Theorems and Methods on Partial Functional Iteration

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***Abstract*— Functional iteration plays a significant role in various fields of applied and theoretical mathematics. Amongst its many applications are the iterative equations of complex numbers in fractal geometry, recursive algorithms in computer science, difference equations for discrete processes in engineering sciences, recursive sequences in Number Theory, and iterative methods for solving equations in Numerical Analysis. In a large number of its applications, functional iteration is non-continuous, and is defined only for discrete values of the iteration variable. In this paper, we use the Taylor series method to define continuous/partial functional iteration and to prove a new theorem that relates the derivative of a function with respect to the independent variable on the one hand, and the function's derivative with the respect to the continuous iteration variable on the other hand. The theorem is shown to be complete and sufficient when given a function *g*(*n*, *x*) to determine if a function *f*(*x*) exists such that *g*(*n*, *x*) is a closed form expression of *f*(*x*) iterated *n* times. We also present the concept of iterators and show that the mapping between functions, *f*(*x*), and their respective iterators, denoted *h*(*x*), is strictly a one-to-one mapping. Both theorems are shown to simplify the study of iterated functions considerably by converting problems or questions that are typically discrete in nature into simple problems of calculus. The theorems can also be used to derive quickly famous results in functional iteration such as the Abel equation and its properties with regard to iterated functions. Further corollaries and lemmas related to continuous functional iteration are also presented.**

***Index Terms*— Iterative Methods, Partial Functional Iteration, Recursion, Functional Equations .**

I. INTRODUCTION

The subject of iteration or recursion plays a very important role in applied and theoretical mathematics. It is often used in computer science to generate compact algorithms without delving into mathematical details [2]. For instance, a simple algorithm that generates Fibonacci sequence can be quickly built using the recursive property of Fibonacci sequence itself without having to build a closed-form formula for generating such sequence. Iterated functions are also a method of constructing fractals, which has led to important implications and applications in several areas including Chaos Theory, computer graphics, art, and even security [5] [9]. In addition, iteration is of vital importance to engineering sciences, where iterative difference equations are used to describe discrete processes and can be used to approximate solutions to ordinary differential equations [6]. In the field of Numerical Analysis, several iterative methods such as Newton's Method can be used, under certain conditions, to calculate the root of an equation [1]. Moreover, iterated functions give rise naturally to the study of recursive sequences in Number Theory, where some of the most famous examples of recursive sequences are Fibonacci sequence and the Logistic Map [7].

Iteration is often synonymous to recursion except in a few places such as in Algorithmics where iteration and recursion refer to two different methods [8] [4]. In this paper, both terms will be used differently, however. A recursive function is defined in this paper to be any function that references itself within its definition. Iterative or iterated functions, on the other hand, are strictly a subset of recursive functions that adhere to a strict formal definition and subsequent properties. In iterated functions, the output of a function is fed back as input to exactly the same function, constituting what is referred to as an *iteration*. To construct a second iteration, the new output is fed back as input to the same function again, and so on. An example of an iterated/recursive function is given in (1). One example of a recursive function, that is not an iterated function, is the well-known Fibonacci function. Note that the Fibonacci function does not satisfy the strict definition of iterated functions given below, and as such it is not an iterated function. Throughout this paper, it is important to distinguish between the two terms because all results and conclusions of this paper are applicable to iterated functions only.

$$f(n) = f(n-1) + k \tag{1}$$

Iteration is typically represented the same way exponentiation is represented; that is, the *n*th iterate of a function *f* is denoted as f^n. This representation might lead to confusion, especially when both iteration and exponentiation are present in the same equation. To avoid this confusion, a different representation of iteration is used in this paper. In this paper, the *n*th iterate of a function f is denoted as ${}^n f$ instead. This representation permits the use of both iteration and exponentiation without causing any potential confusion to readers.

A formal definition of iterated functions can be constructed as follows: Given a set *X* and let *f*: *X*→*X* be a function, the *n*th iterate of the function *f*, denoted as ${}^n f$, satisfies ${}^n f = f \circ {}^{n-1} f = f({}^{n-1} f)$. Using this definition, several important properties of iterated functions can be quickly proven including the following:

Property 1: ${}^a f({}^b f(x)) = {}^{a+b} f(x)$

Property 2: ${}^a({}^b f(x)) = {}^{ab} f(x)$

Property 3: ${}^0 f(x) = Id(x) = x$

Property 4: ${}^1 f(x) = f(x)$

Property 5: ${}^{-1} f(x)$ is the inverse function of $f(x)$.

The third property is very important for iteration to be mathematically consistent. Consider, for instance, the

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function ${}^{n}f(x)$. Using property 1 mentioned above, ${}^{n}f(x)$ can always be rewritten as ${}^{n}f({}^{0}f(x))$. Thus, ${}^{n}f({}^{0}f(x))={}^{n}f(x)$, which implies ${}^{0}f(x)=x$. Note that the constant function $f(x) = k$, although satisfies the formal definition of iterated functions, does not satisfy the properties mentioned above. For instance, it violates property 3 because it is a constant function, and it violates property 5 because it does not have an inverse function. Thus, constant functions are not considered as a subset of iterated functions in this paper.

Some iterated functions can be expressed in closed form in terms of both the independent variable x and the iteration variable n. For instance, the iterated form of the addition function $f(x)=x+k$ is simply ${}^{n}f(x)=x+nk$. Similarly, the multiplication function $f(x)=kx$ leads to the iterated function ${}^{n}f(x)=k^{n}x$. However, many iterated functions cannot be expressed in closed form such as the iterated sine function, ${}^{n}f(x)={}^{n}\sin(x)$, which can only be represented by repeating the sine function n times.

In this paper, we propose new theorems regarding iterated functions and present several examples that illustrate how such theorems simplify the study of continuous/partial functional iteration considerably. The first theorem relates the derivative of an iterated function, ${}^{n}f(x)$, with respect to the independent variable, x, on the one hand, and its derivative with respect to the continuous/partial iteration variable, n, on the other hand. This theorem allows us to determine quickly if an arbitrary function $g(n, x)$ is a closed-form expression of an iterated function ${}^{n}f(x)$. In other words, given an arbitrary function $g(n, x)$, the new theorem permits us to determine if there is a function $f(x)$ such as $g(n,x)={}^{n}f(x)$ and to determine what $f(x)$ is. Furthermore, using this theorem, problems of iterated functions, that typically belong to the realm of discrete mathematics, can be converted into simple problems of calculus. In the second theorem, we present the concept of iterators and show that the mapping between functions, $f(x)$, and their respective iterators, denoted $h(x)$, is strictly a one-to-one mapping. Such theorem provides short-cut solutions to many problems/questions related to functional iteration as will be illustrated in this paper.

The paper starts by showing how to generalize iteration to non-discrete values of the iteration variable using Taylor series, and uses an example, the sine function, to illustrate this method. After that, the same method is used to prove the first theorem that relates a function's derivative with respect to the independent variable x on the one hand, and its derivative with respect to the continuous iteration variable n on the other hand. This theorem is next proved to be complete and sufficient to determine if an arbitrary function of two variables x and n is, in fact, a closed form expression of an iterated function. Next, the iterator, $h(x)$, of a function, $f(x)$, is defined and is shown to be a unique property of any given function. Such *iterator uniqueness theorem*, subsequently, gives rise to a new one-to-one mapping, $f(x)\leftrightarrow h(x)$, in the functions domain. Last, well-known results in functional iteration such as the Abel equation are shown to be easily provable using the new theorems presented in this paper.

II. Defining Partial Functional Iteration

Assume a function $f(x)$ has a Taylor series and that the range of $f(x)$ is a subset of the radius of convergence of its Taylor series, $f(x)$ can be written in the form given in (2) for any value of x that belongs to the radius of convergence, where a_i are arbitrary constants. Because the range of $f(x)$ is a subset of the radius of convergence of its Taylor series, its iterated function ${}^{n}f(x)$ also has a Taylor series with the same radius of convergence. Thus, ${}^{n}f(x)$ can be rewritten in the form given in (3) for any value of x that belongs to the radius of convergence.

$$f(x)=\sum_{i=0}^{\infty}a_i x^i \tag{2}$$

$${}^{n}f(x)=\sum_{i=0}^{\infty}a_i(n)x^i \tag{3}$$

Using the recursive property of iterated functions, equation (4) holds true as well.

$${}^{n}f(x)=\sum_{i=0}^{\infty}a_i\cdot({}^{n-1}f^{\,i}(x))=\sum_{i=0}^{\infty}a_i(n-1)\cdot f(x) \tag{4}$$

Equating equations (3) and (4), the values of $a_i(n)$ can, sometimes, be obtained. For instance, if we assume that $f(x)=\sin(x)$, then

$$f(x)=x-{x^3}/{3!}+{x^5}/{5!}-{x^7}/{7!}\cdots \tag{5}$$

Also,

$$\begin{aligned}&{}^{n}f(x)=a_0(n)+a_1(n)x+a_2(n)x^2+a_3(n)x^3\ldots\\&={}^{n-1}f(f(x))=a_0(n-1)+a_1(n-1)\sin x+a_2(n-1)\sin^2 x\\&+a_3(n-1)\sin^3 x\ldots\cdot\end{aligned} \tag{6}$$

As a result, the values of $a_i(n)$ can be calculated as follows:

$$\begin{aligned}&a_0(n)=0,\quad a_2(n)=0,\quad a_4(n)=0,\quad a_6(n)=0,\ \cdots\\&a_1(n)=a_1(n-1)=1,\\&a_3(n)=a_3(n-1)-(1/3!)a_1(n-1)=a_3(n-1)-1/3!\\&=-n/6\\&a_5(n)=a_5(n-1)-(3/3!)a_3(n-1)+(1/5!)a_1(n-1)\\&=(1/120)(5n^2-4n)\end{aligned}$$

The same procedure can be used to calculate the values of the rest of the coefficients. Consequently, the iterated sine function can be computed using equation (7).

$${}^{n}\sin(x)=x-(n/6)x^3+((5n^2-4n)/120)x^5\cdots \tag{7}$$

The expression in equation (7) can be tested for small values of x as follows. When n=0, we obtain

$${}^{0}\sin(x)=x \tag{8}$$

, which is expected from property 3 mentioned above. Also, when n=1, we obtain

$${}^{1}\sin(x)=x-(1/6)x^3+(1/120)x^5\cdot\cdot \tag{9}$$

, which is indeed the Taylor series for the sine function. Moreover, from equation (7), the *half*-sine function can be computed using equation (10). Note that using equation (10), ${}^{1/2}\sin({}^{1/2}\sin(x))$ is indeed equal to $\sin(x)$ as expected from property 1. Interestingly, the Taylor series in (7) can also be used to compute the inverse sine function, which is a direct result of property 5 of iterated functions mentioned earlier. That is, when n = -1, the iterated sine function in (7) does indeed yield the well-known Taylor series of the inverse sine function. It is vital to keep in mind, however, that equation (7) is valid for small values of x and n when only the first six terms of the Taylor series are used. To achieve a higher degree of precision, further terms should be included.

$$^{1/2}\sin(x) = x - (1/12)x^3 - (1/160)x^5 \cdots \quad (10)$$

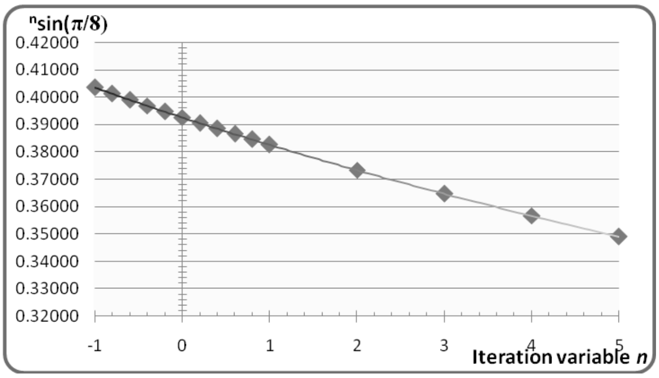

Figure 1: The continuous iterated function $f(x) = {}^n\sin(\pi/8)$ plotted against the continuous iteration variable.

The continuous iterated sine function given in (7) permits us to view how iteration changes the value of ${}^n\sin(x)$, which is plotted in figure 1 for $x=\pi/8$. In this figure, the iterated sine function for integer values of n was computed using the well-known discrete definition of the iterated sine function; e.g. ${}^2\sin(\pi/8)=\sin(\sin(\pi/8)$ and so on. The best fitted curve was plotted using polynomial regression of degree 6. The non-integer values of the iterated sine function, computed using equation 7, were, in turn, inserted into the figure. Note that the latter values do indeed fit into the curve as expected, which implies that equation (7) does produce a consistent definition of the iterated sine function.

Accordingly, functional iteration can be defined for continuous values of the iteration variable using Taylor series. It is important to note that while the method discussed above could be used to define the continuous functional iteration for the sine function, it may not be effective for other functions. However, the Taylor series method allows us to prove important theorems regarding functional iteration such as the theorems discussed next.

III. Iterators and the Derivative of Iterated Functions

Assume a function $f(x)$ has a Taylor series and the range of $f(x)$ is a subset of the radius of convergence of its Taylor series, then ${}^n f(x)$ can always be written in the form given in (3) for any value of x that belongs to the radius of convergence of $f(x)$. The derivative of ${}^n f(x)$ with respect to the continuous iteration variable n is given in (11).

$$\frac{\delta}{\delta n}{}^n f(x) = \sum_{i=0}^{\infty}[\frac{\delta}{\delta n}a_i(n)]\cdot x^i \quad (11)$$

Obtaining the derivative by definition and using the properties of iterated functions, the function $\frac{\delta}{\delta n}{}^n f(x)$ can be computed as shown in (12), which can be rewritten in the form given in (13) by using equation (11) and L'Hospital's rule.

$$\frac{\delta}{\delta n}{}^n f(x) = \lim_{h\to 0}\frac{{}^n f\ {}^h f(x) - {}^n f(x)}{h} \quad (12)$$

$$\frac{\delta}{\delta n}{}^n f(x) = \lim_{h\to 0}\left[\sum_{i=1}^{\infty}[a_i(n)\frac{\delta}{\delta h}{}^h f^i(x)\right]$$
$$= \lim_{h\to 0}\left[\sum_{i=1}^{\infty}[ia_i(n)^h f^{i-1}\frac{\delta}{\delta h}{}^h f(x)\right] \quad (13)$$

Using property 3 of iterated functions, we know that $\lim_{h\to 0}{}^h f(x) = x$. Therefore, equation (13) can be rewritten as shown in (14).

$$\frac{\delta}{\delta n}{}^n f(x) = [\lim_{h\to 0}\frac{\delta}{\delta h}{}^h f(x)]\cdot[\sum_{i=1}^{\infty}ia_i(n)x^{i-1}] \quad (14)$$

Because $\frac{\delta}{\delta x}{}^n f(x) = \sum_{i=1}^{\infty}ia_i(n)x^{i-1}$, equation (14), in turn, implies:

$$\frac{\delta}{\delta n}{}^n f(x) = [\lim_{n\to 0}\frac{\delta}{\delta n}{}^n f(x)]\cdot\frac{\delta}{\delta x}{}^n f(x) \quad (15)$$

Equation (15) relates the derivative of any iterated function ${}^n f(x)$ with respect to the continuous iteration variable n on the one hand, and the function's derivative with respect to the independent variable x on the other hand. It shows that the ratio of the two derivatives at any point (x, n) is a function of the independent variable x only. This latter function will be called the *iterator*, and will be denoted $h(x)$ in this paper. Thus, the iterator of a function $f(x)$ is given by equation (16).

$$h(x) = [\lim_{n\to 0}\frac{\delta}{\delta n}{}^n f(x)] \quad (16)$$

Note that throughout the proof of (15), the only condition that is assumed valid is property 3 of iterated functions, which was shown earlier to be true for mathematical consistency. Thus, the following corollary holds true with regard to all iterated functions.

Corollary 3.1: Any iterated function ${}^n f(x)$, where $f(x)$ has a Taylor series and the range of $f(x)$ is a subset of the radius of convergence of its Taylor series, must meet the following two conditions:

(1) ${}^0 f(x) = x$,

(2) and $\frac{\delta}{\delta n}{}^n f(x) = [\lim_{n\to 0}\frac{\delta}{\delta n}{}^n f(x)]\cdot\frac{\delta}{\delta x}{}^n f(x)$.

This corollary can be tested using well-known closed form expressions of iterated functions, which is shown to be true in table 1. Corollary 3.2, proved next, shows that any function of two variables x and n is a closed-form expression of an iterated function if it satisfies the conditions mentioned in Corollary 3.1.

Table 1: Closed-form expressions of some iterated functions

$f(x)$	${}^n f(x)$	${}^0 f(x)$	$\frac{\delta}{\delta n}{}^n f(x)$ $=[\lim_{n\to 0}\frac{\delta}{\delta n}{}^n f(x)]\cdot\frac{\delta}{\delta x}{}^n f(x)$
$f(x)=x+k$	${}^n f(x)=x+nk$	x	k
$f(x)=kx$	${}^n f(x)=k^n x$	x	$k^n x \ln k$
$f(x)=x^k$	$f(x)=x^{k^n}$	x	$x^{k^n}(\ln x)k^n(\ln k)$

Corollary 3.2: Any function $g(n, x)$ that has a Taylor series with respect to the independent variable x and the range of $g(1, x)$ is a subset of the radius of convergence of its respective Taylor series is an iterated function if it meets the following two conditions:

(1) $g(0,x)=x$,

(2) and $\frac{\delta}{\delta n}g(n,x)=[\lim_{n\to 0}\frac{\delta}{\delta n}g(n,x)]\cdot\frac{\delta}{\delta x}g(n,x)$.

Proof: If the range of $g(1, x)$ is not a subset of its Taylor series radius of convergence, then its Taylor series could diverge during iteration and, thus, its iterated function does not have a Taylor series representation. Otherwise, if $g(n, x)$ has a Taylor series, $g(n, x)$ can be written in the form given in (17).

$$g(n,\ x)=\sum_{i=0}^{\infty}a_i(n)x^i \tag{17}$$

Using equation (17) and the two conditions, equation (18) holds true.

$$\frac{\delta}{\delta n}g(n,x)=\lim_{h\to 0}\left([\frac{\delta}{\delta h}g(h,x)]\cdot\sum_{i=1}^{\infty}[ia_i(n)g^{i-1}(h,x)]\right) \tag{18}$$

Equation (18) can be rewritten as shown in (19), which is equivalent to equation (20).

$$\frac{\delta}{\delta n}g(n,x)=\lim_{h\to 0}\left(\sum_{i=1}^{\infty}\frac{\delta}{\delta h}[a_i(n)g^i(h,x)]\right) \tag{19}$$

$$\frac{\delta}{\delta n}g(n,x)=\lim_{h\to 0}\left(g(n,g(h,x))\right) \tag{20}$$

Using condition 1 and L'Hospital's rule, equation (21) holds true, which, in turn, implies that equation (22) holds true as well.

$$\lim_{h\to 0}\left(\frac{g(n+h,x)-g(n,x)}{h}\right)=\lim_{h\to 0}\left(\frac{g(n,g(h,x))-g(n,x)}{h}\right) \tag{21}$$

$$\lim_{h\to 0}g(n+h,x)=\lim_{h\to 0}g(n,g(h,x)) \tag{22}$$

Equation (22) is equivalent to the definition of iterated functions at the limit $h \rightarrow 0$. Thus, any function $g(n, x)$ that satisfies the conditions stated in the corollary implies that there is a function $f(x)$ such that:

$$[g(n,x)={}^n f(x)\ \wedge\ {}^{n+h}f(x)={}^n f\,{}^h f(x)]\Big|_{h\to 0} \tag{23}$$

Thus, $g(n+m, x)$, where n and m are arbitrary real values, can be rewritten as $g(n+m-h, g(h, x))$, where $h \rightarrow 0$, which, in turn, can be rewritten again as $g(n+m-2h, g(2h, x))$, where $h \rightarrow 0$ using equation (23). The same process can be repeated so that $g(n+m, x)$ is rewritten as $g(n+m-kh, g(kh, x))$, where $h \rightarrow 0$ and $k = m/h$. As a result, $g(n+m, x)$ can always be rewritten as $g(n, g(m, x))$ if $g(n, x)$ satisfies the conditions stated in the corollary. Thus, $g(n, m)$ is an iterated function.

The two corollaries brings us to the following theorem.

Theorem 3.1: Assuming $g(n, x)$ has a Taylor series with respect to the independent variable x and that the range of $g(1, x)$ is a subset of the radius of convergence of its respective Taylor series, then any function $g(n, x)$ is an iterated function *if and only if* it meets the following two conditions:

(1) $g(0,x)=x$,

(2) and $\frac{\delta}{\delta n}g(n,x)=[\lim_{n\to 0}\frac{\delta}{\delta n}g(n,x)]\cdot\frac{\delta}{\delta x}g(n,x)$.

Theorem 3.1 allows us to answer many questions regarding iterated functions quickly. For instance, we can quickly realize that the function $g(n,x)=\cos(n)\cdot x$ is not a closed form expression of an iterated function because both functions $g(n, x)$ and $g(1,x)=\cos(1)\cdot x$ have Taylor series with respect to the independent variable x with infinite radius of convergence, so they both meet the assumptions stated in the theorem, but the function $g(n, x)$ does not satisfy the second condition of Theorem 3.1. As a result, there is no function $f(x)$ that satisfies ${}^n f(x)=\cos(n)\cdot x$.

Lemma 3.1: the only functions $f(x)$ that satisfy ${}^n f(x)=x+y(n)$ are $f(x)=x+k$, where k is an arbitrary constant.

Proof: Since $g(n,x)=x+y(n)$, both functions $g(n, x)$ and $g(1,x)=x+y(1)$ have Taylor series with respect to the independent variable x with infinite radius of convergence so they meet the assumptions stated in the theorem. From the second condition, we know that $\frac{\delta}{\delta n}g(n,x)=k$, where k is an arbitrary constant, which, in turn, implies that $y(n)$ must be of the form: $y(n)=kn$. So, the only functions $f(x)$ that satisfy ${}^n f(x)=x+y(n)$ are $f(x)=x+k$, where k is an arbitrary constant.

Lemma 3.2: Given a function $y(n)$, the function $z(n)$ that makes ${}^n f(x)=z(n)x+y(n)$ an iterated function can be solved by solving the differential equation $\frac{\delta z}{\delta n}+a\,z(n)+b(n)=0$, where $b(n)$ and a are derived from $y(n)$.

Proof: Again, the assumptions of the theorem are satisfied. In order to meet the first condition, $z(0)=1$, and $y(0)=0$. To satisfy the second condition:

$$\frac{\delta}{\delta n}{}^n f(x)=x\frac{\delta z}{\delta n}+\frac{\delta y}{\delta n}=z(n)\left(x\frac{\delta z}{\delta n}(0)+\frac{\delta y}{\delta n}(0)\right)$$

, which can rewritten into:

$$\frac{\delta z}{\delta n}+x^{-1}\frac{\delta y}{\delta n}=z(n)\left(\frac{\delta z}{\delta n}(0)+x^{-1}\frac{\delta y}{\delta n}(0)\right)$$

So, given a specific function $y(n)$, $z(n)$ can be solved by solving the following differential equation:

$$\frac{\delta z}{\delta n}+a\,z(n)+b(n)=0$$

$$b(n)=x^{-1}\frac{\delta y}{\delta n}$$

$$a=-\left(\frac{\delta z}{\delta n}(0)+x^{-1}\frac{\delta y}{\delta n}(0)\right)$$

For instance, suppose that $y(n)$ is given by the following equation:

$$y(n)=2^n-q$$

We know from condition 1 that $y(0)=0$. So, $q=1$. Thus:

$$y(n)=2^n-1$$

And:

$$\frac{\delta y}{\delta n}(0)=\ln 2$$

So, the general solution of $z(n)$ is:

$$z(n)=c_1e^{-an}-\frac{2^n}{x(1+a/\ln 2)},$$

$$a=-(\frac{\delta z}{\delta n}(0)+(\ln 2)x^{-1})$$

Because $z(0)=1$,

$$c_1=\frac{1}{x(1+a/\ln 2)}+1$$

Thus,

$$z(n)=\left(\frac{1}{x(1+a/\ln 2)}+1\right)e^{-an}-\frac{2^n}{x(1+a/\ln 2)},$$

There are two more restrictions, however:

$$a=-\left(\frac{\delta z}{\delta n}(0)+(\ln 2)x^{-1}\right)$$

And $\frac{\delta z}{\delta n}(0)=-a\left(\frac{1}{x(1+a/\ln 2)}+1\right)-\frac{\ln 2}{x(1+a/\ln 2)}$

This gives us a quadratic polynomial. Thus, $\frac{\delta z}{\delta n}(0)=\ln 2$ and $a=-(\ln 2)(1+x^{-1})$. By substituting these values into z(n), we obtain $z(n)=2^n$

Thus, the function ${}^nf(x)=2^nx+2^n-1$ is a closed form expression of an iterated function. The original function $f(x)$ can be obtained using property 4 of iterated functions by simply substituting $n=1$ into the equation, which gives us $f(x)=2x+1$.

IV. The Iterator Uniqueness Theorem

The subsequent theorem shows that the iterator $h(x)$ of a function $f(x)$ is a unique property of that specific function. In other words, the mapping between functions and iterators, $f(x)\leftrightarrow h(x)$, is strictly a one-to-one mapping. This theorem has several implications and applications in iterated functional systems as the subsequent corollaries and lemmas illustrate.

Theorem 4.1: The mapping $f(x)\leftrightarrow[\lim_{n\to 0}\frac{\delta}{\delta n}{}^nf(x)]$ is strictly a one-to-one mapping in the functions domain.

Proof: We know from property 3 of iterated functions that the following equation holds true for all iterated functions.

$${}^0f(x)=x$$

Assume there is two functions, $y(x)$ and $z(x)$, that are strictly different and that their respective iterators are equivalent, then:

$${}^0y(x)={}^0z(x)=x$$

Because the iterators of the two functions are assumed to be equivalent, the following equation holds true for any value x_0:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^hy(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hz(x_0)$$

We also know from property 1 of iterated functions that:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}y(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hy({}^hy(x_0))$$

And similarly:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}z(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hz({}^hz(x_0))$$

Denoting $\lim_{h\to 0}\frac{\delta}{\delta n}{}^hy(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hz(x_0)=x_1$, then:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}y(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hy(x_1)$$

And:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}z(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hz(x_1)$$

Because the two iterators are equivalent, we know that:

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^hy(x_1)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^hz(x_1)$$

Thus,

$$\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}y(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^{2h}z(x_0)$$

The same process can be repeated until $\lim_{h\to 0}\frac{\delta}{\delta n}{}^{kh}y(x_0)=\lim_{h\to 0}\frac{\delta}{\delta n}{}^{kh}z(x_0)$, where $k=1/h$

Thus, ${}^1y(x_0)={}^1z(x_0)$, which contradicts the original assumption that the two functions are strictly different. Consequently, the mapping between functions and iterators is a one-to-one mapping.

Table 2: Examples of iterators of common functions

$f(x)$	$h(x)$
$x+k$	k
kx	$(\ln k)x$
x^k	$(\ln k)(\ln x)x$
$px+q$	$(\ln p)x+\frac{q\ln p}{p-1}$

Corollary 4.1: Given a function $g(n, x)$, the partial differential equation $\frac{\delta g}{\delta n}=h(x)\frac{\delta g}{\delta x}$ where $g(0,x)=x$, has at most one solution.

Proof: If the partial differential equation had two solutions, both solutions would be iterated functions and they both would share the same iterator $h(x)$, which, in turn, violates the iterator uniqueness theorem. Thus, the partial differential equation with the condition that $g(0, x)=x$ has no more than one solution.

Lemma 4.1: The only combination of functions $f(x)$ and $z(x)$ that satisfy the differential equation $\frac{\delta^n f}{\delta n}={}^nz(x)$ is $f(x)=z(x)=e^1x$.

Proof: From the differential equation stated in the lemma, the iterator of $f(x)$ is $h_f(x)=x$. From table 2, such iterator is a unique property of $f(x)=e^1 x$. Thus, the only combination of functions $f(x)$ and $z(x)$ that satisfy the differential equation $\frac{\delta\, {}^n f}{\delta n} = {}^n z(x)$ is $f(x)=z(x)=e^1 x$.

Corollary 4.2: The iterator of $g(x) = {}^m f(x)$ is equal to $m\,h(x)$, where $h(x)$ is the iterator of $f(x)$.

Proof: A direct result of the definition of iterators and the chain rule.

Theorem 4.2: The iterator $h(x)$ of a function $f(x)$ satisfies the autonomous differential equation $\frac{d}{dn} {}^n f(x) = h({}^n f(x))$.

Proof: A direct result of the definition of iterators given in (16) and property 1 of iterated functions.

Corollary 4.3: Given an iterated function ${}^n f(x)$,

$$\lim_{n\to 0}\left(\frac{\delta^2\, {}^n f}{\delta n^2}\right) = h(x)\,h'(x)$$

Proof: Using theorem 4.2:

$$\frac{d}{dn} {}^n f(x) = h({}^n f(x)) \Rightarrow \frac{d^2}{dn^2} {}^n f(x) = h'({}^n f(x)) \frac{d}{dn} {}^n f(x)$$

$$\lim_{n\to 0} \frac{d^2}{dn^2} {}^n f(x) = h'(x) \lim_{n\to 0} \frac{d}{dn} {}^n f(x) = h(x)\,h'(x)$$

Lemma 4.2: ${}^n f(x) = (mn + x^{1-k})^{1/(1-k)}$ is the only class of functions that satisfy the equation $h(x) = mx^k$, where $h(x)$ is the iterator of $f(x)$?

Proof: Using theorem 4.2:

$$\frac{d}{dn} {}^n f(x) = m\ {}^n f^k(x)$$

$${}^n f^{1-k}(x) = (1-k)(mn + C)$$

$${}^n f(x) = ((1-k)(mn+C))^{1/(1-k)}$$

Using property 3 of iterated functions:

$${}^n f(x) = ((1-k)mn + x^{1-k})^{1/(1-k)}$$

This solution is tested for the two conditions given in theorem 3.1, which can be shown to be satisfied. Note that when $m=k=1$, $h(x)=x$, which is the iterator of $f(x)=e^1 x$, whose iterated function is ${}^n f(x) = e^n x$. Thus, the following equation holds true as well:

$$\lim_{j\to 0} (jn + x^j)^{1/j} = e^n x \tag{24}$$

Equation (24) illustrates the accuracy of the iterator uniqueness theorem since it has been proven so far using the theorems presented in this paper only. The equation, however, can be proven differently using the very well-known representation of the natural logarithmic base, *e*, as a limit of sequence.

V. Abel Equation and Iterated Functional Systems

Using theorem 4.2 and property 3 of iterated functions:

$${}^n f(x) = {}^{-1}g(n + g(x)) \tag{25}$$

, where $g(n) = \int h^{-1}(n)\,dn$ and ${}^{-1}g(x)$ is the inverse function of $g(x)$. The function ${}^n f(x)$ in (25) satisfies both conditions stated in theorem 3.1, which implies that the following theorem holds true as well.

Theorem 5.1: A function ${}^n f(x)$ is an iterated function if and only if it can be written in the form ${}^n f(x) = {}^{-1}g(n+g(x))$, for a specific function $g(x)$.

Proof: The function ${}^n f(x)$ in the form given in the theorem satisfies the conditions stated in theorem 3.1. So, ${}^n f(x)$ must be an iterated function. On the other hand, any iterated function satisfies the differential equation given in theorem 4.2, which implies that the iterated function can be written in the form ${}^n f(x) = {}^{-1}g(n+g(x))$, where $g(n) = \int h^{-1}(n)\,dn$, and $h(x)$ is the iterator of $f(x)$.

Using theorem 5.1, the iterator of a function $f(x)$ can be computed by solving the following functional equation:

$$g(f(x)) = 1 + g(x) \tag{26}$$

, where $h(x) = 1/g'(x)$ and $h(x)$ is the iterator of $f(x)$.

Equation (26) is the famous Abel functional equation, which has been known to be closely tied to iterated functions [3]. The relationship between Abel equation and iterated functions can be proven using equation (25) to derive equation (27) for all real values of the iteration variable *n*.

$$g({}^n f(x)) = n + g(x) \tag{27}$$

Thus, to find a closed-form expression of an iterated function ${}^n f(x)$, a solution to Abel's equation in (26) for $g(x)$ needs to be found. The iterated function can then be computed using equation (25). Furthermore, using equation (26) and the fact that $g(n) = \int h^{-1}(n)\,dn$, the iterator $h(x)$ itself of a function $f(x)$ can be determined directly by solving the following functional equation:

$$f'(x) = h(f(x))/h(x) \tag{28}$$

Using corollary 4.2, the iterator of the inverse function, ${}^{-1}f(x)$, is $-h(x)$, where $h(x)$ is the iterator of the function $f(x)$. Thus, equation (28) implies that equation (29) holds true as well.

$${}^{-1}f'(x) = h({}^{-1}f(x))/h(x) \tag{29}$$

Equation (28) reveals a very interesting correlation between iteration and differentiation. To see it, let us assume first that $z=g(x)$. Then, $\lim_{n\to 0} \frac{d}{dn} {}^n f(z) = h(z)$, where $h(x)$ is the iterator of $f(x)$. Thus, $h(f(x))$, where $h(x)$ is the iterator of $f(x)$, is given by the following equation:

$$h(f(x)) = \lim_{n\to 1} \frac{d}{dn} {}^n f(x) \tag{30}$$

As a result, the derivative of a function $f(x)$ is related to iteration by equation (31).

$$f'(x) = \left(\lim_{n\to 1} \frac{d}{dn} {}^n f(x)\right)\left(\lim_{n\to 0} \frac{d}{dn} {}^n f(x)\right)^{-1} \tag{31}$$

In qualitative terms, equation (31) states that the instantaneous derivative of a function $f(x)$ as it transitions towards ${}^{\infty}f(x)$ divided by the instantaneous derivative of the identity function, x, as it also transitions towards ${}^{\infty}f(x)$ is equivalent to the derivative $f'(x)$. Thus, continuous iteration is, interestingly, closely tied to differentiation. It is worth noting that equation (31) can also be proven differently using theorem 3.1. In theorem 3.1, it was shown that any iterated function, ${}^{n}f(x)$, satisfies equation (15). By taking the limit $n \rightarrow 1$ in equation (15), we arrive directly at equation (31).

Last remark, given the iterator $h(x)$ of a function $f(x)$, the iterated function ${}^{n}f(x)$ can be approximated using the approximation shown in (32).

$$ {}^{\Delta n}f(x) = x + h(x)\Delta n \tag{32} $$

To find an approximate value of ${}^{n}f(x)$ using equation (32), we define $y_0=x$. Then, y_i is given by the iterative equation:

$$ y_i = y_{i-1} + h(y_{i-1})\Delta n \tag{33} $$

Then, ${}^{n}f(x) = y_m \mid m = n/\Delta n$. This result is straightforward given the definition of iterators in theorem 4.2 However, a much more interesting question is how to find the kth degree approximation to the iterated function ${}^{n}f(x)$. To do this, we write ${}^{n}f(x)$ in the form shown in (34).

$$ {}^{n}f(x) = \sum_{i=0}^{\infty} b_i(x)\, n^i \tag{34} $$

Because ${}^{0}f(x) = x$, $b_0(x) = x$. Similarly, by definition of iterators, $b_1(x) = h(x)$. So, ${}^{n}f(x)$ can be rewritten in the form shown in (35).

$$ {}^{n}f(x) = x + h(x)n + \sum_{i=2}^{\infty} b_i(x)\, n^i \tag{35} $$

Taking the derivative, $\frac{\delta}{\delta x}{}^{n}f(x) = 1 + h'(x)n + \sum_{i=2}^{\infty} b_i{}'(x)\, n^i$, and using theorem 3.1:

$$ h(x)\frac{\delta}{\delta x}{}^{n}f(x) = \frac{\delta}{\delta n}{}^{n}f(x) $$

$$ h(x) + h'(x)h(x)n + \sum_{i=2}^{\infty} h(x)\, b_i{}'(x)\, n^i = h(x) + \sum_{i=1}^{\infty} i\, b_i(x)\, n^{i-1} $$

Equating terms, we obtain the following recursive definition of the coefficients $b_i(x)$:

- $b_0(x) = x$
- $b_m(x) = \dfrac{b'_{m-1}(x)h(x)}{m}$

Consequently, the kth degree approximation can be computed using equation (34). For instance, the 2nd degree approximation is giving by (36).

$$ {}^{\Delta n}f(x) = x + h(x)\Delta n + \frac{h(x)h'(x)}{2}\Delta n^2 \tag{36} $$

VI. Conclusion

Iterated functions play crucial roles in countless fields of applied and theoretical mathematics. Typically, functional iteration is discrete in nature; defined only for integer values of the iteration variable. However, a systematic approach towards defining continuous iteration can be followed using Taylor series, which gives rise to several new theorems that simplify the study of iterated functions considerably. One theorem shows that the ratio between the derivative of an iterated function with respect to the independent variable and the derivative with respect to the iteration variable is solely determined by the independent variable. In another theorem, such ratio, called the iterator, is proved to be a unique property of any given function. Both theorems and their resultant corollaries can quicken the process of solving problems related to iterated functions. They can also be used to derive famous results such as the Abel equation and its properties with respect to iterated functions.

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