\ L_2\Gamma^{22}>0\\ L_2\Gamma^{12}-\Delta q_1 L_2 A_2^{-1}B_2<L_2\Gamma^{22}\end{cases} \text{or } \mathbf{2b}\begin{cases} L_2\Gamma^{12}-\Delta q_1 L_2 A_2^{-1}B_2<0\\ L_2\Gamma^{22}<0\\ L_2\Gamma^{12}-\Delta q_1 L_2 A_2^{-1}B_2>L_2\Gamma^{22}\end{cases}$$

$$\mathbf{3a}\begin{cases} L_1\Gamma^{21}+\Delta q_1 L_1 A_1^{-1}B_1>0\\ L_1\Gamma^{11}>0\\ L_1\Gamma^{21}+\Delta q_1 L_1 A_1^{-1}B_1<L_1\Gamma^{11}\end{cases} \text{or } \mathbf{3b}\begin{cases} L_1\Gamma^{21}+\Delta q_1 L_1 A_1^{-1}B_1<0\\ L_1\Gamma^{11}<0\\ L_1\Gamma^{21}+\Delta q_1 L_1 A_1^{-1}B_1>L_1\Gamma^{11}\end{cases}$$

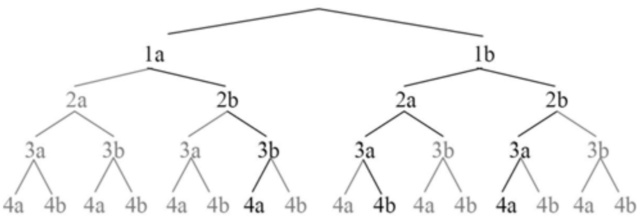
$$\mathbf{4a}\begin{cases} L_2\Gamma^{22}+\Delta q_1 L_2 A_2^{-1}B_2>0\\ L_2\Gamma^{12}>0\\ L_2\Gamma^{22}+\Delta q_1 L_2 A_2^{-1}B_2<L_2\Gamma^{12}\end{cases} \text{or } \mathbf{4b}\begin{cases} L_2\Gamma^{22}+\Delta q_1 L_2 A_2^{-1}B_2<0\\ L_2\Gamma^{12}<0\\ L_2\Gamma^{22}+\Delta q_1 L_2 A_2^{-1}B_2>L_2\Gamma^{12}\end{cases}$$

To simplify notations, restrict us to the case $q_0=1$, $q_1=0$ since case $q_0=0$, $q_1=1$ is treated with the same way and gives the same results. These sixteen possibilities are not all possible. Explain this.

First, we can prove that cases **1a** and **2a** can not be true simultaneously. Indeed, according to **1a**, we have $0<L_1\Gamma^{11}+L_1A_1^{-1}B_1<L_1\Gamma^{21}$. Identically, from case **2a**, we can write $0<L_2\Gamma^{12}+L_2A_2^{-1}B_2<L_2\Gamma^{22}$. Thus, adding these two expressions and using the two last equations of system (9), we finally obtain $LA^{-1}(B+C)+\theta_2<LA^{-1}(B+C)+\theta_1$ *i.e.* $\theta_2<\theta_1$, that is impossible because, by definition, θ_2 is the upper switching threshold and θ_1 the lower so $\theta_2>\theta_1$. Reasoning in the same way, we also obtain this contradiction combining cases **3b** and **4b**, cases **1a** and **4b** and cases **2a** and **3b**.

Moreover, cases **1a** and **3a** are also incompatible because the third inequation in **1a** contradicts the one of **3a**. Same remark can be formulated with cases **2a** and **4a**, cases **1b** and **3b** and cases **2b** and **4b**.

So, removing all these incompatibilities, we can eliminate thirteen of the sixteen possibilities that can be summed up in the following tree (gray branches represent impossi-



ble combinations and black branches the remaining combinations which are **1a-2b-3b-4a**, **1b-2a-3a-4b**, **1b-2b-3a-4a**).

These remarks were made independently of the system parameters values. To continue to restrict the number of possible combinations, let us prove that $B_2=-L_1B_1<0$. Recall that $B_2=\frac{(c+A_1)(c+A_2)p_c}{c(A_1-A_2)}$. We already know that $\frac{p_c}{c(A_1-A_2)}>0$ (since $A_1-A_2=\sqrt{\Delta}>0$, $p_c>0$, $c>0$). So, it remains to study sign of $(c+A_1)(c+A_2)$. We have:

$$(c+A_1)(c+A_2)=c^2+c(A_1+A_2)+A_1A_2,$$

where $A_1A_2=\frac{1}{4}((b+c+d)^2-\Delta)=cd$, $A_1+A_2=-(b+c+d)$. We finally obtain:

$$(c+A_1)(c+A_2)=-c(b+d)+cd=-bc<0,$$

that permits us to conclude $B_2=-L_1B_1<0$. From this, case **1a** implies $L_1\Gamma^{11}>-L_1A_1^{-1}B_1=L_2A_1^{-1}B_2>0$ so case **3b** is not compatible since it implies $L_1\Gamma^{11}<0$. Identically, case **2b** implies $L_2\Gamma^{12}<-L_2A_2^{-1}B_2<0$ that is incompatible with case **4a** (since it implies $L_2\Gamma^{12}>0$).

These two last remarks permit us to remove two combinations on the three remaining. Finally, only one combination is possible: **1b-2a-3a-4b**.

Switchings exist if and only if t_1 exists and σ^i, $i=1,2$ exist. From system (10), as $0>A_1>A_2$ *i.e.* $A_1^{-1}<A_2^{-1}<0$, we obtain:

$$\begin{cases} 0>\ln\left(\frac{L_1\Gamma^{11}+L_1A_1^{-1}B_1}{L_1\Gamma^{21}}\right)>\ln\left(\frac{L_2\Gamma^{12}+L_2A_2^{-1}B_2}{L_2\Gamma^{22}}\right)\\ 0>\ln\left(\frac{L_1\Gamma^{21}-L_1A_1^{-1}B_1}{L_1\Gamma^{11}}\right)>\ln\left(\frac{L_2\Gamma^{22}-L_2A_2^{-1}B_2}{L_2\Gamma^{12}}\right)\end{cases}$$

i.e.

$$\begin{cases} \frac{L_1\Gamma^{21}}{L_1\Gamma^{11}+L_1A_1^{-1}B_1}<\frac{L_2\Gamma^{22}}{L_2\Gamma^{12}+L_2A_2^{-1}B_2}\\ \frac{L_1\Gamma^{11}}{L_1\Gamma^{21}-L_1A_1^{-1}B_1}<\frac{L_2\Gamma^{12}}{L_2\Gamma^{22}-L_2A_2^{-1}B_2}\end{cases}$$

or

$$\begin{cases} L_1\Gamma^{21}(L_2\Gamma^{12}+L_2A_2^{-1}B_2)>L_2\Gamma^{22}(L_1\Gamma^{11}+L_1A_1^{-1}B_1)\\ L_1\Gamma^{11}(L_2\Gamma^{22}-L_2A_2^{-1}B_2)>L_2\Gamma^{12}(L_1\Gamma^{21}-L_1A_1^{-1}B_1),\end{cases} \tag{12}$$

since $L_1\Gamma^{11}+L_1A_1^{-1}B_1<0$, $L_2\Gamma^{12}+L_2A_2^{-1}B_2>0$, $L_1\Gamma^{21}-L_1A_1^{-1}B_1>0$, $L_2\Gamma^{22}-L_2A_2^{-1}B_2<0$.

From the first inequation of system (12) and using the two last equations of system (9), we have:

$$(-L_2\Gamma^{22}+LA^{-1}(B+C)+\theta_1)(L_2\Gamma^{12}+L_2A_2^{-1}B_2)>$$
$$L_2\Gamma^{22}(-L_2\Gamma^{12}-L_2A_2^{-1}B_2+LA^{-1}(B+C)+\theta_2$$

$$\Rightarrow (LA^{-1}(B+C)+\theta_1)(L_2\Gamma^{12}+L_2A_2^{-1}B_2)>$$
$$L_2\Gamma^{22}(LA^{-1}(B+C)+\theta_2).$$

If we suppose $\theta_2+LA^{-1}(B+C)>0$, as $0<L_2\Gamma^{12}+L_2A_2^{-1}B_2<L_2\Gamma^{22}$, we obtain $L_2\Gamma^{22}(\theta_2+LA^{-1}(B+C))>(L_2\Gamma^{12}+L_2A_2^{-1}B_2)(\theta_2+LA^{-1}(B+C))$ that gives us:

$$(\theta_1+LA^{-1}(B+C))(L_2\Gamma^{12}+L_2A_2^{-1}B_2)>$$
$$(\theta_2+LA^{-1}(B+C))(L_2\Gamma^{12}+L_2A_2^{-1}B_2)\ i.e.\ \theta_1>\theta_2$$

since $L_2\Gamma^{12}+L_2A_2^{-1}B_2>0$. This is a contradiction since θ_1 is the lower switching threshold and θ_2 the upper threshold so, by defintion, $\theta_1<\theta_2$. Finally, we conclude that σ^1 exists if and only if $\theta_1+LA^{-1}(B+C)<\theta_2+LA^{-1}(B+C)<0$.

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In the same way, from the second inequation of system (12), we obtain: $(LA^{-1}C+\theta_2)(L_2\Gamma^{22}-L_2A_2^{-1}B_2) > (LA^{-1}C+\theta_1)L_2\Gamma^{12}$. If we suppose $LA^{-1}C+\theta_1>0$, as $L_2\Gamma^{12} < L_2\Gamma^{22}-L_2A_2^{-1}B_2 < 0$, we obtain $L_2\Gamma^{12}(\theta_1 + LA^{-1}C) > (L_2\Gamma^{22}-L_2A_2^{-1}B_2)(\theta_1+LA^{-1}C)$ and:

$$\begin{aligned}&(LA^{-1}C+\theta_2)(L_2\Gamma^{22}-L_2A_2^{-1}B_2) >\\ &(L_2\Gamma^{22}-L_2A_2^{-1}B_2)(\theta_1+LA^{-1}C)\ i.e.\ \theta_2<\theta_1\end{aligned}$$

since $L_2\Gamma^{22}-L_2A_2^{-1}B_2<0$. Yet we get a contradiction with definitions of θ_2 and θ_1. We finally conclude $\theta_2 + LA^{-1}C > \theta_1 + LA^{-1}C > 0$.

Existence of switchings from the second switching is ensured if and only if $-LA^{-1}C < \theta_1 < \theta_2 < -LA^{-1}(B+C)$ which is exactly the same conditions found in dimension one ($-a^{-1}c < \theta_1 < \theta_2 < -a^{-1}(b+c)$ because $L=1$, $A=-a$).

It remains now to search for existence conditions for the first switching. First switching exists if and only if t_1 or $\sigma_1 = t_1 - t_0$ exists. Always considering case $q_0 = 1$, $q_1 = 0$, from system (7) and equation (5), we obtain:

$$\begin{aligned}\sigma_1 &= A_1^{-1}\ln\left(\frac{L_1\Gamma_{1,1}+L_1A_1^{-1}B_1}{L_1\Gamma_{0,1}}\right)\\ &= A_2^{-1}\ln\left(\frac{L_2\Gamma_{1,2}+L_2A_2^{-1}B_2}{L_2\Gamma_{0,2}}\right),\end{aligned} \tag{13}$$

where $\Gamma_{i,j}$, $i=0,1$, $j=1,2$ represent the integration constants components for $n=0$ and $n=1$ (here, we can't consider $\Gamma^1=\Gamma_1$ because Γ^1 is the limit when $n\to+\infty$).

Several cases must be taken into account for choices of $X(t_0)=X_0=(X_{0,1}\ X_{0,2})^t$:

A). $\begin{cases} L_1X_{0,1} < -L_1A_1^{-1}(B_1+C_1)\ i.e.\ L_1\Gamma_{0,1}<0,\\ L_2X_{0,2} < -L_2A_2^{-1}(B_2+C_2)\ i.e.\ L_2\Gamma_{0,2}<0,\end{cases}$

B). $\begin{cases} L_1X_{0,1} > -L_1A_1^{-1}(B_1+C_1)\ i.e.\ L_1\Gamma_{0,1}>0,\\ L_2X_{0,2} < -L_2A_2^{-1}(B_2+C_2)\ i.e.\ L_2\Gamma_{0,2}<0,\end{cases}$

C). $\begin{cases} L_1X_{0,1} > -L_1A_1^{-1}(B_1+C_1)\ i.e.\ L_1\Gamma_{0,1}>0,\\ L_2X_{0,2} > -L_2A_2^{-1}(B_2+C_2)\ i.e.\ L_2\Gamma_{0,2}>0,\end{cases}$

D). $\begin{cases} L_1X_{0,1} < -L_1A_1^{-1}(B_1+C_1)\ i.e.\ L_1\Gamma_{0,1}<0,\\ L_2X_{0,2} > -L_2A_2^{-1}(B_2+C_2)\ i.e.\ L_2\Gamma_{0,2}>0.\end{cases}$

Case C). implies that the system never switches because, from (13), we have $L_1\Gamma_{1,1}+L_1A_1^{-1}B_1>0$, $L_2\Gamma_{1,2}+L_2A_2^{-1}B_2>0$ *i.e.* $L\Gamma_1+LA^{-1}B=\theta_2+LA^{-1}(B+C)>0$ that is in contradiction with the found switchings conditions.

Moreover, from case A)., we directly obtain that σ_1 exists if $0 > L_1\Gamma_{1,1}+L_1A_1^{-1}B_1 > L_1\Gamma_{0,1}$, $0 > L_2\Gamma_{1,2}+L_2A_2^{-1}B_2 > L_2\Gamma_{0,2}$, *i.e.* $\theta_2+LA^{-1}(B+C) > LX_0 + LA^{-1}(B+C)$ *i.e.* $LX_0 < \theta_2 < -LA^{-1}(B+C)$.

From case B). and (13), we have:

$$1 > \frac{L_1\Gamma_{1,1}+L_1A_1^{-1}B_1}{L_1\Gamma_{0,1}} > \frac{L_2\Gamma_{1,2}+L_2A_2^{-1}B_2}{L_2\Gamma_{0,2}} > 0,$$

with $0 < L_1\Gamma_{1,1}+L_1A_1^{-1}B_1 < L_1\Gamma_{0,1}$ and $0 > L_2\Gamma_{1,2}+L_2A_2^{-1}B_2 > L_2\Gamma_{0,2}$. So, we obtain:

$$L_2\Gamma_{0,2}(L_1\Gamma_{1,1}+L_1A_1^{-1}B_1) < L_1\Gamma_{0,1}(L_2\Gamma_{1,2}+L_2A_2^{-1}B_2)$$

$$\Rightarrow \begin{aligned}&(LX_0+LA^{-1}(B+C))(L_1\Gamma_{1,1}+L_1A_1^{-1}B_1) <\\ &L_1\Gamma_{0,1}(\theta_2+LA^{-1}(B+C)).\end{aligned}$$

Since in this case $L_1\Gamma_{0,1} > L_1\Gamma_{1,1}+L_1A_1^{-1}B_1 > 0$ and $\theta_2+LA^{-1}(B+C)<0$, we have $L_1\Gamma_{0,1}(\theta_2+LA^{-1}(B+C)) < (L_1\Gamma_{1,1}+L_1A_1^{-1}B_1)(\theta_2+LA^{-1}(B+C))$ and we conclude:

$LX_0 < \theta_2 < -LA^{-1}(B+C)$ since $L_1\Gamma_{1,1}+L_1A_1^{-1}B_1>0$.

It remains case D). to study. From (3) applied in $t=t_1$, we have:

$$X(t_1)=\begin{pmatrix}\theta_2\\ y(t_1)\end{pmatrix}=\Gamma_1-A^{-1}C.$$

So, we can write $\Gamma_{1,1}-A_1^{-1}C_1=\theta_2$. Thus, σ_1 exists if $0 > \Gamma_{1,1}+A_1^{-1}B_1 > \Gamma_{0,1}$ (since $L_1>0$) and $0 < L_2\Gamma_{1,2}+L_2A_2^{-1}B_2 < L_2\Gamma_{0,2}$. From the first inequality, we obtain $0 > \theta_2+A_1^{-1}(B_1+C_1) > X_{0,1}+A_1^{-1}(B_1+C_1)$ *i.e.* $X_{0,1} < \theta_2 < -LA^{-1}(B+C)$. Then, from the second inequality, as $L_1\Gamma_{1,1}+L_2\Gamma_{1,2}=\theta_2+LA^{-1}C$, we obtain $X_{0,2} > \theta_2(1-L_1)$.

To conclude, we can remark that the first switching condition is the same than the one in dimension one except for case D). (indeed, in dimension one, we have $x_0 < \theta_2 < -a^{-1}(b+c)$). If we consider case $q_0=0$, $q_1=1$, we obtain the same conditions for the existence of σ_{n+1}^i, $i=1,2$ that is to say $-LA^{-1}C < \theta_1 < \theta_2 < -LA^{-1}(B+C)$. And, for the first switching, we obtain $LX_0 > \theta_1 > -LA^{-1}C$ except in one case where we find $X_{0,1} > \theta_1 > -LA^{-1}C$.

We can illustrate these results to the thermostat model given at the beginning of the paragraph. We choose the following numerical values:

R_c	R_m	$Q_c=R_cC_c$	$Q_p=m_pC_p$	P_c	θ_e	q_0
2	1	1000	1000	30	283	1

So, we have $b=\frac{1}{R_cQ_p}=5.10^{-4}$, $c=\frac{1}{R_cQ_c}=5.10^{-4}$, $d=\frac{1}{R_mQ_p}=10^{-3}$, $p_c=\frac{P_c}{Q_c}=0.03$, $\frac{\theta_e}{R_cQ_c}=0.283$. To ensure switchings existence, we must choose $283 = -LA^{-1}C < \theta_1 < \theta_2 < -LA^{-1}(B+C)=313$ and $LX_0 < \theta_2 < -LA^{-1}(B+C)=313$ in cases A). and B). or $X_{0,1} < \theta_2 < -LA^{-1}(B+C)=313$, $X_{0,2} > \theta_2(1-L_1)\approx 171.05$ in case D).

For example, we will illustrate results on case B).

- If we set $X_{0,1}=450$ (we have $L_1X_{0,1}+L_1A_1^{-1}(B_1+C_1)\approx 8.68>0$), $X_{0,2}=100$ ($L_2X_{0,2}+L_2A_2^{-1}B_2\approx$

$-35.49 < 0$), $\theta_1 = 290$, $\theta_2 = 293$, as $LX_0 \approx 286.4 < \theta_2 < -LA^{-1}(B+C)$ and $283 < \theta_1 < \theta_2 < 313$, all conditions are satisfied and the system switches as figure 4 shows.

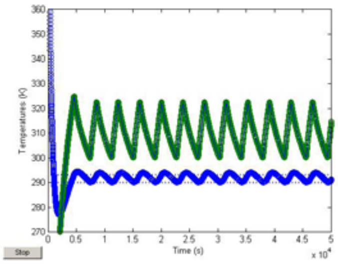

Figure 4: convector and room temperatures simulations for $\theta_1 = 290$ K, $\theta_2 = 293$ K, $X_{0,1} = 450$ K, $X_{0,2} = 100$ K

• Now, if we change value of θ_1, for example $\theta_1 = 282 < -LA^{-1}C$, the first switching exists because the condition is still verified but the other switchings do not exist because the existence conditions on σ^i_{n+1}, $i = 1, 2$ are not satisfied. Figure 5 illustrates this case.

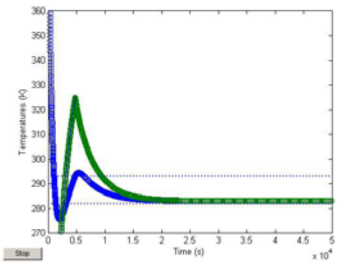

Figure 5: convector and room temperatures simulations for $\theta_1 = 282$ K, $\theta_2 = 293$ K, $X_{0,1} = 450$ K, $X_{0,2} = 100$ K

• Finally, if we change values of $\theta_2 = 285 < LX_0$, $\theta_1 = 284$, the first switching existence condition is not satisfied and the system does not switch (see figure 6).

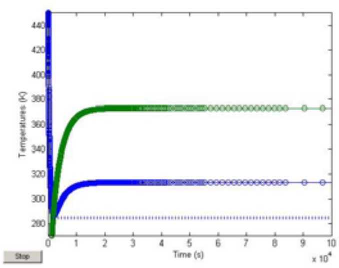

Figure 6: convector and room temperatures simulations for $\theta_1 = 284$ K, $\theta_2 = 285$ K, $X_{0,1} = 450$ K, $X_{0,2} = 100$ K

4 Conclusions and Future Work

This paper permits us to establish conditions on the thresholds to ensure switchings existence. Dimension one and two are treated and we find many similarities between found results.

Moreover, if we add another dimension and if we consider a thermostat with an anticipative resistance controlling a convector located in the same room, it seems that we still have the same conditions for the existence of switchings *i.e.* $-LA^{-1}C < \theta_1 < \theta_2 < -LA^{-1}(B+C)$. An example for some numerical values with $-LA^{-1}C = 283$, $-LA^{-1}(B+C) = 300.3$ is given on figure 7.

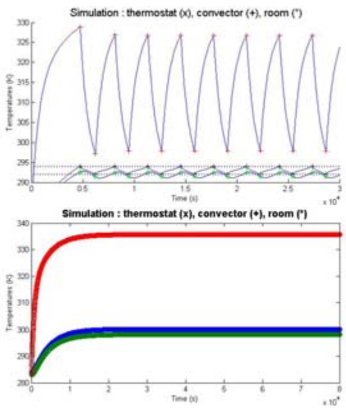

Figure 7: Thermostat, convector and room temperatures simulations for $\theta_1 = 292$ K, $\theta_2 = 294$ K, $X_0 = [283; 283; 283]$ (on the top) and with $\theta_2 = 300.3 > -LA^{-1}(B+C)$ (on the bottom)

For the future, it would be interesting to prove these switchings conditions in dimension three and then to generalize them for any dimension systems belonging to this h.d.s. class.

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