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Interval Oscillation Criteria for Second-Order Delay Differential Equations

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Abstract—**We present new interval oscillation criteria for certain nonlinear delay second order differential equation that are different from some known ones. Our results extend and improve some previous oscillation criteria and handle the cases which are not covered by known results. In particular, several examples that dwell upon the sharp condition of our results are also included.**

Keywords: Oscillation, Interval criteria, Second order, Nonlinear delay equations.

1 Introduction

In this paper, we are concerned with the interval oscillation behavior for the second-order delay differential equation

$$(r(t)\psi(x(t))x'(\tau(t)))' + q(t)f(x(\tau(t))) = 0, t \geq t_0 > 0 \tag{1.1}$$

where
(c_1) $r,\ q,\ \tau \in C([a,\infty), R)$ with $r(t) > 0$ on $[a,\infty)$ for some $a \geq 0$, $\lim_{t\to\infty} \tau(t) = \infty,\ \tau'(t) > 0$;
(c_2) $f,\ \psi \in C(R,R),\ 0 < \psi(x) \leq m$ for some positive constants $m > 0$ and for $x \neq 0$;

By a solution of Eq.(1.1), We mean a function $x \in C^1[T_x,\infty)$, $T_x \geq t_0$, which has the property $r(t)\psi(x(t))x'(\tau(t)) \in C^1[T_x,\infty)$ and satisfies Eq.(1.1). A solution of Eq.(1.1) is called oscillatory if it has arbitrarily large zeros, otherwise it is called non-oscillatory. Finally, Eq.(1.1) is called oscillatory if all its solutions are oscillatory.

The theory of oscillation is an important branch of the qualitative theory of differential equations. It's foundation were laid down by the well-known results regarding zeros of solutions of self-adjoint second-order differential equations published in 1836 by Sturm. Since then, oscillation behavior of solutions to different classes of linear and nonlinear ordinary, functional, partial, discrete, impulsive differential equations have attracted the attention of many researchers.

In the last few decades, there has been increasing interest in obtaining sufficient conditions for different classes of second order differential equations [1-6]. In particular, much work has been down on the following particular cases of Eq.(1.1):

$$x''(t)+q(t)x(t) = 0\ (r(t) \equiv 1,\ \psi(x(t)) \equiv 1,\ f(x(\tau(t))) = x(t)) \tag{1.2}$$

$$(a(t)x'(t))'+q(t)x(t) = 0, \quad (\psi(x(t)) \equiv 1,\ f(x(\tau(t))) = x(t)) \tag{1.3}$$

$$(a(t)x'(t))' + q(t)f(x(t)) = 0, \qquad (\psi(x(t)) \equiv 1,\ \tau(t) \equiv t) \tag{1.4}$$

From the Sturm Separation Theorem, we see that oscillation is only an interval property, i.e. if there exists a sequence of subintervals $[a_i, b_i]$ of $[t_0,\infty)$, as $a_i \to \infty$, such that for each i there exists a solution of Eq.(1.2) that has at least two zeros in $[a_i, b_i]$, then every solution of Eq.(1.2) is oscillatory. El-Sayed [7] established an interval criterion for oscillation of a forced second order equation, but the result is very sharp, because a comparison with equations of constant coefficient is used in the proof. In 1997, Huang [8] and A. Elbert presented some interval criteria for oscillation and non-oscillation of Eq.(1.2). In 2000, Wan-Tong Li and Ravi P. Agarwal [9] obtained several interval criteria for a particular case of Eq.(1.1).

Motivated by the idea of El-Sayed [10], Kong [11], Li and Agarwal [9], by using averaging functions and a generalized Riccati technique, in this paper we extend and improve several earlier interval criteria that of previous authors, that is, criteria given without any restriction on the sign of ρ'.

In the sequel we say that a function $H = H(t,s)$ belongs to a function class X, denoted by $H \in X$, if $H \in C(D,R)$, where $D = \{(t,s), -\infty < s \leq t < \infty\}$ which satisfies

$$H(t,t) = 0, H(t,s) > 0, \quad t > s,$$

and has continuous partial derivatives $\partial H/\partial t$ and $\partial H/\partial s$ on D such that

$$\frac{\partial H}{\partial t} = h_1(t,s)\sqrt{H(t,s)},\ \frac{\partial H}{\partial s} = -h_2(t,s)\sqrt{H(t,s)}.$$

2 Oscillation criteria for increasing f

In this section, we shall deal with the oscillation criteria of Eq.(1.1) under the assumption $(c_1) - (c_2)$ and the

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following assumption:

(c_3) $f'(x)$ exists and $f'(x) \geq \mu$ for some $\mu > 0$ and for all $x \neq 0$.

Theorem2.1 Suppose $(c_1) - (c_3)$ be fulfilled and for any $T \geq t_0$. If there exists $(a, b) \subset [t_0, \infty)$, $c \in (a, b)$ and a positive function $\rho \in C^1([t_0, \infty), R)$ such that

$$\frac{1}{H(c,a)} \int_a^c H(s,a)\rho(s)q(s)ds + \frac{1}{H(b,c)} \int_c^b H(b,s)\rho(s)q(s)ds > \frac{1}{4H(c,a)} \int_a^c \frac{\rho(s)r(s)}{k_1\tau'(s)} Q_1^2(s,a)ds + \frac{1}{4H(b,c)} \int_c^b \frac{\rho(s)r(s)}{k_1\tau'(s)} Q_2^2(b,s)ds. \quad (2.1)$$

where $Q_1(s,t) = h_1(s,t) + \frac{\rho'(s)}{\rho(s)}\sqrt{H(s,t)}$,
$Q_2(t,s) = h_2(t,s) - \frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}$
Then every solution of Eq.(1.1) is oscillatory.

Proof : Suppose to the contrary. Suppose that $x(t)$ be a non-oscillatory solution of Eq.(1.1), say $x(t) \neq 0$ on $[t_0, \infty)$ for some sufficient large $T_0 \geq t_0$. Define

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))x'(\tau(t))}{f(x(\tau(t)))}, \ t \geq t_0 \quad (2.2)$$

Then differentiating (2.2) and making use of Eq.(1.1), assumptions $(c_1) - (c_3)$, we have

$$\omega'(t) = -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - \frac{f'(x(\tau(t)))\tau'(t)}{\rho(t)r(t)\psi(x(t))}\omega^2(t) \leq -\rho(t)q(t) + \frac{\rho'(t)}{\rho(t)}\omega(t) - k_1\frac{\tau'(t)\omega^2(t)}{\rho(t)r(t)} \quad (k_1 = \frac{\mu}{m}) \quad (2.3)$$

Multiplying (2.3) by $H(t,s)$ and integrating it (with t replaced by s) over $[c,t)$ for $t \in [c,b)$, we have for $s \in [c,t)$

$$\int_c^t H(t,s)\rho(s)q(s)ds \leq -\int_c^t H(t,s)\omega'(s)ds$$
$$\int_c^t H(t,s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds$$
$$= H(t,c)\omega(c) - \int_c^t \omega(s)h_2(t,s)\sqrt{H(t,s)}ds$$
$$+ \int_c^t H(t,s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds = H(t,c)\omega(c) -$$
$$\int_c^t \left[\left(\frac{H(t,s)k_1\tau'(s)}{\rho(s)r(s)}\right)^{1/2}\omega(s) - \frac{1}{2}\left(\frac{\rho(s)r(s)}{k_1\tau'(s)}\right)^{1/2}Q_2(t,s)\right]^2 ds$$
$$+ \int_c^t \frac{\rho(s)r(s)}{4k_1\tau'(s)}Q_2^2(t,s)ds \leq H(t,c)\omega(c) + \int_c^t \frac{\rho(s)r(s)}{4k_1\tau'(s)}Q_2^2(t,s)ds \quad (2.4)$$

Letting $t \to b^-$ in (2.4) and dividing it by $H(b,c)$, we obtain

$$\frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)q(s)ds \leq \omega(c) + \frac{1}{H(b,c)}\int_c^b \frac{\rho(s)r(s)}{4k_1\tau'(s)}Q_2^2(b,s)ds \quad (2.5)$$

On the other hand, if we multiply $H(s,t)$ through (2.3) and integrate it (with t replaced by s) over (t,c) for $t \in [a,c)$, we yield for $s \in (t,c]$

$$\int_t^c H(s,t)\rho(s)q(s)ds \leq -\int_t^c H(s,t)\omega'(s)ds + \int_t^c H(s,t)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds$$
$$= -H(t,c)\omega(c) - \int_t^c \omega(s)h_1(s,t)\sqrt{H(s,t)}ds + \int_t^c H(s,t)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds$$
$$\leq -H(c,t)\omega(c) + \int_t^c \frac{\rho(s)r(s)}{4k_1\tau'(s)}Q_1^2(s,t)ds \quad (2.6)$$

Letting $t \to a^+$ in (2.6) and dividing it by $H(c,a)$, we obtain

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)q(s)ds \leq -\omega(c) + \frac{1}{H(c,a)}\int_a^c \frac{\rho(s)r(s)}{4k_1\tau'(s)}Q_1^2(s,a)ds \quad (2.7)$$

Adding (2.5) and (2.7), we have the following inequality

$$\frac{1}{H(c,a)}\int_a^c H(s,a)\rho(s)q(s)ds + \frac{1}{H(b,c)}\int_c^b H(b,s)\rho(s)q(s)ds \leq \frac{1}{4H(c,a)}\int_a^c \frac{\rho(s)r(s)}{k_1\tau'(s)}Q_1^2(s,a)ds + \frac{1}{4H(b,c)}\int_c^b \frac{\rho(s)r(s)}{k_1\tau'(s)}Q_2^2(b,s)ds.$$

which contradict to the condition (2.1), therefore, every solution of Eq.(1.1) is oscillatory. The proof is complete.

Remark 1: In some previous papers, the function $q(t)$ should be positive, but in Theorem 2.1, we can see when $q(t)$ is negative, the conclusion is valid as well.

With the standard yet choice of the $H(t,s)$, $H(t,s) = (t-s)^\lambda$, $t \geq s \geq t_0$, where $\lambda > 1$ is a constant, we obtain the following corollary:

Corollary 2.1 Suppose that the main assumption $(c_1)-(c_3)$ are satisfied, then every solution of Eq.(1.1) is oscillatory provided that there exists a function $\rho\in C^1([t_0,\infty),R)$ such that for each $l\geq t_0$ and for $\lambda>1$, the following two inequalities hold:

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\{\int_l^t (s-l)^\lambda q(s)\rho(s)ds$$
$$-\int_l^t \frac{\rho(s)r(s)}{4k_1\tau'(s)}(s-l)^{\lambda-2}\left[\lambda+\frac{\rho'(s)}{\rho(s)}(s-l)\right]^2 ds\}>0$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\{\int_l^t (t-s)^\lambda q(s)\rho(s)ds$$
$$-\int_l^t \frac{\rho(s)r(s)}{4k_1\tau'(s)}(t-s)^{\lambda-2}\left[\lambda+\frac{\rho'(s)}{\rho(s)}(t-s)\right]^2 ds\}>0$$

Theorem2.2 Suppose $(c_1)-(c_3)$ be fulfilled and $q(t)>0$ for any $T\geq t_0$, there exists a function $\rho\in C^1([t_0,\infty),(0,\infty))$ such that for any $u\in C[a,b]$ satisfying $u'(t)\in L^2[a,b]$ and $u(a)=u(b)=0$, we have

$$\int_a^b\left\{u^2(s)\rho(s)q(s)-\left[\frac{\rho(s)r(s)}{k_1\tau'(s)}u'(s)+\frac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}\right]^2\right\}ds>0 \quad (2.8)$$

then Eq.(1.1) is oscillatory.

Proof :Suppose to the contrary. Suppose that $x(t)$ be a non-oscillatory solution of Eq.(1.1), say $x(t)\neq 0$ on $[t_0,\infty)$ for some sufficient large $T_0\geq t_0$. Similar to the proof of Theorem 2.1, we multiply (2.3) by $u^2(t)$, integrate it with respect to s from a to b and use $u(a)=u(b)=0$, then we get

$$\int_a^b u^2(s)\rho(s)q(s)ds$$
$$\leq-\int_a^b u^2(s)\omega'(s)ds+\int_a^b u^2(s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s)-k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds$$
$$=2\int_a^b \omega(s)u(s)u'(s)ds+\int_a^b u^2(s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s)-k_1\frac{\tau'(s)\omega^2(s)}{\rho(s)r(s)}\right]ds$$
$$=-\int_a^b[\sqrt{\frac{k_1\tau'(s)}{\rho(s)r(s)}}u(s)\omega(s)-\sqrt{\frac{\rho(s)r(s)}{k_1\tau'(s)}}(u'(s)+\tfrac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)})]^2ds$$
$$+\int_a^b\left[\frac{\rho(s)r(s)}{k_1\tau'(s)}u'(s)+\tfrac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}\right]^2ds$$

so

$$\int_a^b\left\{u^2(s)\rho(s)q(s)-\left[\frac{\rho(s)r(s)}{k_1\tau'(s)}u'(s)+\frac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}\right]^2\right\}ds\leq 0$$

which contradicts to the condition (2.8), so every solution of Eq.(1.1) is oscillatory. The proof is complete.

3 Oscillation criteria for non-monotoic f

(c_4) $f(x)$ satisfies $\frac{f(x)}{x}\geq k_2$ for some $k_2>0$ and for all $x\neq 0$

Theorem3.1 Suppose $(c_1)-(c_2)$ and (c_4) be fulfilled and for any $T\geq t_0$. If there exists $(a,b)\subset[t_0,\infty)$, $c\in(a,b)$ and a positive function $\rho\in C^1([t_0,\infty),R)$ such that

$$\frac{1}{H(c,a)}\int_a^c H(s,a)k_2\rho(s)q(s)ds$$
$$+\frac{1}{H(b,c)}\int_c^b H(b,s)k_2\rho(s)q(s)ds$$
$$>\frac{1}{4H(c,a)}\int_a^c\frac{m\rho(s)r(s)}{\tau'(s)}Q_1^2(s,a)ds$$
$$+\frac{1}{4H(b,c)}\int_c^b\frac{m\rho(s)r(s)}{\tau'(s)}Q_2^2(b,s)ds. \quad (3.1)$$

where $Q_1(s,t)=h_1(s,t)+\frac{\rho'(s)}{\rho(s)}\sqrt{H(s,t)}$,
$Q_2(t,s)=h_2(t,s)-\frac{\rho'(s)}{\rho(s)}\sqrt{H(t,s)}$
Then every solution of Eq.(1.1) is oscillatory.

Proof : As above, Suppose that $x(t)$ be a non-oscillatory solution of Eq.(1.1), say $x(t)\neq 0$ on $[t_0,\infty)$ for some sufficient large $T_0\geq t_0$. Let

$$\omega(t)=\rho(t)\frac{r(t)\psi(x(t))x'(\tau(t))}{x(\tau(t))},\ t\geq t_0 \quad (3.2)$$

Then differentiating (3.2), we obtain

$$\omega'(t)=\frac{\rho'(t)}{\rho(t)}\omega(t)-\frac{f(x(\tau(t)))\rho(t)q(t)}{x(\tau(t))}-\frac{\tau'(t)}{\rho(t)r(t)\psi(x(t))}\omega^2(t)$$
$$\leq\frac{\rho'(t)}{\rho(t)}\omega(t)-k_2\rho(t)q(t)-\frac{\tau'(t)}{m\rho(t)r(t)}\omega^2(t) \quad (3.3)$$

Multiplying (3.3) by $H(t,s)$ and integrating it (with t replaced by s) over $[c,t)$ for $t\in[c,b)$, we have for $s\in[c,t)$

$$\int_c^t H(t,s)k_2\rho(s)q(s)ds\leq-\int_c^t H(t,s)\omega'(s)ds$$
$$+\int_c^t H(t,s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s)-\frac{\tau'(s)\omega^2(s)}{m\rho(s)r(s)}\right]ds$$
$$\leq H(t,c)\omega(c)+\int_c^t\frac{m\rho(s)r(s)}{4\tau'(s)}Q_2^2(t,s)ds \quad (3.4)$$

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Letting $t \to b^-$ in (3.4) and dividing it by $H(b,c)$, we obtain

$$\frac{1}{H(b,c)}\int_c^b H(b,s)k_2\rho(s)q(s)ds$$
$$\leq \omega(c) + \frac{1}{H(b,c)}\int_c^b \frac{m\rho(s)r(s)}{4\tau'(s)}Q_2^2(b,s)ds \qquad (3.5)$$

On the other hand, if we multiply $H(s,t)$ through (3.3) and integrate it (with t replaced by s) over (t,c) for $t \in [a,c)$, we yield for $s \in (t,c]$

$$\int_t^c H(s,t)k_2\rho(s)q(s)ds$$
$$\leq -H(c,t)\omega(c) + \int_t^c \frac{m\rho(s)r(s)}{4\tau'(s)}Q_1^2(s,t)ds$$

Letting $t \to a^+$ in (2.6) and dividing it by $H(c,a)$, we obtain

$$\frac{1}{H(c,a)}\int_a^c H(s,a)k_2\rho(s)q(s)ds$$
$$\leq -\omega(c) + \frac{1}{H(c,a)}\int_a^c \frac{m\rho(s)r(s)}{4\tau'(s)}Q_1^2(s,a)ds \qquad (3.6)$$

Adding (3.5) and (3.6), we get the following inequality

$$\frac{1}{H(c,a)}\int_a^c H(s,a)k_2\rho(s)q(s)ds$$
$$+\frac{1}{H(b,c)}\int_c^b H(b,s)k_2\rho(s)q(s)ds$$
$$\leq \frac{1}{4H(c,a)}\int_a^c \frac{m\rho(s)r(s)}{\tau'(s)}Q_1^2(s,a)ds$$
$$+\frac{1}{4H(b,c)}\int_c^b \frac{m\rho(s)r(s)}{\tau'(s)}Q_2^2(b,s)ds$$

which contradict to the condition (3.1), Therefore, every solution of Eq.(1.1) is oscillatory. The proof is complete.
Remark 2:The Remark 1 is also valid in this section.

The following result is analogous to Corollary 2.1 with the assumption (c_3) replaced by (c_4):

Corollary 3.1 Suppose that the main assumption $(c_1)-(c_2)$ and (c_4) are satisfied, then every solution of Eq.(1.1) is oscillatory provided that there exists a function $\rho \in C^1([t_0,\infty),R)$ such that for each $l \geq t_0$ and for $\lambda > 1$, the following two inequalities hold:

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\{\int_l^t (s-l)^\lambda k_2 q(s)\rho(s)ds$$
$$-\int_l^t \frac{m\rho(s)r(s)}{4\tau'(s)}(s-l)^{\lambda-2}\left[\lambda + \frac{\rho'(s)}{\rho(s)}(s-l)\right]^2 ds\} > 0$$

and

$$\limsup_{t\to\infty}\frac{1}{t^{\lambda-1}}\{\int_l^t (t-s)^\lambda k_2 q(s)\rho(s)ds$$
$$-\int_l^t \frac{m\rho(s)r(s)}{4\tau'(s)}(t-s)^{\lambda-2}\left[\lambda + \frac{\rho'(s)}{\rho(s)}(t-s)\right]^2 ds\} > 0$$

Theorem3.2 Suppose $(c_1)-(c_2)$ and (c_4) are satisfied and $q(t) > 0$ for any $T \geq t_0$, there exists a function $\rho \in C^1([t_0,\infty),(0,\infty))$ such that for any $u \in C[a,b]$ satisfying $u'(t) \in L^2[a,b]$ and $u(a) = u(b) = 0$, we have

$$\int_a^b \left\{u^2(s)k_2\rho(s)q(s) - \left[\frac{m\rho(s)r(s)}{\tau'(s)}u'(s) + \frac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}\right]^2\right\}ds$$
$$> 0$$

then Eq.(1.1) is oscillatory.

Proof: Suppose to the contrary. Similar to the proof of Theorem 2.2, we multiply (3.2) by $u^2(t)$, integrate it with respect to s from a to b and use $u(a) = u(b) = 0$, then we get

$$\int_a^b u^2(s)k_2\rho(s)q(s)ds$$
$$\leq -\int_a^b u^2(s)\omega'(s)ds + \int_a^b u^2(s)\left[\frac{\rho'(s)}{\rho(s)}\omega(s) - \frac{\tau'(s)\omega^2(s)}{m\rho(s)r(s)}\right]ds$$
$$= -\int_a^b[\sqrt{\frac{\tau'(s)}{m\rho(s)r(s)}}u(s)\omega(s) -$$
$$\sqrt{\frac{m\rho(s)r(s)}{\tau'(s)}}(u'(s) + \tfrac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)})]^2 ds$$
$$+\int_a^b\left[\frac{m\rho(s)r(s)}{\tau'(s)}u'(s) + \tfrac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}\right]^2 ds$$

so

$$\int_a^b\{u^2(s)k_2\rho(s)q(s) - [\frac{m\rho(s)r(s)}{\tau'(s)}u'(s) + \frac{1}{2}u(s)\frac{\rho'(s)}{\rho(s)}]^2\}ds \leq 0$$

which contradicts to the condition, the proof is completed.

4 Examples

In this section we will show the applications of our oscillation criteria by two examples. We will see that the equation in the example is oscillatory based on the results in Section 2 and Section 3.

Example 1 Consider the nonlinear differential equation

$$[e^t e^{-[x(t)]^2 - ln8}x'(\frac{1}{2}t)]' + e^t x(\frac{1}{2}t)[1 + x^2(\frac{1}{2}t)] = 0, \quad t \geq 1$$

Clearly, $f(x) = x + x^3$, $f'(x) = 1 + 3x^2 \geq 1 = \mu$, $0 < \psi(x) = e^{-[x(t)]^2 - ln8} \leq 1/8$, $r(t) = q(t) = e^t$, $\tau(t) = \frac{1}{2}t$.

Let $u(t) = cost,\ \rho(t) = e^{-t}$, choose k sufficiently large, set $a = 2k\pi + \frac{\pi}{2},\ b = 2k\pi + \frac{3\pi}{2}$, it is easy to verify that

$$\int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} u^2(t)\rho(t)q(t)dt = \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} cos^2(t)dt = \frac{\pi}{2}$$

But

$$\int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} \left[\frac{\rho(t)r(t)}{k_1\tau'(t)}u'(t) + \frac{1}{2}u(t)\frac{\rho'(t)}{\rho(t)}\right]^2 dt$$

$$= \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} \left[\frac{1}{4}(-sint) + \frac{1}{2}cost\cdot(-\frac{1}{2})\right]^2 dt = \frac{\pi}{16} < \frac{\pi}{2}$$

Then equation is oscillatory by Theorem 2.2.

Example 2 Consider the nonlinear differential equation

$$[e^{-\frac{1}{2}t}\cdot\frac{1}{2}\cdot x'(2t)]'+$$

$$e^{-\frac{1}{2}t}[1+x^4(2t)][x(2t)+x^2(2t)sgnx(2t)|sinx(2t)|] = 0,\ \ t\geq 1$$

Clearly, $f(x) = (1+x^4)(x+x^2sgnx\cdot|sinx|)$, it is hard to verify that $f'(x)\geq\mu$ which μ is a non-negative constant, but we can see that $f(x)/x\geq 1 = k_2$, so we consider the Theorems in Section 3.

$$\psi(x(t)) = 1 = m,\ r(t) = q(t) = e^{-\frac{1}{2}t},\ \tau(t) = 2t > t$$

Let $u(t) = cost,\ \rho(t) = e^{\frac{1}{2}t}$, choose k sufficiently large, set $a = 2k\pi + \frac{\pi}{2},\ b = 2k\pi + \frac{3\pi}{2}$, we obtain

$$\int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} u^2(t)k_2(t)\rho(t)q(t)dt = \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} cos^2(t)dt = \frac{\pi}{2}$$

But

$$\int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} \left[\frac{m\rho(t)r(t)}{\tau'(t)}u'(t) + \frac{1}{2}u(t)\frac{\rho'(t)}{\rho(t)}\right]^2 dt$$

$$= \int_{2k\pi+\frac{\pi}{2}}^{2k\pi+\frac{3\pi}{2}} (-\frac{1}{2}sint + \frac{1}{4}cost)^2 dt = \frac{5\pi}{32} < \frac{\pi}{2}$$

Then equation is oscillatory by Theorem 3.2.

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