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A Complex Variable Boundary Element Method for the Flow around Obstacles

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Abstract— **The paper presents an application of the Complex Variable Boundary Element Method (CVBEM) to solve a boundary value problem over two-dimensional multiply connected regions, in fact for the problem of a potential fluid flow around objects. The CVBEM is a powerful numerical tool for solving generally two-dimensional boundary value problems in which appear complex functions, and it represents a numerical application of Cauchy Integral Theorem.**

For solving the boundary integral the problem is reduced at there can be used different kinds of boundary elements. In this paper there are used linear boundary elements, so the geometries involved are approximated by polygonal lines and for the approximation of the unknowns there are used linear basis functions. The CVBEM's advantage over other techniques, pointed out by the present paper, is the fact that when this method is applied the approximation exactly solves the equation, so using this method good approximations can be found. A computer code based on this method is developed and numerical results are obtained for some particular cases.

Index Terms—**complex boundary element method, fluid flow, linear boundary element, multiply connected domain**

I. Introduction

By use of the Cauchy integral equation for complex variable analytic functions it is obtained an advanced mathematical approach for solving two-dimensional potential problems as those that arise when we study a fluid flow around one or more objects. The theoretical bases of this method where put around 1983 by Hromadka and its collaborators [1], [2].

The advantage of this method over the other methods that can be used to solve the same problems comes from the fact that the numerical application in this case is analytic and so the approximation exactly solves the equation, while the other numerical techniques develop only inexact approximations for the equation.

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The paper presents an application of the complex variable boundary element method (CVBEM) for solving a boundary value problem over two-dimensional triply connected regions, in fact for the problem of a potential fluid flow around two objects.

The application of CVBEM for solving problems over two-dimensional multiply connected regions has a great practical importance in computational fluid dynamics because, for example, there can be developed streamlines within a river with flows past bridge piers, and so it can be used to design the bridge pier alignment so as to minimize the disturbance. This method can be also successfully applied in other kind of problems of continuum mechanics as heat conduction [3], cracks, etc

II. The CVBEM Numerical Statement

Let us consider a uniform steady potential bi-dimensional river flow of an inviscid fluid past some arbitrary obstacles, first we consider only two, of boundary Γ_1, Γ_2. We want to determine the perturbation induced by the presence of the obstacles and the action exerted by the fluid on them applying the CVBEM. Using dimensionless variables, we have:

$$\Delta\varphi(x, y) = 0 \text{ on } \Omega, \quad (1)$$

where $\varphi(x, y)$ is the perturbation potential, Ω is he fluid domain, a multiply connected domain enclosed by boundaries $\Gamma^*, \Gamma_1, \Gamma_2$ ($\Gamma = \Gamma^* \cup \Gamma_1 \cup \Gamma_2$),

and the boundary conditions: $grad\varphi \cdot \bar{n} = 0$ across the flow boundaries on Γ^* and on $\Gamma_1 \cup \Gamma_2$, where $\bar{n}(n_x, n_y)$ is the outward unit normal at the corresponded boundary, and by defining an arbitrarily chosen potential drop between the upstream and downstream boundaries, noted φ_1 and φ_2.

Using the complex variable $z = x + iy$, the perturbation potential $f(z) = \varphi(z) + i\psi(z)$, where $\psi(z)$ is the stream function, φ and ψ being related by the Cauchy-Riemann equations $\frac{\partial\varphi}{\partial x} = \frac{\partial\psi}{\partial y}$ $\frac{\partial\varphi}{\partial y} = -\frac{\partial\psi}{\partial x}$, real-valued functions that are harmonic functions for $z : \Delta\varphi = 0, \Delta\psi = 0$, we get a holomorphic function f

We consider an approximation of the problem boundary Γ as a polygonal line $\Gamma = \bigcup_{k=1}^{N} \Gamma_k$, where Γ_k is a straight line segment with nodal points at the end-points, noted z_k, z_{k+1}, situated on the real boundary.

We choose m nodal points $z_j,\ j=\overline{1,m+1}$, on the outer curve Γ^*, $z_{m+1}=z_1$, numbered in a counterclockwise direction, n nodes $z_j,\ j=\overline{m+2,m+n+2}$, on the inner curve Γ_1, $z_{m+n+2}=z_{m+2}$, located in a clockwise direction and p nodes $z_j,\ j=\overline{m+n+3,m+n+p+3}$, on the other inner curve Γ_2, $z_{m+n+3}=z_{m+n+p+3}$, located in a clockwise direction too.

So we have: $\Gamma^*=\bigcup_{k=1}^{m}\Gamma_k$, $\Gamma_1=\bigcup_{k=m+2}^{m+n+1}\Gamma_k$, $\Gamma_2=\bigcup_{k=m+n+3}^{m+n+p+2}\Gamma_k$.

The next step in using the CVBEM is to develop a continuous approximation of the unknown $f(z)$ on Γ by the global trial function $F(z)=\sum_{\substack{k=1\\k\neq m+1\\k\neq m+n+2}}^{m+n+p+2}N_k(z)F_k$, $z\in\Gamma$,

where $N_k(z)$ is a continuous function representing the influence of over elements that have z_k as nodal point, so over Γ_{k-1} and Γ_k.

The approximation we construct is $\tilde{f}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{F(\zeta)}{\zeta-z}d\zeta$, $z\in\Omega$, the integral been taken in the counterclockwise direction.

Because $F(z)$ is continuous on Γ, $\tilde{f}(z)$ is analytic in Ω as an extension of the following theorem given in [1] to multiply connected regions, and so its real and imaginary parts satisfy Lapalce equation over Ω.

Theorem 1.

Let Γ be a simple closed contour with finite length L and simply connected interior Ω. Let $h(\zeta)$ be a continuous function on Γ. Then $\tilde{w}(z)$ is analytic in Ω, where $\tilde{w}(z)$ is defined by the contour integral $\tilde{w}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{h(\zeta)}{\zeta-z}d\zeta$.

III. LINEAR BASIS FUNCTION

We get the following discretized form:

$$\tilde{f}(z)=\frac{1}{2\pi i}\int_{\bigcup_{\substack{k=1\\k\neq m+1\\k\neq m+n+2}}^{m+n+p+2}\Gamma_k}\frac{F(\zeta)}{\zeta-z}d\zeta$$

In this paper we consider on each boundary element a linear approximation for $F(z)$. After some calculus we get for the nodal point j the following linear basis function:

$$N_j(z)=\begin{cases}\dfrac{z-z_{j-1}}{z_j-z_{j-1}}, z\in\Gamma_{j-1}\\[2mm] \dfrac{z_{j+1}-z}{z_{j+1}-z_j}, z\in\Gamma_j\\[2mm] 0,\ z\notin\Gamma_{j-1}\bigcup\Gamma_j\end{cases}$$

We further get:

$$\int_{\Gamma_k}\frac{F(\zeta)}{\zeta-z_0}d\zeta=$$

$$=\frac{z_{j+1}F_j-z_jF_{j+1}}{z_{j+1}-z_j}\int_{\Gamma_k}\frac{d\zeta}{\zeta-z_0}+\frac{F_{f+1}-F_j}{z_{j+1}-z_j}\int_{\Gamma_j}\frac{\zeta d\zeta}{\zeta-z_0}.$$

The two integrals from the right side of the above relation can be analytically evaluated:

$$\int_{\Gamma_j}\frac{d\zeta}{\zeta-z_0}=ln(\zeta-z_0)\Big|_{z_j}^{z_{j+1}}=$$

$$=\ln\frac{z_{j+1}-z_0}{z_j-z_0}=\ln\left|\frac{z_{j+1}-z_0}{z_j-z_0}\right|+i\theta(j+1,j)=\ln_j(z_0)$$

$$\int_{\Gamma_k}\frac{\zeta d\zeta}{\zeta-z_0}=(z_{j+1}-z_j)+z_0\int_{\Gamma_j}\frac{d\zeta}{\zeta-z_0}=$$

$$=(z_{j+1}-z_j)+z_0ln(\zeta-z_0)\Big|_{z_j}^{z_{j+1}}=(z_{j+1}-z_j)+,$$

$$+z_0\ln\left|\frac{z_{j+1}-z_0}{z_j-z_0}\right|+i\theta(j+1,j)$$

where $\theta(j+1,j)$ is the central angle between straight line segment joining points z_j and z_{j+1} to central point $z_0\in\Omega$.

So we deduce:

$$\int_{\Gamma_k}\frac{F(\zeta)}{\zeta-z_0}d\zeta=F_{j+1}-F_j+F_{j+1}\frac{z_0-z_j}{z_{j+1}-z_j}l_j(z_0)-$$

$$-F_j\frac{(z_0-z_{j+1})}{z_{j+1}-z_j}l_j(z_0)$$

Finally we get:

$$2\pi i\tilde{f}(z_0)=\sum_{\substack{j=1\\j\neq m+1\\j\neq m+n+2}}^{m+n+p+2}(F_{j+1}-F_j)+$$

$$+\sum_{\substack{j=1\\j\neq m+1\\j\neq m+n+2}}^{m+n+p+2}\frac{[F_{j+1}(z_0-z_j)-F_j(z_0-z_{j+1})]}{z_{j+1}-z_j}l_j(z_0)$$

Because the first term cancels we deduce:

$$2\pi i\tilde{f}(z_0)=\sum_{\substack{j=1\\j\neq m+1\\j\neq m+n+2}}^{m+n+p+2}\frac{[F_{j+1}(z_0-z_j)-F_j(z_0-z_{j+1})]}{z_{j+1}-z_j}l_j(z_0)$$

and further

$$2\pi i\tilde{f}(z_0)=\sum_{\substack{j=1\\ j\neq m+1\\ j\neq m+n+2}}^{m+n+p+2} A_j(z_0)F_j,$$

with $A_j(z_0)=\frac{(z_0-z_{j-1})}{z_j-z_{j-1}}l_{j-1}(z_0)-\frac{(z_0-z_{j+1})}{z_{j+1}-z_j}l_j(z_0)$,

when $j\neq 1, m+2, m+n+3$.

For $j=1$,

$$A_1(z_0)=\frac{(z_0-z_m)}{z_1-z_m}l_m(z_0)-\frac{(z_0-z_2)}{z_2-z_1}l_1(z_0),$$

for $j=m+2$,

$$A_{m+2}(z_0)=\frac{(z_0-z_{m+n+1})}{z_{m+2}-z_{m+n+1}}l_{m+n+1}(z_0)-\\ -\frac{(z_0-z_{m+3})}{z_{m+3}-z_{m+2}}l_{m+2}(z_0)$$

for $j=m+n+3$,

$$A_{m+n+3}(z_0)=\frac{(z_0-z_{m+n+p+2})}{z_{m+n+3}-z_{m+n+p+2}}l_{m+n+p+2}(z_0)-\\ -\frac{(z_0-z_{m+n+p+2})}{z_{m+n+p+2}-z_{m+n+3}}l_{m+n+3}(z_0),$$

From the above relation we can write the complex function $\tilde{f}(z_0)$ in terms of nodal values of F, in fact in terms of F_j, so:

$$\tilde{f}(z_0)=\tilde{\varphi}\begin{pmatrix} z_0,\Phi_1,...,\Phi m,\Phi_{m+2},...,\Phi_{m+n+1},...,\\ \Phi_{m+n+p+2},\Psi_1,...,\Psi_m,\Psi_{m+2},...,\Psi_{m+n+1},...,\Psi_{m+n+p+2}\end{pmatrix}+\\ +i\tilde{\psi}\begin{pmatrix} z_0,\Phi_1,...,\Phi_m,\Phi_{m+2},...,\Phi_{m+n+1},...,\\ \Phi_{m+n+p+2},\Psi_1,...,\Psi_m,\Psi_{m+2},...,\Psi_{m+n+1},...,\Psi_{m+n+p+2}\end{pmatrix} \qquad (*)$$

where z_0 is in Ω.

As we can see the global function is continuous on Γ, and we also have: $F(z_j)=F_j=\Phi_j+i\Psi_j$, the nodal values for the approximation function. We also have the nodal value of the solution function for the complex potential, $f_j=\varphi_j+i\psi_j$, where $f_j=f(z_j)$, and φ_j,ψ_j are the values of the state and the stream functions.

For given values of $F_j=\Phi_j+i\Psi_j$ at each z_j the above relation gives $\tilde{f}$ an analytic function in Ω, and $\mathrm{Re}(\tilde{f})$ and $\mathrm{Im}(\tilde{f})$ both satisfy the Laplace equation in Ω. If $\tilde{f}(z)=f(z)$ on Γ, then $\tilde{f}(z)=f(z)$ in Ω, and so $\tilde{f}(z)$ is the solution to the original boundary value problem.

We need to evaluate, using a limit process the value of $\tilde{f}(z_0)$ for $z_0\in\Gamma$.

IV. The limit process and the expressions of the coefficients

Concerning the calculation of the coefficients, it is performed by imposing effectively $z_0\to z_i\in\Gamma$ in the previous expressions of A_j. Except the elements Γ_{i-1} and Γ_i which become singular, this implies a simple replacement of z_0 with z_i. With regard to the coefficients coming from the singular integral, we do as in [4], we shall use the evaluation of a principal value (in the Cauchy sense) of a singular integral of the type $\int_\Gamma \frac{f(\xi)}{(\xi-z)}d\xi$ and the equality $\lim_{z\to z_p}(z-z_p)\ln(z-z_p)=0$ (see [5]).

So we get:

$$A_{ji}=A_j(z_i)=\frac{(z_i-z_{j-1})}{z_j-z_{j-1}}l_{j-1}(z_i)-\frac{(z_i-z_{j+1})}{z_{j+1}-z_j}l_j(z_i)$$

$$\ln_j(z_i)=\ln\frac{z_{j+1}-z_i}{z_j-z_i},\ \ln_{j-1}(z_i)=\ln\frac{z_j-z_i}{z_{j-1}-z_i},$$

for $j\neq 1, m+2, m+n+3$, and $i\neq j-1, i\neq j, i\neq j+1$

$$A_{jj}=\lim_{z_0\to z_j}A_j(z_0)=\ln\frac{z_{j+1}-z_j}{z_{j-1}-z_j},$$

$$A_{jj-1}=\lim_{z_0\to z_{j-1}}A_j(z_0)=-\frac{z_{j-1}-z_{j+1}}{z_{j+1}-z_j}\ln\frac{z_{j+1}-z_{j-1}}{z_j-z_{j-1}}$$

$$A_{jj+1}=\lim_{z_0\to z_{j+1}}A_j(z_0)=\frac{z_{j+1}-z_{j-1}}{z_j-z_{j-1}}\ln\frac{z_j-z_{j+1}}{z_{j-1}-z_{j+1}}$$

Similarly we get the other coefficients:

$$A_{1i}=A_1(z_i)=\frac{(z_i-z_m)}{z_1-z_m}l_m(z_i)-\frac{(z_i-z_2)}{z_2-z_1}l_1(z_i),$$

for $i\neq m, i\neq 1, i\neq 2$

$$A_{11}=\lim_{z_0\to z_1}A_1(z_0)=\ln\frac{z_2-z_1}{z_m-z_1}$$

$$A_{1m}=-\frac{(z_m-z_2)}{z_2-z_1}\ln\frac{z_2-z_m}{z_1-z_m},$$

$$A_{12}=\frac{(z_2-z_m)}{z_1-z_2}\ln\frac{z_1-z_2}{z_m-z_2}$$

$$A_{m+2,i}=A_{m+2}(z_i)=\frac{(z_i-z_{m+n+1})}{z_{m+2}-z_{m+n+1}}l_{m+n+1}(z_i)-\\ -\frac{(z_i-z_{m+3})}{z_{m+3}-z_{m+2}}l_{m+2}(z_i),$$

for $i\neq m+n+1, i\neq m+2, i\neq m+3$

$$A_{m+2,m+2}=\ln\frac{z_{m+3}-z_{m+2}}{z_{m+n+1}-z_{m+2}}$$

$$A_{m+2,m+n+1} = -\frac{\left(z_{m+n+1}-z_{m+3}\right)}{z_{m+3}-z_{m+2}}\ln\frac{z_{m+3}-z_{m+n+1}}{z_{m+2}-z_{m+n+1}}$$

$$A_{m+2,m+3} = \frac{\left(z_{m+3}-z_{m+n+1}\right)}{z_{m+2}-z_{m+3}}\ln\frac{z_{m+2}-z_{m+3}}{z_{m+n+1}-z_{m+3}}$$

$$A_{m+n+3,i} = A_{m+n+3}(z_i) = \frac{\left(z_i - z_{m+n+p+2}\right)}{z_{m+n+3}-z_{m+n+p+2}} l_{m+n+p+2}(z_i) - $$
$$-\frac{\left(z_i - z_{m+n+p+2}\right)}{z_{m+n+p+2}-z_{m+n+3}} l_{m+n+3}(z_i)$$

for $i \neq m+n+p+2, i \neq m+n+3, i \neq m+n+4$,

$$A_{m+n+3,m+n+3} = \ln\frac{z_{m+n+4}-z_{m+n+3}}{z_{m+n+p+2}-z_{m+n+3}}$$

$$A_{m+n+3,m+n+p+2} =$$
$$= -\frac{\left(z_{m+n+p+2}-z_{m+n+4}\right)}{z_{m+n+4}-z_{m+n+3}}\ln\frac{z_{m+n+4}-z_{m+n+p+2}}{z_{m+n+3}-z_{m+n+p+2}}$$

$$A_{m+n+3,m+n+4} =$$
$$= -\frac{\left(z_{m+n+4}-z_{m+n+p+2}\right)}{z_{m+n+3}-z_{m+n+4}}\ln\frac{z_{m+n+3}-z_{m+n+4}}{z_{m+n+p+2}-z_{m+n+4}}.$$

All the coefficients are so evaluated and they depend only on the nodal points. We consider in the above relations $z_0 = z_i$, $i = \overline{1, m+n+p+2}, i \neq m+1, i \neq m+n+2$.

As *i* takes all these values we obtain a system of *m+n+p* relations, in terms of complex numbers of the following form:

$$2\pi i\tilde{f}(z_i) = \sum_{\substack{j=1\\ j\neq m+1\\ j\neq m+n+2}}^{m+n+p+2} A_j(z_i)F_j = \sum_{\substack{j=1\\ j\neq m+1\\ j\neq m+n+2}}^{m+n+p+2} A_{ji}F_j,$$

Using the complex expression of F_j and $\tilde{f}_j = \tilde{f}(z_j)$: $F_j = \Phi_j + i\Psi_j$ and $\tilde{f}_j = \tilde{\varphi}_j + i\tilde{\psi}_j$, we deduce:

$$2\pi i(\tilde{\varphi}_i + i\tilde{\psi}_i) = \sum_{\substack{j=1\\ j\neq m+1\\ j\neq m+n+2}}^{m+n+p+2} A_{ji}\left(\Phi_j + i\Psi_j\right) \quad (**)$$

If $\varphi(x,y)$ and $\psi(x,y)$ are known continuously on Γ, and $F_j = \Phi_j + i\Psi_j = \varphi_j + i\psi_j$ for all the nodes than $f(z) = \tilde{f}(z)$ on $\Omega \cup \Gamma$. Generally $\varphi(x,y)$ and $\psi(x,y)$ are known only on portions of Γ. If there are N nodes let suppose that there are N_1 nodes where we know $\varphi(x,y)$ and N_2 nodes where we know $\psi(x,y)$, $N = N_1 + N_2$. The next step is to impose in the above relations the boundary conditions: $\varphi_i = \tilde{\varphi}_i$ for all the nodes where the potential is known and $\psi_i = \tilde{\psi}_i$ for the nodes where the stream function is known. Doing so we generate implicit expressions of the unknown nodal values as functions of all the unknown variables, so m equations of m unknowns which can be solved using the computer. The computer is also used for getting the coefficients of the matrix involved. The evaluated nodal values enclosed to the original set of known nodal values completely define $\tilde{f}(z)$ on $\Omega \cup \Gamma$.

Taking into account that $A_{ji} = a_{ji} + ib_{ji}$ ($a_{ji} = \mathrm{Re}(A_{ji}), b_{ji} = \mathrm{Im}(A_{ji})$) and isolating the real and the imaginary parts in system (**) we obtain the following linear system of equations, in terms of real unknowns and coefficients:

$$\begin{cases} -2\pi\tilde{\psi}_i = a_{ji}\varphi_j - b_{ji}\psi_j \\ 2\pi\tilde{\varphi}_i = b_{ji}\varphi_j + a_{ji}\psi_j \end{cases}$$

$i, j = \overline{1, m+n+1}, i \neq m+1, j \neq m+1$

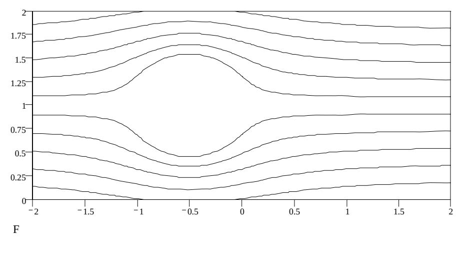
Imposing that: $\varphi_i = \tilde{\varphi}_i$ for the N_1 nodes where the potential is known and $\psi_i = \tilde{\psi}_i$ for the nodes where the stream function is known, and after solving the system we obtained the other unknown values for both functions.

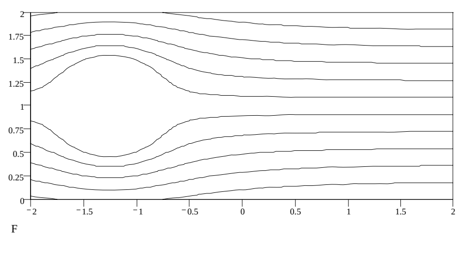
So all the nodal values are then known. By replacing them in relation (*) we get the analytic function in Ω, $\tilde{f}$, which satisfies relation $\tilde{f}(z) = f(z)$ on Γ and therefore the relation $\tilde{f}(z) = f(z)$ in Ω. So $\tilde{f}(z)$ is the solution to the original boundary value problem.

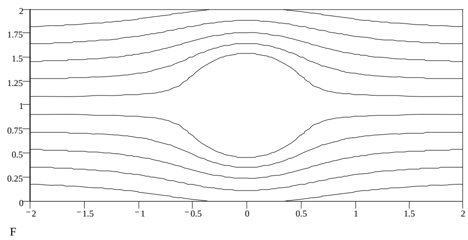
V. Numerical results

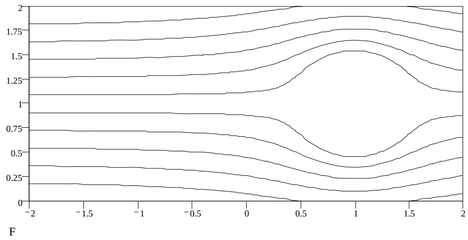
The problem of the evaluation of the system coefficients, and also that of finding its solution can be easily solved with a computer code made in MATHCAD.

Numerical results can be obtained for any shape for the two obstacles, but in order to make a checking and to validate the computer code we consider a particular case, the problem of a potential flow between two plane parallel walls around a circle, because it is a problem with a known solution. It has a great importance because it offers us the possibility to make a comparison between the exact solution and the numerical one. A computer code in MATHCAD is made in order to find the numerical solutions for different positions of the obstacle, and they are represented in the graphics below.

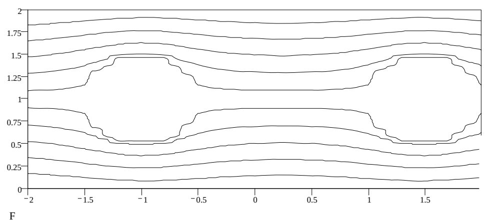


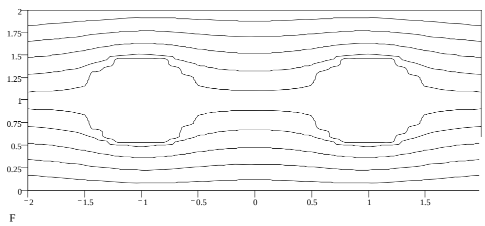


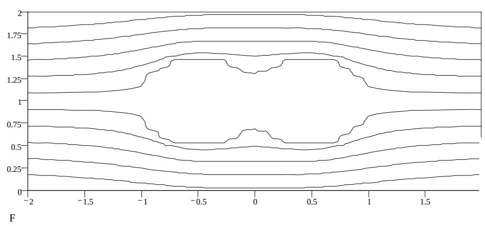




In the following figure there are represented the numerical results obtained for different position of two circular obstacles situated between the walls. Both have the same radius, and their centers are situated at the same distances from the walls, but different distances between their centers have been considered.







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