Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

ADFE Simulation for Anisotropic Convection Diffusion Problems with Moving Boundaries

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Abstract—**Alternating direction finite element (ADFE) simulation for moving boundary anisotropic convection diffusion problems is studied. Through the coordinate transformation of the spatial variants, a new domain independent of the time is obtained on which two ADFE algorithms are designed by introducing a small implicit viscous term and approaching the anisotropic diffusion explicitly. Theoretical analysis show that both algorithms have the optimal H^1 and L^2 norm spacial convergency, while their precisions for the temporal variant are $O(\Delta t)$ and $O((\Delta t)^2)$ respectively. Numerical tests are made on three-dimensional model problem to verified the efficiency of the algorithms.**

Keywords: moving boundary, alternating direction finite element method, anisotropic diffusion problem, coordinate transformation, numerical analysis

1 Introduction

Many practical problems such as fluid flows and fire combustion in science and engineering have unfixed developing domain [1]-[3], and their simulation requires the accurate tracking of moving boundaries. It presents a challenge to numerical methods since not only is the domain shape irregular but it changes during the computation. Usually rather heavy computation is needed, especially in the more complicated multi-dimensional case. Alternating direction (AD) method is an efficient way to deal with multi-dimensional problems, since it can reduce their solving procedure to a series of simple one-dimensional problems, hence greatly eliminates the calculation. But it is difficult to be applied to the simulation for moving boundary problems since the irregularity and persistent variation of the boundary may bring much complexity to its practical realization. However, for a variety class of moving boundary problems, AD procedure is feasible. In [3], alternating direction finite element (ADFE) method for moving boundary problems is studied for the first time, coordinate transformation [4] is used, algorithm construction and theoretical analysis for 2-dimensional isotropic diffusion are provided. ADFE method [3], [5], [6] can keep both the advantages of AD method (efficiency) and finite element method (high accuracy). In many scientific fields, such as magnetohydrodynamic, radiative hydrodynamic, geological formations, oil reservation and transfer, thermal properties of structural materials and crystals, image processing, plasma physics, etc., anisotropic diffusion often occurs and has widely application [7]. In many cases, these anisotropic diffusion problems appear with moving boundaries. In this paper, we study the ADFE schemes for d-dimensional ($d \geq 2$) anisotropic convection diffusion problems with moving boundaries. Firstly, coordinate transformation is used, with which the floating practical domain is changed to a fixed one (with boundaries independent of time) called as computational domain, then AD splitting is achieved and ADFE calculations are done on this computational domain. During the calculation procedure, AD splitting is made only once, and the corresponding evaluations can be turned back to the original domain whenever and wherever needed.

The idea of the paper can also be extended to other related problems with moving boundaries and other numerical schemes, for example, AD finite difference (ADFD), etc., as long as they are efficient to solve the derived problems after the coordinate transformation.

Consider the moving boundary anisotropic convection diffusion problem

$$\begin{aligned} & u_t - \sum_{i,j=1}^{d} \frac{\partial}{\partial x_j}\left(a_{ij}(u)\frac{\partial u}{\partial x_i}\right) + \sum_{i=1}^{d} b_i(u)\frac{\partial u}{\partial x_i} = f(u), \\ & \qquad\qquad x \in \Omega(t), t \in J, \qquad (1.1) \\ & u(x,0) = u_0(x), \qquad x \in \Omega(0), \\ & u(x,t) = 0, \qquad x \in \partial\Omega(t), t \in J, \end{aligned}$$

where $\Omega(t) = \{x = (x_1, x_2, \cdots, x_d), x_i \in (s_{i,1}(t), s_{i,2}(t)), i = 1, 2, \cdots, d\} \subset R^d$ ($d \geq 2$ is the dimension of the space), $J = [0, T]$. $a_{ij}(u) = a_{ij}(x,t,u)$, $b_i(u) = b_i(x,t,u)$, $f(u) = f(x,t,u)$, $u_0(x)$, $s_{i,1}(t)$, $s_{i,2}(t)$ $(i, j = 1, 2, \cdots, d)$ are given functions with proper smoothness. Suppose that there exist positive constants s_*, s^*, a_* and a^* such that $s_* \leq s_{i,2}(t) - s_{i,1}(t) \leq s^* (i = 1, 2, \cdots, d)$ and for all $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_d) \in R^d$, $a_*|\gamma|^2 \leq \sum_{i,j=1}^{d} (a_{ij}(x,t,\phi)$

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$\gamma_i, \gamma_j) \le a^*|\gamma|^2$, $\forall x \in \Omega(t), t \in J, \phi \in R$, where $|\gamma| = \sqrt{\gamma_1^2 + \gamma_2^2 + \cdots + \gamma_d^2}$.

In this paper, the ADFE simulation for (1.1) is studied. Firstly, two ADEF schemes are proposed in Section 2, then corresponding approximation and stability properties are derived in Section 3, and the optimal H^1 and L^2 norm spacial and $O(\Delta t)$ and $O((\Delta t)^2)$ temporal convergency is obtained. Finally, in Section 4, numerical tests on both schemes for 3-dimensional model problem are given, which demonstrate the validity of our ADFE simulation, also further discussion is presented.

Here and below, K will be a generic positive constant, and may be different each time it appears. ϵ is an arbitrarily small constant.

2 ADFE Schemes

Denote $(\phi, \psi)_t = \int_{\Omega(t)} \phi\psi dx$, the variational form of (1.1) can be described as: finding $u \in L^2(J, H_0^1(\Omega(t)))$, $u_t \in L^2(J, H^{-1}(\Omega(t)))$ such that

$$\begin{aligned} &(u_t, v)_t - \sum_{i,j=1}^{d} (a_{ij}(u)\tfrac{\partial u}{\partial x_i}, \tfrac{\partial v}{\partial x_j})_t \\ &+ \sum_{i=1}^{d} (b_i(u)\tfrac{\partial u}{\partial x_i}, v)_t = (f(u), v)_t, \forall v \in H_0^1(\Omega(t)), t \in J, \\ &(u(x,0), v)_0 = (u_0, v)_0, \ \forall v \in H_0^1(\Omega(0)). \qquad (2.1) \end{aligned}$$

Introduce the following coordinate transformation relations: $y_i = \frac{x_i - s_{i,1}(t)}{s_{i,2}(t) - s_{i,1}(t)}$, namely, $x_i = s_{i,1}(t) + [s_{i,2}(t) - s_{i,1}(t)]y_i$, $i = 1, 2, \cdots, d$; then $y_i \in [0,1]$, and the moving domain $\Omega(t)$ is transformed into the fixed computational domain $\hat{\Omega} = (0,1) \times (0,1) \cdots \times (0,1)$. Denote $y = (y_1, y_2, \cdots, y_d)$, $\hat{u} = \hat{u}(y,t) = u(x,t)$; $\hat{a}_{ij}(\hat{u}) = \hat{a}_{ij}(y,t,\hat{u}) = a_{ij}(x,t,u)$, $\hat{b}_i(\hat{u}) = \hat{b}_i(y,t,\hat{u}) = b_i(x,t,u)$, $\hat{f}(\hat{u}) = \hat{f}(y,t,\hat{u}) = f(x,t,u)$, and $p_{ij}(\hat{u}) = p_{ij}(y,t,\hat{u}) = \frac{\hat{a}_{ij}(\hat{u})}{[s_{i,2}(t) - s_{i,1}(t)][(s_{j,2}(t) - s_{j,1}(t)]}$, $q_i(\hat{u}) = q_i(y,t,\hat{u}) = \frac{1}{s_{i,2}(t) - s_{i,1}(t)}\{\hat{b}_i(\hat{u}) - \dot{s}_{i,1}(t) - [\dot{s}_{i,2}(t) - \dot{s}_{i,1}(t)]y_i\}$, where $\dot{\phi}(t) = \frac{d\phi}{dt}$, $i, j = 1, 2, \cdots, d$.

One sees from the coercive assumption on a_{ij} and $s_{i,1}, s_{i,2}$ that there exist positive constants p_* and p^* such that for all $\gamma \in R^d$, $p_*|\gamma|^2 \le \sum_{i,j=1}^{d} (p_{ij}(y,t,\hat{\phi})\ \gamma_i, \gamma_j) \le p^*|\gamma|^2$, $\forall y \in \hat{\Omega}, t \in J, \hat{\phi} \in R$.

Denote $(\hat{\phi}, \hat{\psi}) = \int_{\hat{\Omega}} \hat{\phi}\hat{\psi} dy$. The variational form (2.1) is equivalent to finding $\hat{u} \in L^2(J, H_0^1(\hat{\Omega}))$, $\hat{u}_t \in L^2(J, H^{-1}(\hat{\Omega}))$ such that

$$\begin{aligned} &(\hat{u}_t, \hat{v}) - \sum_{i,j=1}^{d} (p_{ij}(\hat{u})\tfrac{\partial \hat{u}}{\partial y_i}, \tfrac{\partial \hat{v}}{\partial y_j}) \\ &+ \sum_{i=1}^{d} (q_i(\hat{u})\tfrac{\partial \hat{u}}{\partial y_i}, \hat{v}) = (\hat{f}(\hat{u}), \hat{v}), \ \forall \hat{v} \in H_0^1(\hat{\Omega}), t \in J, \\ &(\hat{u}(y,0), \hat{v}) = (\hat{u}_0, \hat{v}), \ \forall \hat{v} \in H_0^1(\hat{\Omega}). \qquad (2.2) \end{aligned}$$

Divide $\hat{\Omega}$ into $M_1 \times M_2 \times \cdots \times M_d$ small equal intervals and denote $h_i = 1/M_i$, $i = 1, 2, \cdots, d$, $h = max\{h_1, h_2, \cdots, h_d\}$. Let $\alpha_i \in \{0, 1\}, i = 1, 2, \cdots, d$; $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$ and $D^\alpha \phi = \frac{\partial^{|\alpha|}\phi}{\partial y_1^{\alpha_1} \partial y_2^{\alpha_2} \cdots \partial y_d^{\alpha_d}}$, obviously, $D^0\phi = \phi$; denote $H = \{\phi \mid \phi,\ D^\alpha \phi \in L^2(\Omega)$ for $|\alpha| = 1, 2, \cdots, d$; and $\|D^\alpha \phi\| \le Kh^{j-|\alpha|}\|\phi\|_j$ for $j = 0, 1, 2, \cdots, |\alpha|$ and $|\alpha| = 2, \cdots, d.\}$. Let $\otimes$ denote the tensor product operator, and let $\mu_i = span\{\gamma_1^i(y_i), \gamma_2^i(y_i), ..., \gamma_{M_i}^i(y_i)\} \subset H_0^1([0,1])$, $i = 1, 2, \cdots, d$; let $\mu = \mu_1 \otimes \mu_2 \otimes \cdots \otimes \mu_d = span(N_1, N_2, \cdots, N_l) \subset H$ be k degree finite dimensional space.

Divide $[0, T]$ into L small equal intervals, denote $\Delta t = \frac{T}{L}$, and $t_n = n\Delta t$. Let $d_t\phi^n = \frac{\phi^{n+1} - \phi^n}{\Delta t}$, $\partial_t \phi^n = \frac{\phi^{n+1} - \phi^{n-1}}{2\Delta t}$, and $\partial_{tt}\phi^n = \frac{\phi^{n+1} - 2\phi^n + \phi^{n-1}}{(\Delta t)^2}$.

Denote $p_{ij}^n(\hat{U}) = p_{ij}(y, t_n, \hat{U}^n)$, $q_i^n(\hat{U}) = q_i(y, t_n, \hat{U}^n)$, $\hat{f}^n(\hat{U}) = \hat{f}(y, t_n, \hat{U}^n)$, $i, j = 1, 2, \cdots, d$. Let λ be a properly selected positive constant. By introducing a small implicit viscous term and approaching the anisotropic diffusion explicitly, we propose two ADFE discrete schemes as follows:

(1) Let $\lambda > \frac{1}{2}p^*$, finding $\hat{U}^n \in \mu$ such that

$$\begin{aligned} &(d_t\hat{U}^n, \hat{v}) + \sum_{i,j=1}^{d} (p_{ij}^n(\hat{U})\tfrac{\partial \hat{U}^n}{\partial y_i}, \tfrac{\partial \hat{v}}{\partial y_j}) \\ &+ \sum_{i=1}^{d} (q_i^n(\hat{U})\tfrac{\partial \hat{U}^n}{\partial y_i}, \hat{v}) + \lambda\Delta t(\nabla d_t\hat{U}^n, \nabla\hat{v}) \\ &+ \sum_{|\alpha|=2}^{d} (\lambda\Delta t)^{|\alpha|}(D^\alpha d_t\hat{U}^n, D^\alpha \hat{v}) \\ = \ &(f^n(\hat{U}), \hat{v}), \qquad \forall \hat{v} \in \mu. \qquad (2.3) \end{aligned}$$

(2) Let $\lambda > \frac{1}{4}p^*$, finding $\hat{U}^n \in \mu$ such that

$$\begin{aligned} &(\partial_t\hat{U}^n, \hat{v}) + \sum_{i,j=1}^{d} (p_{ij}^n(\hat{U})\tfrac{\partial \hat{U}^n}{\partial y_i}, \tfrac{\partial \hat{v}}{\partial y_j}) \\ &+ \sum_{i=1}^{d} (q_i^n(\hat{U})\tfrac{\partial \hat{U}^n}{\partial y_i}, \hat{v}) + \lambda(\Delta t)^2(\nabla \partial_{tt}\hat{U}^n, \nabla\hat{v}) \\ &+ \tfrac{1}{2}\Delta t \sum_{|\alpha|=2}^{d} (2\lambda\Delta t)^{|\alpha|}(D^\alpha \partial_{tt}\hat{U}^n, D^\alpha \hat{v}) \\ = \ &(f^n(\hat{U}), \hat{v}), \qquad \forall \hat{v} \in \mu. \qquad (2.4) \end{aligned}$$

Let $\hat{U}^n = \sum_{j=1}^{l} \Gamma_j^n N_j$; set $C_{irm} = (\gamma_r^i(y_i), \gamma_m^i(y_i))$, $A_{irm} = ((\gamma_r^i(y_i))', (\gamma_m^i(y_i))')$, for $r, m = 1, ..., M_i$; let $C_i = (C_{irm})_{r,m}$, $A_i = (A_{irm})_{r,m}$ be $M_i \times M_i$ matrices, let I_j be $M_j \times M_j$ unit matrix, $i = 1, 2, \cdots, d$. If the nodes are numbered in x_1 direction first, then x_2 and so on, and finally x_d, then equivalently, (2.3) and (2.4) can be respectively written into the following AD solving series of unknown vectors:

$$\begin{aligned} &(C_1 + \lambda\Delta t A_1) \otimes I_2 \otimes I_3 \cdots \otimes I_d \Upsilon_1^{n+1} = \Delta t\Phi^n, \\ &\cdots, \\ &I_1 \otimes \cdots \otimes I_{j-1} \otimes (C_j + \lambda\Delta t A_j) \otimes I_{j+1} \otimes \cdots \\ &\otimes I_d \Upsilon_j^{n+1} = \Upsilon_{j-1}^{n+1}, \end{aligned}$$

$$\begin{array}{l}\cdots,\\ I_1 \otimes I_2 \otimes \cdots \otimes I_{d-1} \otimes (C_d + \lambda \Delta t A_d)\Upsilon_d^{n+1} = \Upsilon_{d-1}^{n+1},\\ \Gamma^{n+1} = \Gamma^n + \Upsilon_d^{n+1}. \qquad (2.5)\end{array}$$

$$\begin{array}{l}(C_1 + 2\lambda \Delta t A_1) \otimes I_2 \otimes I_3 \cdots \otimes I_d \Upsilon_1^{n+1} = 2\Delta t \Psi^n,\\ \cdots,\\ I_1 \otimes \cdots \otimes I_{j-1} \otimes (C_j + 2\lambda \Delta t A_j) \otimes I_{j+1} \otimes \cdots\\ \otimes I_d \Upsilon_j^{n+1} = \Upsilon_{j-1}^{n+1},\\ \cdots,\\ I_1 \otimes I_2 \otimes \cdots \otimes I_{d-1} \otimes (C_d + 2\lambda \Delta t A_d)\Upsilon_d^{n+1} = \Upsilon_{d-1}^{n+1},\\ \Gamma^{n+1} = 2\Gamma^n - \Gamma^{n-1} + \Upsilon_d^{n+1}. \qquad (2.6)\end{array}$$

where $\Phi^n = -\sum_{i,j=1}^{d}(p_{ij}^n(\hat{U})\frac{\partial \hat{U}^n}{\partial y_i}, \frac{\partial \hat{v}}{\partial y_j}) - \sum_{i=1}^{d}(q_i^n(\hat{U})\frac{\partial \hat{U}^n}{\partial y_i}, \hat{v}_i)$ $+(\hat{f}^n(\hat{U}), \hat{v})$ and $\Psi^n = \Phi^n - \frac{1}{2}(d_t\hat{U}^{n-1}, \hat{v})$, Γ^{n+1} is the expected unknown vector.

Since C_i and A_i are independent of time, the AD decomposition in (2.5) and (2.6) only need to be manipulated once, and can be used at each time step, hence the calculation is highly economic.

3 Approximation and Stability Properties

We define the Ritz projector of the solution $\hat{u}$ into space μ as $\tilde{u}$, which satisfies

$$\begin{array}{l}\sum_{i,j=1}^{d}(p_{ij}(\hat{u})\frac{\partial(\hat{u}-\tilde{u})}{\partial y_i}, \frac{\partial \hat{v}}{\partial y_j}) + \sum_{i=1}^{d}(q_i(\hat{u})\frac{\partial(\hat{u}-\tilde{u})}{\partial y_i}, \hat{v})\\ +\kappa(\hat{u}-\tilde{u}, \hat{v}) = 0, \quad \forall \hat{v}, \in \mu \qquad (3.1)\end{array}$$

where κ is a proper positive constant. Set $\hat{u}-\tilde{u} = \eta$, then similarly as [8], we can get the following approximation properties:

$$\begin{array}{rl} & \|\eta_t\|_{L^2(L^2)} + \|\eta\|_{L^\infty(L^2)} + h\|\eta\|_{L^\infty(H^1)}\\ = & O(h^{k+1}). \qquad (3.2)\end{array}$$

Denote $\xi^n = \tilde{u}^n - \hat{U}^n$, then $\hat{u}^n - \hat{U}^n = \xi^n + \eta^n$. We have the following approximation property.

Theorem 1 For $\lambda > \frac{1}{2}p*$, $k \geq 1$, if

$$\begin{array}{rl} & \|\xi^0\| + (\Delta t)^{\frac{1}{2}}\|\nabla \xi^0\| + \sum_{|\alpha|=2}^{d}(\Delta t)^{\frac{|\alpha|}{2}}\|D^\alpha \xi^0\|\\ = & O(h^{k+1} + \Delta t) \qquad (3.3)\end{array}$$

satisfies, then for the ADFE scheme (2.3), there is

$$\begin{array}{rl} & \max_{0\leq n\leq N}\|\hat{u}^n - \hat{U}^n\| + h(\sum_{n=0}^{N}\Delta t\|\nabla(\hat{u}^n - \hat{U}^n)\|^2)^{\frac{1}{2}}\\ = & O(h^{k+1} + \Delta t).\end{array}$$

Theorem 2 For $\lambda > \frac{1}{4}p*$, $k \geq 3 - \frac{6}{d+1}$, if

$$\begin{array}{rl} & \|\xi^0\|_1 + \|\xi^1\|_1 + \sum_{|\alpha|=2}^{d}(\Delta t)^{\frac{|\alpha|+1}{2}}\| D^\alpha d_t \xi^0\|\\ = & O(h^{k+1} + (\Delta t)^2) \qquad (3.4)\end{array}$$

satisfies, then for the ADFE scheme (2.4), there is

$$\begin{array}{l}(\Delta t\sum_{n=1}^{N-1}\|\partial_t(\hat{u}^n - \hat{U}^n)\|^2)^{\frac{1}{2}} + \max_{0\leq n\leq N}\|\hat{u}^n - \hat{U}^n\|\\ +h\max_{0\leq n\leq N}\|\nabla(\hat{u}^n - \hat{U}^n)\| = O(h^{k+1} + (\Delta t)^2).\end{array}$$

From these theorems, we see for ADFE scheme (2.3) discrete $L^\infty(L^2)$ and $L^2(H^1)$ approximation norm are all optimal and for (2.4) discrete $L^\infty(L^2)$ and $L^\infty(H^1)$ approximation norm are all optimal on the computational domain.

Proof of Theorem 1: Subtracting (2.4) from (2.2), and noting the relation (3.1), we derive the error equation:

$$\begin{array}{rl} & [(d_t\xi^n, \hat{v}) + \sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}(D^\alpha d_t \xi^n, D^\alpha \hat{v})]\\ & +[\sum_{i,j=1}^{d}(p_{ij}^n(\hat{U})\frac{\partial \xi^n}{\partial y_i}, \frac{\partial \hat{v}}{\partial y_j}) + \lambda\Delta t(\nabla d_t \xi^n, \nabla \hat{v})]\\ = & (d_t\hat{u}^n - \hat{u}_t^n - d_t\eta^n - \sum_{i=1}^{d}[q_i^n(\hat{u}) - q_i^n(\hat{U})]\frac{\partial \tilde{u}^n}{\partial y_i}\\ & -\sum_{i=1}^{d}q_i^n(\hat{U})\frac{\partial \xi^n}{\partial y_i} + \kappa\eta^n + [f^n(\hat{u}) - f^n(\hat{U})], \hat{v})\\ & -\sum_{i,j=1}^{d}([p_{ij}^n(\hat{u}) - p_{ij}^n(\hat{U})]\frac{\partial \tilde{u}^n}{\partial y_i}, \frac{\partial \hat{v}}{\partial y_j})\\ & +[\lambda\Delta t(\nabla d_t\tilde{u}^n, \nabla \hat{v})\\ & +\sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}(D^\alpha d_t \tilde{u}^n, D^\alpha \hat{v})]. \qquad (3.5)\end{array}$$

Taking $\hat{v} = \xi^{n+1}$ as a test function, multiplying (3.5) by $2\Delta t$ and summing for $n = 0, 1, ..., N-1$ $(1 \leq N \leq L)$, and denoting the derived relation as $\sum_{i=1}^{2}P_i^N = \sum_{i=1}^{3}Q_i^N$, estimating these terms one by one, we show for the left hand, there are

$$\begin{array}{rl} P_1^N = & \sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}(\|D^\alpha \xi^N\|^2 - \|D^\alpha \xi^0\|^2)\\ & +\|\xi^N\|^2 - \|\xi^0\|^2. \qquad (3.6)\\ P_2^N = & 2\Delta t\sum_{n=0}^{N-1}[(\lambda I\nabla \xi^{n+1}, \nabla \xi^{n+1})\\ & +([P^n(\hat{U}) - \lambda I]\nabla \xi^n, \nabla \xi^{n+1})]\\ \geq & 2\Delta t[\sum_{n=1}^{N-1}(\lambda - \frac{1}{2}\|p^{n-1}(\hat{U}) - \lambda I\| - \frac{1}{2}\|p^n(\hat{U})\\ & -\lambda I\|)\|\nabla \xi^n\|^2 - \frac{1}{2}\|p^0(\hat{U}) - \lambda I\|\|\nabla \xi^0\|^2\\ & +(\lambda - \frac{1}{2}\|p^{N-1}(\hat{U}) - \lambda I\|)\|\nabla \xi^N\|^2],\end{array}$$

where $p^n(\hat{U}) - \lambda I$ is a $d \times d$ matrix whose elements are $p_{ii}^n(\hat{U}) - \lambda$ and $p_{ij}^n(\hat{U})$ $(i, j = 1, 2, \cdots, d$ and $i \neq j)$ respectively. Since $\lambda > \frac{1}{2}p^*$, for any small positive ϵ, there exist positive constant p_0 such that $\lambda - \frac{1}{2}\|P^n(\hat{U}) - \lambda I\| - \frac{1}{2}\|P^{n+1}(\hat{U}) - \lambda I\| - \epsilon \geq p_0 > 0$, where $\|p^n(\hat{U}) - \lambda I\|$ is the ordinary metric norm of $p^n(\hat{U}) - \lambda I$, hence

$$\begin{array}{rl} & P_2^N - \epsilon\Delta t\sum_{n=0}^{N}\|\nabla \xi^n\|^2\\ \geq & 2p_0\Delta t\sum_{n=1}^{N}\|\nabla \xi^n\|^2 - K\Delta t\|\nabla \xi^0\|^2. \qquad (3.7)\end{array}$$

Now we turn our attention to each Q_i^n on the right side. We have

$$Q_1^N \le K\Delta t\sum_{n=0}^{N-1}\|\xi^n\|^2+\epsilon\Delta t\sum_{n=0}^{N-1}\|\nabla\xi^n\|^2 +K[(\Delta t)^2+\|\eta\|^2_{L^2(L^2)}+\|\eta_t\|^2_{L^2(L^2)}]. \quad (3.8)$$

$$Q_2^N \le K\Delta t\sum_{n=0}^{N-1}\|\xi^n\|^2+\epsilon\Delta t\sum_{n=0}^{N-1}\|\nabla\xi^n\|^2 +K\|\eta\|^2_{L^2(L^2)}. \quad (3.9)$$

$$Q_3^N \le K\Delta t\sum_{n=0}^{N-1}\sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}\|D^\alpha\xi^n\|^2 +K(\Delta t)^2+\epsilon\Delta t\sum_{n=0}^{N-1}\|\nabla\xi^n\|^2. \quad (3.10)$$

where (3.10) is valid for $k\ge 1$.

Combining relations (3.5)-(3.10) and manipulating the above inequality with the standardization of the constant coefficients, we see

$$\begin{aligned}&\|\xi^N\|^2+\Delta t\sum_{n=0}^{N}\|\nabla\xi^n\|^2+\sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}\|D^\alpha\xi^N\|^2\\ \le\ & K[\|\xi^0\|^2+\Delta t\|\nabla\xi^0\|^2+\sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}\|D^\alpha\xi^0\|^2\\ &+(\Delta t)^2+\|\eta\|^2_{L^2(L^2)}+\|\eta_t\|^2_{L^2(L^2)}]+K\Delta t\sum_{n=0}^{N-1}\|\xi^n\|^2\\ &+K\Delta t\sum_{n=0}^{N-1}\sum_{|\alpha|=2}^{d}(\lambda\Delta t)^{|\alpha|}\|D^\alpha\xi^n\|^2.\end{aligned}$$

Using Gronwall's lemma, we deduce

$$\begin{aligned}&\|\xi^N\|^2+\Delta t\sum_{n=0}^{N}\|\nabla\xi^n\|^2+\sum_{|\alpha|=2}^{d}(\Delta t)^{|\alpha|}\|D^\alpha\xi^N\|^2\\ \le\ & K[\|\xi^0\|^2+\Delta t\|\nabla\xi^0\|^2+\sum_{|\alpha|=2}^{d}(\Delta t)^{|\alpha|}\|D^\alpha\xi^0\|^2\\ &+(\Delta t)^2+\|\eta\|^2_{L^2(L^2)}+\|\eta_t\|^2_{L^2(L^2)}]. \quad (3.11)\end{aligned}$$

Summarizing estimates (3.11) and (3.2), we accomplish the proof of Theorem 1. ¶

Proof of Theorem 2: Subtracting (2.4) from (2.2), and noticing (3.1), we get the error equation:

$$\begin{aligned}&[(\partial_t\xi^n,\hat v)+\tfrac12\Delta t\sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}(D^\alpha\partial_{tt}\xi^n,D^\alpha\hat v)]\\ &+[\sum_{i,j=1}^{d}(p_{ij}^n(\hat U)\tfrac{\partial\xi^n}{\partial y_i},\tfrac{\partial\hat v}{\partial y_j})+\lambda(\Delta t)^2(\nabla\partial_{tt}\xi^n,\nabla\hat v)]\\ =\ &(\partial_t\hat u^n-\hat u_t^n-\partial_t\eta^n-\sum_{i=1}^{d}[q_i^n(\hat u)-q_i^n(\hat U)]\tfrac{\partial\tilde u^n}{\partial y_i}\\ &-\sum_{i=1}^{d}q_i^n(\hat U)\tfrac{\partial\xi^n}{\partial y_i}+\kappa\eta^n+[f^n(\hat u)-f^n(\hat U)],\hat v)\\ &-\sum_{i,j=1}^{d}([p_{ij}^n(\hat u)-p_{ij}^n(\hat U)]\tfrac{\partial\tilde u^n}{\partial y_i},\tfrac{\partial\hat v}{\partial y_j})\\ &+[\lambda(\Delta t)^2(\nabla\partial_{tt}\tilde u^n,\nabla\hat v)\\ &+\tfrac12\Delta t\sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}(D^\alpha\partial_{tt}\tilde u^n,D^\alpha\hat v)]. \quad (3.12)\end{aligned}$$

Taking $\hat v=\partial_t\xi^n$ as a test function, multiplying (3.12) by $2\Delta t$ and summing for $n=1,2,...,N-1$ $(2\le N\le L)$, denoting the derived relation as $\sum_{i=1}^{2}L_i^N=\sum_{i=1}^{3}R_i^N$, estimating these terms in turn, noting that $\partial_{tt}\xi^n=(d_t\xi^n-d_t\xi^{n-1})/\Delta t$, $\partial_t\xi^n=(d_t\xi^n+d_t\xi^{n-1})/2$, we see for the left hand, there are

$$L_1^N = \tfrac{\Delta t}{2}\sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}(\|D^\alpha d_t\xi^{N-1}\|^2-\|D^\alpha d_t\xi^0\|^2) +2\Delta t\sum_{n=1}^{N-1}\|\partial_t\xi^n\|^2. \quad (3.13)$$

$$\begin{aligned}L_2^N =\ &\lambda(\|\nabla\xi^N\|^2+\|\nabla\xi^{N-1}\|^2-\|\nabla\xi^1\|^2-\|\nabla\xi^0\|^2)\\ &+2\Delta t\sum_{n=1}^{N-1}([P^n(\hat U)-2\lambda I]\nabla\xi^n,\nabla\partial_t\xi^n)\\ :=\ &L_{2,1}^N+L_{2,2}^N. \quad (3.14)\end{aligned}$$

Using summation by parts, we know

$$\begin{aligned}|L_{2,2}^N|\le\ &\tfrac12\|p^{N-1}(\hat U)-2\lambda I\|(\|\nabla\xi^{N-1}\|^2+\|\nabla\xi^N\|^2)\\ &+\tfrac12\|p^0(\hat U)-2\lambda I\|(\|\nabla\xi^0\|^2+\|\nabla\xi^1\|^2)\\ &+K\Delta t\sum_{n=1}^{N-1}\|\nabla\xi^n\|^2, \quad (3.15)\end{aligned}$$

where $\|p^n(\hat U)-2\lambda I\|$ is the ordinary metric norm of $d\times d$ matrix $p^n(\hat U)-2\lambda I$ whose elements are $p_{ii}^n(\hat U)-2\lambda$ and $p_{ij}^n(\hat U)$ $(i,j=1,2,\cdots,d$ and $i\ne j)$ respectively.

As to the right side, firstly,

$$R_1^N \le K[(\Delta t)^4+\|\eta\|^2_{L^2(L^2)}+\|\eta_t\|^2_{L^2(L^2)}] +K\Delta t\sum_{n=1}^{N-1}\|\xi^n\|_1^2+\epsilon\Delta t\sum_{n=1}^{N-1}\|\partial_t\xi^n\|^2. \quad (3.16)$$

Secondly, using summation by parts, and noting

$$\begin{aligned}&|[p_{ij}^{n+1}(\hat u)-p_{ij}^{n+1}(\hat U)]-[p_{ij}^{n-1}(\hat u)-p_{ij}^{n-1}(\hat U)]|\\ \le\ &K_0\Delta t(|\xi^{n+1}|+|\eta^{n+1}|+|\xi^{n-1}|+|\eta^{n-1}|\\ &+|\partial_t\xi^n|+|\partial_t\eta^n|),\end{aligned}$$

we obtain

$$\begin{aligned}R_2^N\le\ &K[\|\xi^1\|_1^2+\|\xi^0\|_1^2+\|\eta^1\|^2+\|\eta^0\|^2\\ &+\|\eta\|^2_{L^2(L^2)}+\|\eta_t\|^2_{L^2(L^2)}]+K\Delta t\sum_{n=1}^{N-1}\|\xi^n\|_1^2\\ &+\epsilon[\|\nabla\xi^N\|^2+\Delta t\sum_{n=1}^{N-1}\|\partial_t\xi^n\|^2]. \quad (3.17)\end{aligned}$$

$$\begin{aligned}R_3^N\le\ &K\Delta t\sum_{n=1}^{N-1}\|\nabla\xi^n\|^2+\epsilon[\|\nabla\xi^N\|^2+\|\nabla\xi^{N-1}\|^2]\\ &+K[(\Delta t)^4+\|\nabla\xi^0\|^2+\|\nabla\xi^1\|^2]. \quad (3.18)\end{aligned}$$

where (3.18) stands for $k\ge 3-\frac{6}{d+1}$.

Now combine relations (3.12)-(3.18). It implies that

$$\begin{aligned}&2\Delta t\sum_{n=1}^{N-1}\|\partial_t\xi^n\|^2+\tfrac{\Delta t}{2}\sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}\|D^\alpha d_t\xi^{N-1}\|^2\\ &+[\lambda-\tfrac12\|p^{N-1}(\hat U)-2\lambda I\|](\|\nabla\xi^N\|^2+\|\nabla\xi^{N-1}\|^2)\\ \le\ &K[(\Delta t)^4+\|\xi^0\|_1^2+\|\xi^1\|_1^2+\|\eta\|^2_{L^\infty(L^2)}+\|\eta_t\|^2_{L^2(L^2)}]\\ &+\tfrac{\Delta t}{2}\sum_{|\alpha|=2}^{d}(2\lambda\Delta t)^{|\alpha|}\|D^\alpha d_t\xi^0\|^2+K\Delta t\sum_{n=1}^{N-1}\|\xi^n\|_1^2\\ &+[\tfrac12\|p^0(\hat U)-2\lambda I\|+\lambda](\|\nabla\xi^0\|^2+\|\nabla\xi^1\|^2).\end{aligned}$$

Noting $\lambda > \frac{1}{4}p^*$, manipulating this relation with the standardization of the constant coefficients, and then using Gronwall's lemma, we deduce

$$\begin{aligned}&\Delta t \sum_{n=1}^{N-1} \|\partial_t \xi^n\|^2 + \|\xi^N\|_1^2 + \|\xi^{N-1}\|_1^2 \\ &+ \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|+1} \|D^\alpha d_t \xi^{N-1}\|^2 \\ \leq\ & K[(\Delta t)^4 + \|\eta\|^2_{L^\infty(L^2)} + \|\eta_t\|^2_{L^2(L^2)} \\ &+\|\xi^0\|_1^2 + \|\xi^1\|_1^2 + \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|+1} \|D^\alpha d_t \xi^0\|^2]. \quad (3.19)\end{aligned}$$

Summarizing (3.19) and (3.2), we derive Theorem 2. ¶

Taking the test function $\hat{v} = \hat{U}^{n+1}$ in (2.3), $\hat{v} = \partial_t \hat{U}^n$ in (2.4), and using analogous reasoning procedures as in the proof of Theorem 1 and Theorem 2 respectively, we obtain the stability result on the computational domain.

Theorem 3 Under the condition of Theorem 1, there is

$$\begin{aligned}&\|\hat{U}^N\|^2 + \Delta t \sum_{n=0}^{N} \|\nabla \hat{U}^n\|^2 + \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|} \|D^\alpha \hat{U}^N\|^2 \\ \leq\ & K[\|\hat{U}^0\|^2 + \Delta t \|\nabla \hat{U}^0\|^2 + \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|} \|D^\alpha \hat{U}^0\|^2 \\ &+\Delta t \sum_{n=0}^{N-1} \|f^n(\hat{U})\|^2].\end{aligned}$$

Theorem 4 Under the condition of Theorem 2, there is

$$\begin{aligned}&\Delta t \sum_{n=1}^{N-1} \|\partial_t \hat{U}^n\|^2 + \|\nabla \hat{U}^N\|^2 + \|\nabla \hat{U}^{N-1}\|^2 \\ &+ \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|+1} \|D d_t \hat{U}^{N-1}\|^2 \\ \leq\ & K[\|\nabla \hat{U}^0\|^2 + \|\nabla \hat{U}^1\|^2 + \sum_{|\alpha|=2}^{d} (\Delta t)^{|\alpha|+1} \|D d_t \hat{U}^0\|^2 \\ &+\Delta t \sum_{n=1}^{N-1} \|f^n(\hat{U})\|^2].\end{aligned}$$

Hence from Theorem 1-4, we conclude that ADFE schemes (2.3) and (2.4) are both uniquely solvable, and have optimal H^1 and L^2 norm convergence properties on the computational domain. Since the transformation is invertible between the computational domain and the practical domain, by using the equivalent norm property (Lemma 2.2 in [4]), we get the same conclusion for the corresponding approximation solution U and the exact solution u of (2.1) on the practical domain $\Omega(t)$.

4 Numerical Results and Discussion

To start procedures (2.3) and (2.4), perfect initial values are needed to satisfy (3.3) and (3.4) respectively. In fact, these conditions are easy to fulfill. For example, letting

$$(\hat{U}^0, \hat{v}) = (\hat{u}^0, \hat{v}), \quad \forall \hat{v} \in \mu.$$

or $\hat{U}^0 = \tilde{u}^0$, naturally for $k \geq 1$, (3.3) stands; letting $\hat{U}^0 = \tilde{u}^0$, and defining $\hat{U}^1$ as

$$\begin{aligned}&(\tfrac{\hat{U}^1 - \hat{U}^0}{\Delta t}, \hat{v}) + \sum_{i,j=1}^{d} (p_{ij}(\hat{U}^{\frac{1}{2}}) \tfrac{\partial \hat{U}^{\frac{1}{2}}}{\partial y_i}, \tfrac{\partial \hat{v}}{\partial y_j}) \\ &+ \sum_{i=1}^{d} (q_i(\hat{U}^{\frac{1}{2}}) \tfrac{\partial \hat{U}^{\frac{1}{2}}}{\partial y_i}, \hat{v}) = (\hat{f}(\hat{U}^{\frac{1}{2}}), \hat{v}), \quad \forall \hat{v} \in \mu.\end{aligned}$$

where $\hat{U}^{\frac{1}{2}} = \frac{\hat{U}^0 + \hat{U}^1}{2}$, then (3.4) is available for $k \geq 1$.

Now we present some numerical examples for the ADFE simulation studied here to show the efficiency of our schemes. Consider 3-dimensional model problem (1.1) with the following characters. We slightly change the former notation $x = (x_1, x_2, \cdots, x_d)$ into $X = (x, y, z)$ now to write the formulas in a more popular way.

$$\begin{aligned}s_{i,1}(t) =\ & -2 + \cos(t),\ s_{i,2}(t) = 2 + \cos(t),\ i = 1, 2, 3; \\ a_{11}(u) =\ & 0.5\cos[c(u)] + 3.5 + 0.5\sin[\pi(x - \cos(t))], \\ a_{22}(u) =\ & 0.5\cos[c(u)] + 3.5 + 0.5\sin[\pi(y - \cos(t))], \\ a_{33}(u) =\ & 0.5\cos[c(u)] + 3.5 + 0.5\sin[\pi(z - \cos(t))], \\ a_{12}(u) =\ & a_{21}(u) = 0.5\sin[c(u)] + 0.5\sin[\pi(x - \cos(t))] \\ & +0.5\sin[\pi(y - \cos(t))], \\ a_{13}(u) =\ & a_{31}(u) = 0.5\sin[c(u)] + 0.5\sin[\pi(x - \cos(t))] \\ & +0.5\sin[\pi(z - \cos(t))], \\ a_{23}(u) =\ & a_{32}(u) = 0.5\sin[c(u)] + 0.5\sin[\pi(y - \cos(t))] \\ & +0.5\sin[\pi(z - \cos(t))], \\ b_1(u) =\ & \sin[c(u)] - \sin(t) - 0.5\pi\cos[\pi(x - \cos(t))] \\ & -1.5\pi\cos[\pi(y - \cos(t))] \\ & -1.5\pi\cos[\pi(z - \cos(t))], \\ b_2(u) =\ & \sin[c(u)] - \sin(t) - 1.5\pi\cos[\pi(x - \cos(t))] \\ & -0.5\pi\cos[\pi(y - \cos(t))] \\ & -1.5\pi\cos[\pi(z - \cos(t))], \\ b_3(u) =\ & \sin[c(u)] - \sin(t) - 1.5\pi\cos[\pi(x - \cos(t))] \\ & -1.5\pi\cos[\pi(y - \cos(t))] \\ & -0.5\pi\cos[\pi(z - \cos(t))], \\ f(u) =\ & (-\pi^2\{12 + 0.5\sin[\pi(x - \cos(t))] \\ & +0.5\sin[\pi(y - \cos(t))] \\ & +0.5\sin[\pi(z - \cos(t))]\} - 1)u,\end{aligned}$$

$$u(x, y, z, 0) = -\sin(\pi x)\sin(\pi y)\sin(\pi z),$$

where $c(u) = e^{-t}\sin[\pi(x - \cos(t))]\sin[\pi(y - \cos(t))]$ $\sin[\pi(z - \cos(t))] - u$. Its exact solution can be expressed as

$$\begin{aligned}u(x, y, z, t) =\ & e^{-t}\sin[\pi(x - \cos(t))]\sin[\pi(y - \cos(t))] \\ & \sin[\pi(z - \cos(t))].\end{aligned}$$

Numerical tests are given with both ADFE scheme (2.3) and (2.4) after coordinate transformation on the linear finite element space ($k = 1$). Let's denote UERRMAX= $\max_{0 \leq n \leq N} \|\hat{u}^n - \hat{U}^n\|$, UTERRL2= $(\Delta t \sum_{n=1}^{N-1} \|\partial_t(\hat{u}^n - \hat{U}^n)\|^2)^{\frac{1}{2}}$, UPERRMAX= $h \max_{0 \leq n \leq N} \|\nabla(\hat{u}^n - \hat{U}^n)\|$ and UPERRL2= $h(\Delta t \sum_{n=0}^{N} \|\nabla(\hat{u}^n - \hat{U}^n)\|^2)^{\frac{1}{2}}$. Let $M_i = 13$, hence $h_i = 1/13$ ($i = 1, 2, 3$). For (2.3), set $J =$

$[0,1]$, $L = 100$, hence $\Delta t = 1/100$, correspondingly, $h_1^2+h_2^2+h_3^2+\Delta t = 0.02775147928994083$. The upper error bounds given by numerical calculation on the computational domain are UERRMAX= 0.01104986816058213 and UPERRL2= 0.02304097874653152 which are accordant with $O(h_1^2 + h_2^2 + h_3^2 + \Delta t)$. For (2.4), set $J = [0,2]$, $L = 50$, hence $\Delta t = 1/40$ and $h_1^2 + h_2^2 + h_3^2 + \Delta t^2$ =0.01935147928994083. Note that for $d = 3$, $k = 1$ is less than $3 - \frac{6}{d+1} = \frac{3}{2}$. The computational error bounds are UTERRL2= 0.02423442011282067, UERRMAX= 0.02247563471115946 and UPERRMAX= 0.05941795483815188, almost accordant with $O(h_1^2+h_2^2+h_3^2 + \Delta t^2)$.

Figs. 1- 4 show the absolute error $\hat{u}^n - \hat{U}^n$ on the X-planes slices on the computational domain at different time, from which it can be seen that after certain time, the error decreases with time developing. Similar results are got on the Y-planes and Z-planes slices. Figs. 5 and 6 respectively illustrate the development of various errors between the exact solution of (2.2) and the approximation solutions got from (2.3) and (2.4) along with time advancing. Numerical results verify the theoretical conclusion.

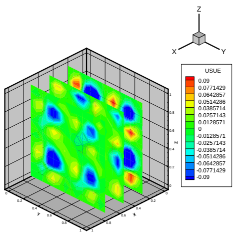

Figure 1: Absolute error at $t = 0.25$ on X-planes slices

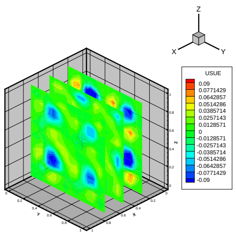

Figure 2: Absolute error at $t = 0.5$ on X-planes slices

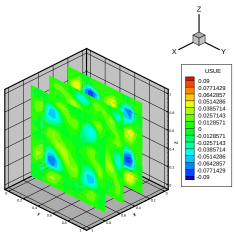

Figure 3: Absolute error at $t = 0.75$ on X-planes slices

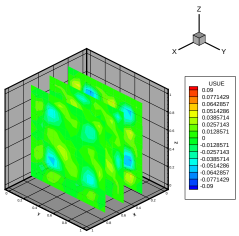

Figure 4: Absolute error at $t = 1$ on X-planes slices

By transforming the domain with perfect coordinate transformation, the ADFE method can be applied to solve a class of moving boundary problems, for example, parabolic integro-differential problems and so on. For certain more complicated boundaries whose movements depend on both the temporal and the spatial variants

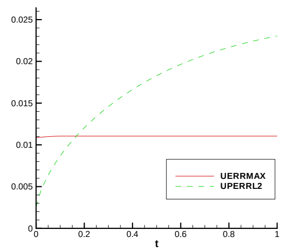

Figure 5: Errors development for scheme (2.3)

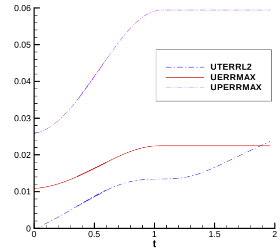

Figure 6: Errors development for scheme (2.4)

with enough smoothness, this strategy can also be considered. Other efficient numerical procedures, such as ADFD method, etc., can also be used to solve moving boundary problems under coordinate transformation.

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