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A New Model Updating Method for Quadratic Eigenvalue Problems

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Abstract—**In this paper, we consider two finite element model updating problem which incorporate the measured modal data into the analytical finite element model, producing an adjusted model on the damping and stiffness, that closely match the experimental modal data. We develop an efficient numerical algorithm for solving this problem. The new algorithm is direct methods which require $O(nk^2)$ flops. Here n is the dimension of the coefficient matrices defining the analytical model and k is the number of measured eigenpairs.**

Keywords: quadratic eigenvalue problem, Model updating

1 Introduction

Vibrating systems, such as automotives, bridges, highways and buildings are usually described by distributed parameters. However, due to the lack of viable computational methods to handle distributed parameter systems, a finite element method is generally used to discretize such systems to an analytical finite element model (see [1, Chap 2] for details), namely, a second-order differential equation

$$M_a\ddot{\mathbf{q}}(t) + C_a\dot{\mathbf{q}}(t) + K_a\mathbf{q}(t) = \mathbf{f}(t). \qquad (1.1)$$

Here M_a, C_a and $K_a \in \mathbb{R}^{n\times n}$ are all symmetric and represent the analytical mass, damping and stiffness matrices, respectively (with M_a being symmetric positive definite, or $M_a > 0$), $\mathbf{q}(t)$ is the $n \times 1$ vector of positions and $\mathbf{f}(t)$ is the $n \times 1$ vector of external force. It is known that solving the homogeneous equation (1.1) (i.e. $\mathbf{f}(t) \equiv \mathbf{0}$) corresponds to solving the quadratic eigenvalue problem (QEP)

$$Q_a(\lambda)\mathbf{x} = (\lambda^2 M_a + \lambda C_a + K_a)\mathbf{x} = \mathbf{0} \qquad (1.2)$$

by letting $\mathbf{q}(t) = e^{\lambda t}\mathbf{x}$. The scalar λ and the associated vector $\mathbf{x}$ in (1.2) are called, respectively, eigenvalues and eigenvectors of the quadratic pencil $Q_a(\lambda)$. Note that the QEP (1.2) has $2n$ finite eigenvalues because the leading M_a is nonsingular.

In the finite element model (1.2) for structural dynamics, the analytical mass and stiffness matrices are, in general, clearly defined by physical parameters and evaluated by static tests. However, the analytical damping matrix for precise dissipative effects is not well understood, because it is a purely dynamics property that cannot be measured statically and must be determined by dynamic testing. This makes the process of modeling and experimental verification difficult. A common simplification is to assume proportional damping, which seems to be sufficient where damping levels are lower than 10% of critical [2]. Two new methods for damping matrix identification are developed by [3] which produce accurate representative damping matrices. They serve to integrate the theory and practical application of damping matrix identification. Therefore, it is assumed in this paper that acceptable models of the analytical mass, damping and stiffness matrices are available. It is our objective to incorporate the measured modal data into the finite element model, aiming to produce an adjusted finite element model on the mass, damping and stiffness with modal properties that closely match the experimental modal data.

Finite element model updating (FEMU) problems have emerged in the 1990's as a significant subject to the design, construction, and maintenance of mechanical systems. Model updating, at its most ambitious form, attempts to correct errors in a finite element model. It uses measured data such as natural frequencies, damping ratios, mode shapes and frequency response functions, which can usually be obtained by vibration test. In the past decade, a number of approaches to the FEMU problem are proposed (see [1, 4], and references therein). For example, Baruch [5], Baruch and Bar-Itzak [6], Bermann [7], Bermann and Nagy [8] and Wei [9, 10, 11] proposed various updating methods to correct the analytical mass and stiffness matrices of undamped systems (i.e. $C_a = 0$). In Datta [12], Datta et al. [13], and Datta and Sarkissian [14], studies are undertaken toward a nonsymmetric feedback design problem for second-order control system. That consideration eventually leads to a partial eigenstructure assignment problem for the QEP. A new symmetric feedback design for the QEP using symmetric eigenstructure assignment are recently developed in [15].

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The FEMU problem for damped systems was first proposed by Friswell, Inman and Pilkey [2]. They considered the mass matrix to be exact, and updated the damping and stiffness matrices by using the measured modal data as a reference. Following the basic idea of [5, 6], they minimized the difference between the analytical and updated damping/stiffness matrices, subject to the constraints that the eigenmatrix equation is satisfied and the damping/stiffness matrices are symmetric. That is, the FEMU problem proposed by [2] can be formulated by the following constrained optimization problem:

Problem FEMU. *Find $n \times n$ real matrices C and K to minimize the objective function*

$$J = \frac{1}{2}\nu \|M_a^{-\frac{1}{2}}(C - C_a)M_a^{-\frac{1}{2}}\|_F^2 + \frac{1}{2}\|M_a^{-\frac{1}{2}}(K - K_a)M_a^{-\frac{1}{2}}\|_F^2 \tag{1.3a}$$

subject to

$$M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0, \tag{1.3b}$$

$$C^\top = C, \qquad K^\top = K. \tag{1.3c}$$

Here M_a, C_a and K_a are respectively the analytical mass, damping and stiffness matrices, $\nu > 0$ is a weighting parameter, and C and K are respectively the updated damping and stiffness matrices. The measured eigenvalue matrix Λ and the associated eigenvector matrix Φ satisfy

$$\Lambda = \mathrm{diag}(\lambda_1^{[2]}, \ldots, \lambda_\ell^{[2]}, \lambda_{2\ell+1}, \ldots, \lambda_k) \in \mathbb{R}^{k\times k} \tag{1.4a}$$

with $k \ll n$ and

$$\lambda_j^{[2]} = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}, \quad \beta_j \neq 0,$$

for $j = 1, \ldots, \ell$, and

$$\Phi = [\boldsymbol{\varphi}_{1R}, \boldsymbol{\varphi}_{1I}, \ldots, \boldsymbol{\varphi}_{\ell R}, \boldsymbol{\varphi}_{\ell I}; \boldsymbol{\varphi}_{2\ell+1}, \ldots, \boldsymbol{\varphi}_k] \in \mathbb{R}^{n\times k}. \tag{1.4b}$$

Throughout this paper, we assume that Λ in (1.4a) has only simple eigenvalues and Φ in (1.4b) is of full column rank.

For Problem FEMU, Friswell, Inman and Pilkey [2, 3] proposed a updating method by using the Lagrange multiplier method to solve (1.3). The solutions C and K are given by

$$C = C_a - \frac{2}{\nu}M_a\mathrm{Re}(\Gamma_\Lambda\Lambda\Phi^\top + \Phi\Lambda\Gamma_\Lambda^\top)M_a \tag{1.5}$$

and

$$K = K_a - 2M_a\mathrm{Re}(\Gamma_\Lambda\Phi^\top + \Phi\Gamma_\Lambda^\top)M_a, \tag{1.6}$$

where $\Gamma_\Lambda \in \mathbb{C}^{n\times k}$ solves linear equation

$$2M_a\mathrm{Re}(\Gamma_\Lambda\Phi^\top + \Phi\Gamma_\Lambda^\top)M_a\Phi + \frac{2}{\nu}M_a\mathrm{Re}(\Gamma_\Lambda\Lambda\Phi^\top + \Phi\Lambda\Gamma_\Lambda^\top)M_a\Phi\Lambda = M_a\Phi\Lambda^2 + C_a\Phi\Lambda + K_a\Phi. \tag{1.7}$$

There are two weaknesses for the method. Firstly, the solution Γ_Λ in (1.7) is, in general, complex while the updated matrices C and K are expected to be real symmetric. Secondly, the dimension n of coefficient matrices in the finite element model (1.2) is usually quite large. It is impractical to solve the large and dense linear system (1.7), which requires $O(n^3k^3)$ flops.

In Section 3, we develop an efficient algorithm for solving Problem FEMU in (1.3). The new algorithm is a direct method, which avoids the Lagrange multiplier method in [2, 3], requiring only $O(n^2k)$ flops. In practice, M_a, C_a and K_a are usually sparse with $O(n)$ nonzero entries, and the computational cost is then reduced to $O(nk^2)$ flops.

2 Solving a PD-IQEP.

For a given matrix pair $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ $(k \leq n)$, where Λ and Φ are defined by (1.4a) and (1.4b), respectively, we now consider the partially described inverse quadratic eigenvalue problem (PD-IQEP):

Find a general form of symmetric matrices M, C and K, with M being positive definite, that satisfy the equation

$$M\Phi\Lambda^2 + C\Phi\Lambda + K\Phi = 0, \tag{2.1a}$$

$$M^\top = M > 0, \quad C^\top = C, \quad K^\top = K. \tag{2.1b}$$

A general solution to the PD-IQEP is given in [15] as follows:

Theorem 2.1. *Let Φ have the QR-factorization*

$$\Phi = Q\begin{bmatrix} R \\ 0 \end{bmatrix} \equiv [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}, \tag{2.2}$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal with $Q_1 \in \mathbb{R}^{n\times k}$ and $R \in \mathbb{R}^{k\times k}$ is nonsingular, and let $S = R\Lambda R^{-1}$. Then the general solution to the PD-IQEP defined by (2.1a) *and* (2.1b) *is given by*

$$M = Q\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}Q^\top, \ C = Q\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}Q^\top,$$

$$K = Q\begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}Q^\top. \tag{2.3}$$

Here the $n \times n$ symmetric positive definite matrix $\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, the $(n-k) \times (n-k)$ symmetric submatrices C_{22} and K_{22}, and the $(n-k) \times k$ submatrix

$C_{21} = C_{12}^\top$ *can be arbitrarily chosen. The symmetric submatrices* C_{11} *and* K_{11} *and the submatrices* K_{21} *and* K_{12} *satisfy*

$$C_{11} = -(M_{11}S + S^\top M_{11} + R^{-\top} D R^{-1}), \tag{2.4a}$$
$$K_{11} = S^\top M_{11} S + R^{-\top} D \Lambda_m R^{-1}, \tag{2.4b}$$
$$K_{21} = K_{12}^\top = -(M_{21}S^2 + C_{21}S), \tag{2.4c}$$

with

$$D = \operatorname{diag}\left(\begin{bmatrix} \xi_1 & \eta_1 \\ \eta_1 & -\xi_1 \end{bmatrix}, \ldots, \begin{bmatrix} \xi_\ell & \eta_\ell \\ \eta_\ell & -\xi_\ell \end{bmatrix}, \ \xi_{2\ell+1}, \ldots, \xi_k \right) \tag{2.5}$$

and ξ_i *and* η_i *being arbitrary real numbers.*

In the rest of this paper we will utilize this result to develop an efficient algorithm for solving problems FEMU described in Section 1.

3 Solving Problem FEMU

To solve Problem FEMU, we first solve two optimization problems. Let D and R be given in (2.5) and (2.2), respectively. We denote

$$R^{-1} = [\mathbf{r}_1, \ldots, \mathbf{r}_k] = \begin{bmatrix} r_{11} & \cdots & r_{1k} \\ & \ddots & \vdots \\ 0 & & r_{kk} \end{bmatrix}. \tag{3.1}$$

Problem I. *Given* $A = [\mathbf{a}_1, \ldots, \mathbf{a}_k]$, $B = [\mathbf{b}_1, \ldots, \mathbf{b}_k] \in \mathbb{R}^{k\times k}$ *and* $\nu > 0$*, let*

$$\mathbf{x} = (\xi_1,\ \eta_1, \ldots,\ \xi_\ell,\ \eta_\ell,\ \xi_{2\ell+1}, \ldots,\ \xi_k)^\top \tag{3.2}$$

be constructed from the matrix D *in* (2.5)*. Find* $\mathbf{x}^*$ *to minimize*

$$f(\mathbf{x}) = \nu\|A + R^{-\top} D R^{-1}\|_F^2 + \|B - R^{-\top}\Lambda^\top D R^{-1}\|_F^2 = \sum_{j=1}^k f_j(\mathbf{x}), \tag{3.3a}$$

where

$$f_j(\mathbf{x}) = \nu\|\mathbf{a}_j + R^{-\top} D\mathbf{r}_j\|_2^2 + \|\mathbf{b}_j - R^{-\top}\Lambda^\top D\mathbf{r}_j\|_2^2. \tag{3.3b}$$

Solution: Note that

$$\begin{bmatrix} \xi & \eta \\ \eta & -\xi \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u & v \\ -v & u \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix}.$$

The vector $D\mathbf{r}_j$ in (3.3b) can be rewritten as

$$D\mathbf{r}_j = \Gamma_j \mathbf{x}, \qquad j = 1, \ldots, k, \tag{3.4}$$

where

$$\Gamma_j = \operatorname{diag}\left(\begin{bmatrix} r_{1j} & r_{2j} \\ -r_{2j} & r_{1j} \end{bmatrix}, \ldots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \ldots, r_{k,j} \right) \in \mathbb{R}^{k\times k}. \tag{3.5}$$

Substituting (3.4) into (3.3b), and then differentiating $f_j(\mathbf{x})$, we have

$$\begin{aligned} \nabla f_j(\mathbf{x}) &= \left(\frac{\partial f_j}{\partial x_1}, \ldots, \frac{\partial f_j}{\partial x_k} \right)^\top \\ &= 2\nu \left(R^{-\top}\Gamma_j\right)^\top \left(\mathbf{a}_j + R^{-\top}\Gamma_j \mathbf{x}\right) \\ &\quad - 2\left(R^{-\top}\Lambda^\top\Gamma_j\right)^\top \left(\mathbf{b}_j - R^{-\top}\Lambda^\top\Gamma_j\mathbf{x}\right). \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} \nabla f(\mathbf{x}) &= \sum_{j=1}^k \nabla f_j(\mathbf{x}) \\ &= 2\sum_{j=1}^k \Big[\nu\left(R^{-\top}\Gamma_j\right)^\top \mathbf{a}_j + \nu\Gamma_j^\top \left(R^\top R\right)^{-1}\Gamma_j\mathbf{x} \\ &\quad - \Gamma_j^\top \Lambda R^{-1}\mathbf{b}_j + \Gamma_j^\top \Lambda \left(R^\top R\right)^{-1}\Lambda^\top\Gamma_j\mathbf{x} \Big]. \end{aligned} \tag{3.6}$$

Setting $\nabla f(\mathbf{x}) = \mathbf{0}$, we derive the linear equation for $\mathbf{x}$:

$$G\mathbf{x} = \mathbf{b}, \tag{3.7}$$

where

$$G = \sum_{j=1}^k \left[\nu\Gamma_j^\top \left(R^\top R\right)^{-1}\Gamma_j + \Gamma_j^\top\Lambda\left(R^\top R\right)^{-1}\Lambda^\top\Gamma_j \right] \tag{3.8a}$$

and

$$\mathbf{b} = \sum_{j=1}^k \left(\Gamma_j^\top \Lambda R^{-1}\mathbf{b}_j - \nu\Gamma_j^\top R^{-1}\mathbf{a}_j \right). \tag{3.8b}$$

Since the function $f(\mathbf{x})$ in (3.3a) must have an optimum, the linear system of (3.7) is consistent, and therefore, $\mathbf{x} = \mathbf{x}^*$ is solvable.

Problem II. *Given* $E,\ F \in \mathbb{R}^{(n-k)\times k}$, $\nu > 0$ *and* $S = R\Lambda R^{-1}$ *as in Theorem 2.1, minimize*

$$g(X) = \nu\|E - X\|_F^2 + \|F + XS\|_F^2 \tag{3.9}$$

for $X = [x_{ij}] \in \mathbb{R}^{(n-k)\times k}$*.*

Solution: Differentiating (3.9) yields

$$\begin{aligned} \frac{\partial g}{\partial x_{ij}} &= -2\nu\operatorname{tr}\left[(E - X)^\top \mathbf{e}_i\mathbf{e}_j^\top\right] + 2\operatorname{tr}\left[(F + XS)^\top\mathbf{e}_i\mathbf{e}_j^\top S\right], \\ &= -2\nu\mathbf{e}_i^\top(E - X)\mathbf{e}_j + \mathbf{e}_i^\top(F + XS)S^\top\mathbf{e}_j, \end{aligned}$$

and so we have

$$\nabla g(X) = 2\left[-\nu E + \nu X + FS^\top + XSS^\top\right]. \quad (3.10)$$

By solving $\nabla g(X) = 0$, we get

$$X = (\nu E - FS^\top)(\nu I + SS^\top)^{-1}. \quad (3.11)$$

We now return to Problem FEMU. Let

$$C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, \quad K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}, \quad (3.12a)$$

$$C := M_a^{-\frac{1}{2}} C M_a^{-\frac{1}{2}}, \quad K := M_a^{-\frac{1}{2}} K M_a^{-\frac{1}{2}}, \quad (3.12b)$$

$$\Phi := M_a^{\frac{1}{2}} \Phi, \quad M := M_a^{-\frac{1}{2}} M_a M_a^{-\frac{1}{2}} = I. \quad (3.12c)$$

Then it follows from (2.3)–(2.5) and $Q = [Q_1, Q_2]$ that Problem FEMU becomes

$$\begin{aligned}\min\Bigg\{&\frac{1}{2}\nu \left\| Q^\top C_a Q - \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \right\|_F^2 \\ &+ \frac{1}{2} \left\| Q^\top K_a Q - \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} \right\|_F^2 \Bigg\} \\ &= \frac{1}{2}\left[f(D) + 2g(C_{21}) + h(C_{22}, K_{22})\right], \quad (3.13)\end{aligned}$$

where

$$\begin{aligned}f(D) &= \nu\|A + R^{-\top} D R^{-1}\|_F^2 + \|B - R^{-\top}\Lambda^\top D R^{-1}\|_F^2, \\ g(C_{21}) &= \nu\|E - C_{21}\|_F^2 + \|F + C_{21} S\|_F^2, \\ h(C_{22}, K_{22}) &= \nu\|C_{22} - Q_2^\top C_a Q_2\|_F^2 + \|K_{22} - Q_2^\top K_a Q_2\|_F^2,\end{aligned}$$

with

$$A = Q_1^\top C_a Q_1 + S + S^\top, \ B = Q_1^\top K_a Q_1 - S^\top S, \quad (3.14a)$$

$$E = Q_2^\top C_a Q_1, \quad F = Q_2^\top K_a Q_1. \quad (3.14b)$$

Clearly, (3.13) achieves its minimal value if and only if

$$\min f(D), \quad \min g(C_{21}), \quad \min h(C_{22}, K_{22})$$

are achieved. Obviously, $h(C_{22}, K_{22})$ is minimized if and only if

$$C_{22} = Q_2^\top C_a Q_2, \qquad K_{22} = Q_2^\top K_a Q_2. \quad (3.15)$$

The optimization problems $\min f(D)$ and $\min g(C_{21})$ can be solved via Problems I and II, with the matrices A, B, E and F defined by (3.14).

In summary, we have the following algorithm.

Algorithm I. For a given $\nu > 0$, an analytical quadratic pencil $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$ and a matrix pair $(\Lambda, \Phi) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ as defined in (1.4), we seek the symmetric solutions C and K to Problem FEMU.

1) Set $C_a := M_a^{-\frac{1}{2}} C_a M_a^{-\frac{1}{2}}, K_a := M_a^{-\frac{1}{2}} K_a M_a^{-\frac{1}{2}}$, $\Phi := M_a^{\frac{1}{2}}\Phi$;

2) Compute the QR-factorization of Φ : $\Phi = [Q_1, Q_2]\begin{bmatrix} R \\ 0 \end{bmatrix}$ and $S = R\Lambda R^{-1}$;

3) Compute $C_{22} = Q_2^\top C_a Q_2$ and $K_{22} = Q_2^\top K_a Q_2$;

4) Solve $G\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = (\xi_1, \eta_1, \cdots, \xi_\ell, \eta_\ell, \xi_{2\ell+1}, \cdots, \xi_k)^\top$, where

$$G = \sum_{j=1}^k \Gamma_j^\top \left[\nu\left(R^\top R\right)^{-1} + \Lambda\left(R^\top R\right)^{-1}\Lambda^\top\right]\Gamma_j,$$

$$\mathbf{b} = \sum_{j=1}^k \Gamma_j^\top\left(\Lambda R^{-1}\mathbf{v}_j - \nu R^{-1}\mathbf{u}_j\right),$$

$$\Gamma_j = \mathrm{diag}\Bigg(\begin{bmatrix} r_{1,j} & r_{2,j} \\ -r_{2,j} & r_{1,j} \end{bmatrix}, \cdots, \begin{bmatrix} r_{2\ell-1,j} & r_{2\ell,j} \\ -r_{2\ell,j} & r_{2\ell-1,j} \end{bmatrix}, r_{2\ell+1,j}, \cdots, r_{k,j}\Bigg),$$

$$[\mathbf{u}_1, \cdots, \mathbf{u}_k] = Q_1^\top C_a Q_1 + S + S^\top,$$

$$[\mathbf{v}_1, \cdots, \mathbf{v}_k] = Q_1^\top K_a Q_1 - S^\top S,$$

$$(r_{1,j}, \cdots, r_{k,j})^\top = R^{-1}\mathbf{e}_j;$$

5) Form D as in (2.5) and compute

$$\begin{aligned}C_{11} &= -\left(S + S^\top + R^{-\top} D R^{-1}\right), \\ K_{11} &= S^\top S + R^{-\top} D \Lambda R^{-1}, \\ C_{21} &= Q_2^\top\left(\nu C_a Q_1 - K_a Q_1 S^\top\right)(\nu I + SS^\top)^{-1}, \\ K_{21} &= -C_{21} S;\end{aligned}$$

6) Compute

$$C = M_a^{\frac{1}{2}} Q \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}},$$

$$K = M_a^{\frac{1}{2}} Q \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix} Q^\top M_a^{\frac{1}{2}},$$

where $Q = [Q_1, Q_2]$.

Note that the linear system in step 4 is solvable, because the cost function has global minimizer.

Remark 3.1. (i) In a finite element model, the analytical matrices M_a, C_a and K_a are usually very large and sparse. Matrix M_a is, in general, diagonal or banded and therefore easily invertible. In practice, the number of measured eigenpairs is much less than the dimension of the finite element model, i.e., $k \ll n$. The orthogonal matrix $Q = [Q_1, Q_2]$ in step 2 of Algorithm I can be computed and stored in the form of a diagonal matrix plus a low rank updating by Householder transformations. Suppose the multiplication of the sparse matrix C_a or K_a to a vector needs $O(n)$ flops. Then, the computational cost of Algorithm I is $O(nk^2)$ flops. Obviously, if the analytical matrices are all dense, then the computational cost of Algorithm I will increased to $O(n^2k)$ flops.

(ii) Using Algorithm I to solve Problem FEMU in (1.3) is different from using (1.5)–(1.7). The latter needs to solve a large (and possibly dense) $nk \times nk$ linear system in (1.7), which is impractical when n is very large.

4 Numerical results

A set of pseudo simulation data was provided by the Boeing Company for testing. After a model reduction technique, we get three symmetric analytical matrices M_a, C_a and K_a with dimension 42 and M_a being positive definite. The 2-norms of M_a, C_a and K_a are 3.9057×10^8, 1.2250×10^8 and 2.0326×10^8, respectively.

Test 1. Since $M_a > 0$, the quadratic pencil $Q_a(\lambda) = \lambda^2 M_a + \lambda C_a + K_a$ has 84 finite eigenvalues. We first compute all 84 eigenpairs of $Q_a(\lambda)$ by solving a generalized eigenvalue problem of a linearization of $Q_a(\lambda)$. Then the measured eigenpairs $(\Lambda_a, \Phi_a) \in \mathbb{R}^{14\times 14} \times \mathbb{R}^{42\times 14}$ are chosen from those 84 computed eigenpairs of $Q_a(\lambda)$ so that eigenvalues of Λ_a are nearest to the original. Actually, the relative residual is estimated by

$$\frac{\|M_a\Phi_a\Lambda_a^2 + C_a\Phi_a\Lambda_a + K_a\Phi_a\|_F}{\|M_a\Phi_a\Lambda_a^2\|_F + \|C_a\Phi_a\Lambda_a\|_F + \|K_a\Phi_a\|_F} = 4.0671 \times 10^{-10}.$$

Intuitively, the optimal solutions C and K for Problem FEMU should be very close to C_a and K_a, respectively. We use Algorithm I to solve Problem FEMU with $\nu = 1$, the relative errors of the updated matrices are estimated by

$$\frac{\|C - C_a\|_{F_a}}{\kappa_1} \simeq 10^{-10}, \qquad \frac{\|K - K_a\|_{F_a}}{\kappa_1} \simeq 10^{-10},$$

where $\|\cdot\|_{F_a} = \|M_a^{-\frac{1}{2}}(\cdot)M_a^{-\frac{1}{2}}\|_F$ and $\kappa_1 = \max\{\|C_a\|_{F_a}, \|K_a\|_{F_a}\}$. The relative residual of (Λ_a, Φ_a) is estimated by

$$\frac{\|M_a\Phi_a\Lambda_a^2 + C\Phi_a\Lambda_a + K\Phi_a\|_F}{\|M_a\Phi_a\Lambda_a^2\|_F + \|C\Phi_a\Lambda_a\|_F + \|K\Phi_a\|_F} = 5.4135 \times 10^{-14}.$$

Test 2. Consider the given measured eigenvalues

$$\begin{aligned}\{\lambda_{mj}\}_{j=1}^{14} \\ = \{&-0.60939 \pm 37.365\iota, -0.73496 \pm 36.707\iota, \\ &-2.8832 \pm 31.992\iota, -1.8907 \pm 61.437\iota, \\ &-1.9112 \pm 54.181\iota, -2.2785 \pm 39.639\iota, \\ &-5.0387, -4.3416\}. \end{aligned} \quad (4.1)$$

The eigenpairs of the experimental model are used to create the experimental modal data. It is assumed that only the fundamental mode characteristics are experimentally determined and only s ($s \le 42$) components of eigenvector are measured. Suppose now we are given the measured mode shapes $\mathbf{v}_j \in \mathbb{R}^s$, $j = 1, \ldots, 14$. The measured eigenvectors $\boldsymbol{\varphi}_j$ is estimated by

$$\boldsymbol{\varphi}_j = D_j \widetilde{D}_j^{\dagger} \mathbf{v}_j, \quad j = 1, \ldots 14, \quad (4.2)$$

where D_j is defined by $D_j = [\lambda_{mj}^2 M_a + \lambda_{mj} C_a + K_a]^{-1} B_a$ with the control influence matrix $B_a \in \mathbb{R}^{n\times t}$ ($t \le s$). The matrix $\widetilde{D}_j$ consists of the first s rows of D_j, and the superscript " † " denotes the pseudo inverse. We first construct the eigenmatrix pair (Λ, Φ) associated with (4.1) and (4.2) as in (1.4). Then we use Algorithm I to compute the updated matrices C and K with $\nu = 0.1$, 1.0 and 10, respectively. The numerical results are shown in Table 4.1.

Table 4.1 relative residuals and optimal values

ν	0.1	1.0	10
r_1	1.4725×10^{-14}	1.4826×10^{-14}	1.4859×10^{-14}

Here, the relative residual is defined by

$$r_1 = \frac{\|M_a\Phi\Lambda^2 + C\Phi\Lambda + K\Phi\|_F}{\|M_a\Phi\Lambda^2\|_F + \|C\Phi\Lambda\|_F + \|K\Phi\|_F}.$$

From the accurate relative residuals in Tables 4.1, we see that the new proposed methods have high efficiency and reliability.

5 Conclusions

In this paper, we have developed two efficient numerical algorithms for finite element model updating problems. The new algorithm compute symmetric updated damping and stiffness, that closely match the experimental modal data. The new algorithm is direct method which are highly efficient and reliable, according to our numerical experiments. The algorithm produce encouraging results and interesting insight in a simple pseudo test suit provided by the Boeing Company.

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