Proceedings of the World Congress on Engineering 2009 Vol II
WCE 2009, July 1 - 3, 2009, London, U.K.

AGEI Method For Diffusion Equations

Bin Zheng[1] *, Qinghua Feng[2] *

Abstract—**In this paper, we present a high order implicit scheme for one dimension heat conduction equations. The scheme is proved to be unconditionally stable. Based on the scheme a class of parallel alternating group explicit iterative method (AGEI) is constructed, and convergence analysis for the method is done. Numerical experiments show that the method is effective in computation.**

Keywords: heat conduction equation, parallel computation, iterative method, alternating group

1 Preface

In this paper, we will consider the following initial boundary value problem:

$$\begin{cases} \dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial x^2},\ 0\le x\le 1,\ 0\le t\le T \\ u(x,0)=f(x), \\ u(0,t)=g_1(t),\ u(1,t)=g_2(t). \end{cases} \tag{1.1}$$

In the last twenty years, researches on parallel numerical methods are getting more and more popular. D. J. Evans presented an AGE method for diffusion equations in [1] originally. The AGE method is soon developed to solve other partial differential equations by many authors such as in [2-5]. The developed methods are all intrinsically parallel, and can obtain absolute stability in computation. But we notice that almost all the methods have no more than two order accuracy for spatial step.

In the section 2 of this paper, we will present a symmetry three time level implicit scheme with accuracy of order four in spatial step and order two in time step. Then an AGEI method will be constructed based on the scheme. In section 3 and 4, convergence analysis and stability analysis for the AGEI method are given respectively. In section 5, results of numerical experiments for the AGEI method are presented.

2 The Construction Of AGEI Method

The domain $\Omega:(0,1)\times(0,T)$ will be divided into $(m\times N)$ meshes with spatial step size h=$\frac{1}{m}$ in x direction and the time step size $\tau=\frac{T}{N}$. Grid points are denoted by (x_i,t_n) or (i, n), $x_i=ih(i=0,\ 1,\cdots,m),\ t_n=n\tau(n=0,\ 1,\cdots,\frac{T}{\tau})$. The numerical solution of (1.1) is denoted by u_i^n, while the exact solution $u(x_i,\ t_n)$. If we approach (1.1) at $(i,\ n)$ with center-difference scheme:

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}=\frac{1}{4h^2}[(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})$$

$$+2(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})] \tag{2.1}$$

Applying Taylor's formula to the scheme at (x_i,t_n), then it follows $(\frac{\partial u}{\partial t})_i^n+\frac{\tau^2}{6}(\frac{\partial^3 u}{\partial t^3})_i^n=(\frac{\partial^2 u}{\partial x^2})_i^n+\frac{h^2}{12}(\frac{\partial^4 u}{\partial x^4})_i^n+O(\tau^2+h^4)$.

Considering $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^4 u}{\partial x^4}$, we have $(\frac{\partial u}{\partial t})_i^n+\frac{\tau^2}{6}(\frac{\partial^3 u}{\partial t^3})_i^n=(\frac{\partial^2 u}{\partial x^2})_i^n+\frac{h^2}{12}(\frac{\partial^2 u}{\partial t^2})_i^n+O(\tau^2+h^4)$. We approach $(\frac{\partial^2 u}{\partial t^2})_i^n$ with $\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{\tau^2}$. Combining with (2.1) we have the following scheme:

$$\frac{u_i^{n+1}-u_i^{n-1}}{2\tau}=\frac{1}{4h^2}[(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1})+2(u_{i+1}^{n}-2u_i^{n}+u_{i-1}^{n})$$

$$+(u_{i+1}^{n-1}-2u_i^{n-1}+u_{i-1}^{n-1})]-\frac{h^2}{12}(\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{\tau^2}) \tag{2.2}$$

The truncation error of (2.2) can easily be obtained as $O(\tau^2+h^4)$.

Let $r=\frac{\tau}{2h^2}$, then we rewrite (2.2) as below:

$$-ru_{i-1}^{n+1}+(1+\frac{1}{12r}+2r)u_i^{n+1}-ru_{i+1}^{n+1}=2ru_{i-1}^n$$

$$+(\frac{1}{6r}-4r)u_i^n+2ru_{i+1}^n+ru_{i-1}^{n-1}+(1-\frac{1}{12r}-2r)u_i^{n-1}+ru_{i+1}^{n-1}. \tag{2.3}$$

Let $U^n=(u_1^n,\ u_2^n,\ \cdots,\ u_{m-1}^n)^T,\ p=1+\frac{1}{12r},\ q=1-\frac{1}{12r}$, then from (2.1) we have $AU^{n+1}=F^n$. here $F^n=BU^n+CU^{n-1}+[2ru_0^n+ru_0^{n-1}+ru_0^{n+1},0,\cdots,0,2ru_m^n+ru_m^{n-1}+ru_m^{n+1}]^T$.

$$A=\begin{pmatrix} p+2r & -r & & & \\ -r & p+2r & -r & & \\ & \cdots & \cdots & \cdots & \\ & & -r & p+2r & -r \\ & & & -r & p+2r \end{pmatrix}$$

*School of Science, Shandong University of Technology, Zibo, 255049, People's Republic of China, E-mail[1]: zhengbin2601 @ 126.com, E-mail[2]: fqhua @ sina.com

$$B=\begin{pmatrix} \frac{1}{6r}-4r & 2r & & & \\ 2r & \frac{1}{6r}-4r & 2r & & \\ & \cdots & \cdots & \cdots & \\ & & 2r & \frac{1}{6r}-4r & 2r \\ & & & 2r & \frac{1}{6r}-4r \end{pmatrix}$$

$$C=\begin{pmatrix} q-2r & r & & & \\ r & q-2r & r & & \\ & \cdots & \cdots & \cdots & \\ & & r & q-2r & r \\ & & & r & q-2r \end{pmatrix}$$

A, B, C are all $(m-1)\times(m-1)$ matrixes.

The alternating group iterative method will be constructed in four cases as follows:
(1)$m=4k+1,\ k$ is an integer.
Let $A=\frac{1}{2}(G_1+G_2)$,
here $G_1=diag(G_{11},\cdots,G_{11})_{(m-1)\times(m-1)}$,
$G_2=diag(G_{21},G_{11},\cdots,G_{11},G_{21})_{(m-1)\times(m-1)}$.

$$G_{11}=\begin{pmatrix} p+2r & -r & 0 & 0 \\ -r & p+2r & -2r & 0 \\ 0 & -2r & p+2r & -r \\ 0 & 0 & -r & p+2r \end{pmatrix}$$

$$G_{21}=\begin{pmatrix} p+2r & -r \\ -r & p+2r \end{pmatrix}$$

Then the alternating group explicit iterative method (AGEI1) can be constructed as below:

$$\begin{cases} (\rho I+G_1)U^{n+1}_{k+\frac{1}{2}}=(\rho I-G_2)U^{n+1}_{k}+\widetilde{F}^n \\ (\rho I+G_2)U^{n+1}_{k+1}=(\rho I-G_1)U^{n+1}_{k+\frac{1}{2}}+\widetilde{F}^n \end{cases} \quad k=0,1,\cdots \tag{2.4}$$

Here $\widetilde{F}^n=2F^n$.
(2)$m=4k+3,\ k$ is an integer.
Let $A=\frac{1}{2}(\overline{G}_1+\overline{G}_2)$,
here $\overline{G}_1=diag(G_{11},\cdots,G_{11},G_{21})_{(m-1)\times(m-1)}$,
$\overline{G}_2=diag(G_{21},G_{11},\cdots,G_{11})_{(m-1)\times(m-1)}$
Then the AGEI2 method can be constructed as below:

$$\begin{cases} (\rho I+\overline{G}_1)U^{n+1}_{k+\frac{1}{2}}=(\rho I-\overline{G}_2)U^{n+1}_{k}+\widetilde{F}^n \\ (\rho I+\overline{G}_2)U^{n+1}_{k+1}=(\rho I-\overline{G}_1)U^{n+1}_{k+\frac{1}{2}}+\widetilde{F}^n \end{cases} \quad k=0,1,\cdots \tag{2.5}$$

(3)$m=2k+1,\ k$ is an integer.
Let $A=\frac{1}{2}(H_1+H_2)$,
here $H_1=diag(H_{11},\cdots,H_{11})_{(m-1)\times(m-1)}$,
$H_2=diag(H_{21},H_{11},\cdots,H_{11},H_{21})_{(m-1)\times(m-1)}$

$$H_{11}=\begin{pmatrix} p+2r & -2r \\ -2r & p+2r \end{pmatrix},\ H_{21}=p+2r$$

Then the AGEI3 method can be constructed as below:

$$\begin{cases} (\rho I+H_1)U^{n+1}_{k+\frac{1}{2}}=(\rho I-H_2)U^{n+1}_{k}+\widetilde{F}^n \\ (\rho I+H_2)U^{n+1}_{k+1}=(\rho I-H_1)U^{n+1}_{k+\frac{1}{2}}+\widetilde{F}^n \end{cases} \quad k=0,1,\cdots \tag{2.6}$$

(4)$m=2k,\ k$ is an integer.
Let $A=\frac{1}{2}(\overline{H}_1+\overline{H}_2)$,
here $\overline{H}_1=diag(H_{11},\cdots,H_{11},H_{21})_{(m-1)\times(m-1)}$,
$\overline{H}_2=diag(H_{21},H_{11},\cdots,H_{11})_{(m-1)\times(m-1)}$ Then the AGEI4 method can be constructed as below:

$$\begin{cases} (\rho I+\overline{H}_1)U^{n+1}_{k+\frac{1}{2}}=(\rho I-\overline{H}_2)U^{n+1}_{k}+\widetilde{F}^n \\ (\rho I+\overline{H}_2)U^{n+1}_{k+1}=(\rho I-\overline{H}_1)U^{n+1}_{k+\frac{1}{2}}+\widetilde{F}^n \end{cases} \quad k=0,1,\cdots \tag{2.7}$$

3 Convergence Analysis of AGEI Method

Lemma 1[6] Let $\theta>0$, and $G+G^T$ is nonnegative, then $(\theta I+G)^{-1}$exists, and

$$\|(\theta I+G)^{-1}\|_2\le\theta^{-1} \tag{3.1}$$

Lemma 2[6] On the conditions of Lemma 1, we have:

$$\|(\theta I-G)(\theta I+G)^{-1}\|_2\le 1 \tag{3.2}$$

Theorem 1 The alternating group explicit iterative method (2.4) is convergent.

Proof: From the construction of the matrixes we can see that $G_1,\ G_2,\ (G_1+G_1^T),\ (G_2+G_2^T)$ are all nonnegative matrixes. Then we have

$$\|(\rho I-G_1)(\rho I+G_1)^{-1}\|_2\le 1,\ \|(\rho I-G_2)(\rho I+G_2)^{-1}\|_2\le 1$$

From (2.2), we obtain $U^{n+1}_{k+1}=GU^{n+1}_{k}+(\rho I+G_2)^{-1}[(\rho I-G_1)(\rho I+G_1)^{-1}\widetilde{F}^n+\widetilde{F}^n]$. here $G=(\rho I+G_2)^{-1}(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)$ is growth matrix.

Let $\widetilde{G}=(\rho I+G_2)G(\rho I+G_2)^{-1}=(\rho I-G_1)(\rho I+G_1)^{-1}(\rho I-G_2)(\rho I+G_2)^{-1}$, then $\rho(G)=\rho(\widetilde{G})\le\|\widetilde{G}\|_2\le 1$, which shows the AGEI1 method given by (2.4) is convergent.

Analogously we have:

Theorem 2 The alternating group explicit iterative method (2.5)-(2.7) are also convergent.

4 Stability Analysis

Lemma 3[7] if b and c are real numbers, and $\lambda_1,\ \lambda_2$ are the roots of $\lambda^2-b\lambda-c=0$, then we have $|\lambda_i|<1,\ i=1,2$ if and only if $|b|<1-c<2$.

We use fourier method to analyze the stability of (2.3), we let $w_i^n = (u_i^n, u_i^{n-1})^T$, then from (2.3) we have

$$\begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix} w_{i-1}^{n+1} + \begin{pmatrix} p+2r & 0 \\ 0 & 1 \end{pmatrix} w_i^{n+1}$$

$$+ \begin{pmatrix} -r & 0 \\ 0 & 0 \end{pmatrix} w_{i+1}^{n+1} = \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix} w_{i-1}^{n}$$

$$+ \begin{pmatrix} p-q-4r & q-2r \\ 1 & 0 \end{pmatrix} w_i^n + \begin{pmatrix} 2r & r \\ 0 & 0 \end{pmatrix} w_{i+1}^n$$

Let $w_i^n = v^n e^{i\alpha x_i}$, then we have

$$\begin{pmatrix} p+2r-2rcos(\alpha h) & 0 \\ 0 & 1 \end{pmatrix} v^{n+1}$$

$$= \begin{pmatrix} p-q-4r+4rcos(\alpha h) & q-2r+2rcos(\alpha h) \\ 1 & 0 \end{pmatrix} v^n$$

Furthermore

$$v^{n+1} = \begin{pmatrix} \frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})} & \frac{1-\frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})} \\ 1 & 0 \end{pmatrix} v^n$$

$$= Tv^n$$

Let λ be the eigenvalue of T, then we have

$$\lambda^2 - \frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})}\lambda - \frac{1-\frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})} = 0$$

From lemma 3, the stability of (2.3) can be obtained under the condition $|\frac{\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})}| \leq 1 - \frac{1-\frac{1}{12r} - 4rsin^2(\frac{\alpha h}{2})}{p+4rsin^2(\frac{\alpha h}{2})} < 2$, that is, $|\frac{1}{6r} - 8rsin^2(\frac{\alpha h}{2})| \leq \frac{1}{6r} + 8rsin^2(\frac{\alpha h}{2}) < 2 + \frac{1}{6r} + 8rsin^2(\frac{\alpha h}{2})$, which is obviously true. So we can get the following theorem:

Theorem 3 The scheme (2.3) is unconditionally stable.

5 Numerical Experiments

We consider the following initial boundary value problem of diffusion equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \ 0 \leq x \leq 1, \ 0 \leq t \leq T \\ u(x,0) = sin(\pi x), \\ u(0,t) = 0, \ u(1,t) = 0. \end{cases} \quad (4.1)$$

The exact solution for the problem is $u(x,t) = e^{-\pi^2 t} sin(2\pi x)$. Let $||E_1||_\infty = max|u_i^n - u(x_i,t_n)|$, $||E_2||_\infty = max|(u_i^n - u(x_i,t_n))/u(x_i,t_n)|$, $i = 1, 2, \cdots, m-1$. We use the iterative error 1×10^{-10} to control the process of iterativeness, and the results of numerical experiments are listed in the following two tables:

Table 1: The numerical results at

$m = 16, \ \tau = 10^{-4}, \ \rho = 1$

	$t = 100\tau$	$t = 200\tau$
$\|\|E_1\|\|_\infty$	8.906×10^{-4}	8.066×10^{-4}
$\|\|E_2\|\|_\infty$	9.872×10^{-2}	9.869×10^{-2}
iterative times	27.7	27.85

Table 2: The numerical results at

$m = 16, \ \tau = 10^{-4}, \ \rho = 1$

	$t = 500\tau$	$t = 1000\tau$
$\|\|E_1\|\|_\infty$	5.994×10^{-4}	3.654×10^{-4}
$\|\|E_2\|\|_\infty$	9.860×10^{-2}	9.847×10^{-2}
iterative times	27.94	27.489

Table 3: The numerical results at

$m = 16, \ \tau = 10^{-5}, \ \rho = 1$

	$t = 1000\tau$	$t = 2000\tau$
$\|\|E_1\|\|_\infty$	1.260×10^{-4}	1.522×10^{-4}
$\|\|E_2\|\|_\infty$	1.396×10^{-2}	1.863×10^{-2}
iterative times	173.27	172.614

Table 4: The numerical results at

$m = 16, \ \tau = 10^{-5}, \ \rho = 1$

	$t = 5000\tau$	$t = 10000\tau$
$\|\|E_1\|\|_\infty$	2.173×10^{-4}	2.911×10^{-4}
$\|\|E_2\|\|_\infty$	3.576×10^{-2}	7.844×10^{-2}
iterative times	170.488	166.89

From Table 1 and Table 2 we can see that the numerical solution for the AGEI method can converge to the exact solution excellently, and the computation error won't accumulate when time steps increases, which accords with the conclusion of convergence and stability analysis. Furthermore, the AGEI method has the obvious property of parallelism.

6 Conclusions

In this paper, we present an alternating group explicit iterative(AGEI) method by using a special three time level implicit scheme with accuracy order $O(\tau^2 + h^4)$. Then the convergence analysis and stability analysis are done. The AGEI method is suitable for parallel computation in solving large equation set. Furthermore, and the construction of the AGEI method can also be applied to other partial differential equations.

References

[1] D. J. Evans , A. R. B. Abdullah , Group Explicit Method for Parabolic Equations [J]. Inter. J. Comput. Math. . 14 (1983) 73-105.

[2] Z. B. lin, S. X. min, Alternating Segment Crank-Nicolson Scheme [J], Compu. Phi. 1 (1995) 115-120.

[3] R. Tavakoli, P. Davami, New stable group explicit finitedifference method for solution of diffusion equation, Appl. Math. Comput. 181 (2006) 1379-1386.

[4] G. W. Yuan, L. J. Shen, Y. L. Zhou, Unconditional stability of parallel alternating difference schemes for semilinear parabolic systems, Appl. Math. Comput. 117 (2001) 267-283.

[5] C. N. Dawson, T. F. Dupont, Explicit/implicit conservative Galerkin domain decomposition procedures for parabolic problems, Math. Comp. 58 (197) (1992) 21-34.

[6] T. Z. fu, F. X. fang, A new explicit method for convection-diffusion equation, J. of engi. math. 17 (2000) 65-69.

[7] H. J. Gan, Iterative methods for linear algebra equation set, Science press(china), 1991.