Ruin Probability with Constant Interest Force and Negatively Dependent Insurance Risks

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Abstract—**In this paper, under the assumption that the claimsize is Negatively dependent subexponentially distributed and the constant interest force is considered, a simple asymptotics of ruin probability for renewal risk model within finite horizon is obtained. The results obtained extended the corresponding results of related papers.**

Keywords: Ruin probability with finite horizon; renewal risk model; Negatively dependent; subexponential class

1 The model

Consider a nonstandard renewal risk model, in which the individual claim size, X_n, $n \geq 1$, form a sequence of identically distributed, not necessarily independent, and non-negative random variables (r.v.s) with a common distribution (d.f.) $F(x) = 1 - \overline{F}(x) = P(X \leq x)$ for $x \in [0, \infty)$ and a finite mean $\mu = EX_1$. The inter-occurrence times $\theta_n, n = 1, 2, ...$, are another sequence of independent, identically distributed (i.i.d.) nonnegative random variables with mean $E\theta_1 = 1/\lambda$. The random variables $\sigma_k = \sum_{i=1}^{k} \theta_i$, $k = 1, 2, ...$ constitute a renewal counting process

$$N(t) = \sup\left\{n \geq 1 : \sum_{i=1}^{n} \theta_i \leq t\right\} \tag{1.1}$$

with mean $\lambda(t) = EN(t)$. In the case that θ_n has an exponential distribution, the renewal model is then called the compound Poisson model. Risk reserve process is defined by

$$U_\delta(t) = ue^{\delta t} + c\int_0^t e^{\delta(t-s)}ds - \sum_{i=1}^{N(t)} X_i e^{\delta(t-\sigma_i)}, \tag{1.2}$$

where $u > 0$ is initial capital, $c > 0$ is premium rate and $\delta > 0$ is the constant interest force. $\{X_n, n \geq 1\}$, $\{N(t), t \geq 0\}$ and $\{\theta_n\}$ are assumed to be mutually independent. Denote by

$$\psi(u, T) = P(U_\delta(t) < 0 \ \text{ for some } \ T \geq t > 0),$$

the probability of ruin within time T. If $T = \infty$, then $\psi(u, \infty)$ is called ultimate ruin probability. All limit relationships in this paper, unless otherwise stated, are for $u \to \infty$. $A \sim B$ and $A \gtrsim B (A \lesssim B)$ respectively mean that

$$\lim_{u\to\infty} \frac{A}{B} = 1 \ \ and \ \ \lim_{u\to\infty} \frac{A}{B} \geq (\leq) 1.$$

2 Preliminaries

In this paper, we pay attention to the claims with heavy-tailed claims, which have been the focus of many references in insurance and finance, for the facts that they have close relationship with large claims; see Embrechts et al. (1997) and Goldie & Klüppelberg (1998) for a nice review. We say that a distribution F belongs to the Pareto-like distribution class $\mathcal{R}_{-\alpha}$ if there is some $\alpha > 0$ such that $\overline{F}(x) = x^{-\alpha}L(x)$, $x > 0$, where $L(x)$ is a slowly varying function as $x \to \infty$ and index $-\alpha < 0$. So called extended regular varying class, $ERV(-\alpha, -\beta)$ is defined, if for some $0 < \alpha \leq \beta < \infty$ and for any $y > 1$, $y^{-\beta} \leq \liminf_{u\to\infty} \frac{\overline{F}(uy)}{\overline{F}(u)} \leq \limsup_{u\to\infty} \frac{\overline{F}(uy)}{\overline{F}(u)} \leq y^{-\alpha}$. Long-tailed distribution class, denoted by $\mathcal{L}$, is defined if for each $F \in \mathcal{L}$, $\lim_{x\to\infty} \frac{\overline{F}(x+t)}{\overline{F}(x)} = 1$, for any t (or, equivalently, for $t = 1$). Dominated class $\mathcal{D}$ is defined, if for each $F \in \mathcal{D}$, satisfying $\limsup_{x\to\infty} \frac{\overline{F}(xy)}{\overline{F}(x)} < \infty$, for any fixed $0 < y < 1$. The most important heavy-tailed class may be $\mathcal{S}$, usually called subexponential class. By definition, a d.f. F belongs to $\mathcal{S}$ iff $\lim_{x\to\infty} \frac{\overline{F^{*n}}(x)}{\overline{F}(x)} = n$, for any n, where F^{*n} denotes the n-fold convolution of F, with convention that F^{*0} is a d.f. degenerate at 0. These heavy-tailed classes have the properties below (see Embrechts et al. (1997)):

$$\mathcal{R}_{-\alpha} \subset ERV(-\alpha, -\beta) \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}. \tag{2.1}$$

The asymptotic behavior of the ultimate ruin probability is an important topic in the area of risk theory. A very famous asymptotic relation was established Embrechts and Veraverbeke (1982). They obtained the following result when the integrated tail distribution of the cliam is subexponentially distributed $\psi(u) \sim \frac{1}{\mu}\int_u^\infty \overline{F}(y)\mathrm{d}y$. Ruin probability under the constant interest force in a continuous time risk model has been extensively investigated. In the classical risk model, Klüppelberg and Stadtmuller (1998) obtained $\psi(u) \sim \frac{\lambda}{\alpha r}\overline{F}(u)$, when the claimsize is of

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regularly varying with index $-\alpha$, and r is constant interest force. Asmussen (1998) and Asmussen et al. (2002) obtained a more general result: $\psi(u) \sim \frac{\lambda}{r}\int_u^\infty \frac{\overline{F}(y)}{y}\mathrm{d}y$, where the claimsize is assumed to be in $\mathcal{S}^*$, an important subclass of $\mathcal{S}$. In the Poisson case, Tang (2005a) obtained (2.3) for subexponential claims. In the renewal case, Tang (2005b) proved that $\psi(u) \sim \frac{\mathrm{E}e^{-r\alpha\theta_1}}{1-\mathrm{E}e^{-r\alpha\theta_1}}\overline{F}(u)$. We emphasize that the methods used in the references mentioned above greatly depends on the i.i.d. assumption on the claims. Specially, the methods of Klüppelberg and Stadtmuller (1998); Tang (2005a) depends upon the regular variation assumption of the claimsize distribution. Jiang (2008) extended some results to the risky case. See also Jiang (2004, 2008). Dufresne and Gerber (1991) first researched the ruin probability for perturbed compound Poisson process. Most recently, Chen and Ng (2007) obtained a simple asymptotic formula when the claims are pairwise negatively dependent with distribution of ERV class. In this paper, with quite different methods, we aim to extend their results to the case that the pairwise ND claims belong to $\mathcal{L}\cap\mathcal{D}$.

Two random variables, X_1 and X_2, are called negatively Dependent (ND) if for all real numbers x_1 and x_2, $P(X_1 \le x_1, X_2 \le x_2) \le P(X_1 \le x_1)P(X_2 \le x_2)$, or, equivalently, $P(X_1 \ge x_1, X_2 \ge x_2) \le P(X_1 \ge x_1)P(X_2 \ge x_2)$. See Lehmann (1966). We say that a sequence of random variables $\{X_1, X_2, ...\}$ is pairwise ND if for all positive integers $i \neq j$ the random variables X_i and X_j are ND. We can easily construct pairs of r.v.s that are ND but not independent through a simple mechanism provided by Farlie-Gumbel-Morgenstern family of distributions.

3 Main Result and Some Necessary lemmas

The following theorem is the main result of this paper:

Theorem 1. *In the renewal risk model introduced in section 1. If $F \in \mathcal{L}\cap\mathcal{D}$, then the finite time ruin probability up to time T satisfies*

$$\psi(u,T) \sim \int_0^T \overline{F}(ue^{\delta s})dm(s), \tag{3.1}$$

where $m(x)$ is the renewal function of the process, i.e., $m(x) = EN(x)$.

To complete theorem 2.1, some lemmas in the following are needed.

Lemma 1. *If F is subexponential, the tail of its n-fold convolution is bounded by F's tail in the following way: for any $\varepsilon > 0$, there exists an $A(\varepsilon) > 0$ such that, uniformly for all $n \ge 1$ and all $x \ge 0$,*

$$\overline{F^{n*}}(x) \le A(\varepsilon)(1+\varepsilon)^n\overline{F}(x), \tag{3.2}$$

see Embrechts *et al.* (1997).

The following result can be found in Cline and Samorodnitsky (1994).

Lemma 2. *Let X and Y be two independent random variables with distributions F and G. The distribution of product X and Y is denoted by H.*

(1) If $F \in \mathcal{L}$ and G doesn't degenerate to zero, and for any fixed $a > 0$, it holds that $\lim_{x\to\infty}\frac{\overline{G}(x/a)}{\overline{H}(x)} = 0$, then $XY \in \mathcal{L}$.

(2) If $F \in \mathcal{D}$ and $P(Y > 0) > 0$, then $H \in \mathcal{D}$.

(3) If $X \in \mathcal{S}$ and Y is bounded and doesn't degenerate to zero, then $H \in \mathcal{S}$.

The following Lemma is the extension of Tang and Tsitsiashvili (2003) on i.i.d. claim assumption:

Lemma 3. *Let $\{X_i, i \ge 1\}$ be pairwise ND r.v.s with common distribution F. σ_n is defined in section 1. Then for any positive integer n_0, we have*

$$P(\sum_{i=1}^{n_0} X_i e^{-\delta\sigma_i} > u) \sim \sum_{i=1}^{n_0} P(X_i e^{-\delta\sigma_i} > u). \tag{3.3}$$

4 Proof of Theorem 1

Obviously, ruin probability defined by (1.3) is equivalent to the following

$$\begin{aligned} &\psi(u;T)\\ = \; &P(e^{-\delta t}U_\delta(t) < 0,\\ &for\ \ some\ \ T \ge t > 0 | U_\delta(0) = u) \end{aligned} \tag{4.1}$$

The finite-time ruin probability $\psi(u;T)$ satisfying that

$$\psi(u;T) \ge P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \ge u + \frac{c}{\delta}). \tag{4.2}$$

Similarly, we have that

$$\psi(u;T) \le P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \ge u). \tag{4.3}$$

If we can prove that relations

$$\begin{aligned} &P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \ge u)\\ \sim \; &\int_0^T \overline{F}(ue^{\delta s})dm(s)\\ \sim \; &P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \ge u + \frac{c}{\delta}) \end{aligned} \tag{4.4}$$

are true, then Theorem 1 will be finished. From Lemma 1, for any fixed $\varepsilon > 0$, there exists a constant $C(\varepsilon) > 0$ such that

$$\begin{aligned} & P(\sum_{i=1}^{k} X_i e^{-\delta t} \geq u) \\ \leq \quad & C(\varepsilon)(1+\varepsilon)^k P(X_1 e^{-\delta t} \geq u) \end{aligned} \tag{4.5}$$

holds for all $k = 1, 2, ..., t \geq 0$ and $u \geq 0$. Rewrite

$$\begin{aligned} & P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \geq u) \\ = \quad & (\sum_{k=1}^{N_0} + \sum_{k=N_0+1}^{\infty}) P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u, \\ & N(T) = k). \end{aligned} \tag{4.6}$$

For all integer $k \geq N_0$, pay attention to the condition that $N(T) = k$, we have

$$\begin{aligned} & \sum_{k=N_0+1}^{\infty} P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq x, N(T) = k) \\ \leq \quad & A(\varepsilon) \int_0^T \overline{F}(e^{\delta t} u) \cdot \\ & \sum_{k=N_0+1}^{\infty} (1+\varepsilon)^k P(N(T-t) = k) dF_{\theta_1}(t) \\ \leq \quad & A(\varepsilon) E[(1+\varepsilon)^{N(T)} I(N(T) \geq N_0)] \cdot \\ & \int_0^T P(X_1 e^{-\delta t} \geq x) dF_{\theta_1}(t). \end{aligned} \tag{4.7}$$

Especially, we can choose $\varepsilon > 0$ and $N_0 > 0$ such that $A(\varepsilon)E[(1+\varepsilon)^{N(T)} I(N(T) \geq N_0)]$ is smaller than any arbitrarily given number, say, $\eta_0 > 0$. In fact

$$\begin{aligned} & E[(1+\varepsilon)^{N(T)} I(N(T) \geq N_0)] \\ \leq \quad & \sum_{k=N_0}^{\infty} ((1+\varepsilon) E e^{-\theta_1})^k e^T, \end{aligned} \tag{4.8}$$

here we have used Chebyxev inequality. We choose ε small enough such that $(1+\varepsilon)Ee^{-\theta_1} < 1$, and then choose N_0 large enough so that $A(\varepsilon)E[(1+\varepsilon)^{N(T)} I(N(T) \geq N_0)] < \eta_0$. Thus, for this $\eta_0 > 0$ and the same integer N_0, there exists a number $u_1 > 0$, for all $u > u_1$, we have

$$\begin{aligned} & \sum_{k=N_0+1}^{\infty} P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ \leq \quad & \eta_0 \int_0^T P(X_1 e^{-\delta s} \geq u) dm(s). \end{aligned} \tag{4.9}$$

By Lemma 2 and 3, we can get for $k = 1, 2, ..., N_0$,

$$\begin{aligned} & P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq x, N(T) = k) \\ \sim \quad & \sum_{i=1}^{k} P(X_i e^{-\delta\sigma_i} \geq u, N(T) = k). \end{aligned} \tag{4.10}$$

Therefore

$$\begin{aligned} & \sum_{k=1}^{N_0} P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ \sim \quad & \sum_{k=1}^{N_0} \sum_{i=1}^{k} P(X_i e^{-\delta\sigma_i} \geq u, N(T) = k). \end{aligned} \tag{4.11}$$

In other words, for the same $\eta_0 > 0$ and the same N_0, there exists $u_2 > 0$, for all $u > u_2$, it holds that

$$\begin{aligned} & \sum_{k=1}^{N_0} P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ \leq \quad & (1+\eta_0) \sum_{k=1}^{\infty} \sum_{i=1}^{k} P(X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ = \quad & (1+\eta_0) \int_0^T P(X_1 e^{-\delta s} \geq u) dm(s). \end{aligned} \tag{4.12}$$

Thus, for $u > \max\{u_1, u_2\}$, we get

$$\begin{aligned} & P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \geq u) \\ \leq \quad & (1+2\eta_0) \int_0^T P(X_1 e^{-\delta s} \geq u) dm(s). \end{aligned} \tag{4.13}$$

On the other hand

$$\begin{aligned} & P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \geq u) \\ = \quad & \sum_{k=1}^{\infty} P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ \geq \quad & \sum_{k=1}^{N_0} (1-\eta_0) \sum_{i=1}^{k} P(X_i e^{-\delta\sigma_i} \geq u, N(T) = k) \\ \geq \quad & (1-2\eta_0) \int_0^T P(X_1 e^{-\delta s} \geq u) dm(s). \end{aligned} \tag{4.14}$$

By the arbitrariness of η_0, we have

$$\begin{aligned} & P(\sum_{i=1}^{N(T)} X_i e^{-\delta\sigma_i} \geq u) \\ \sim \quad & \int_0^T P(X_1 e^{-\delta s} \geq u) dm(s). \end{aligned} \tag{4.15}$$

From Lemma 2, we know that every $X_i e^{-\sigma_i}$ still belongs to $\mathcal{L} \cap \mathcal{D}$. Hence

$$\begin{aligned} & P(\sum_{i=1}^{k} X_i e^{-\delta\sigma_i} \geq u + \frac{c}{\delta}) \\ \sim \quad & \sum_{i=1}^{k} P(X_i e^{-\delta\sigma_i} \geq u), \end{aligned} \tag{4.16}$$

for all the $1 \leq k \leq N_0$. Thus Theorem 1 is completed.

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