On a Nonlinear Nonlocal Cauchy Problem

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Abstract -We prove the existence of integral solutions to a nonlinear time-dependent functional differential equation with a nonlocal initial condition. The approach relies on the theory of m-accretive operators and compactness methods.

Index Terms-Compact evolution operator, evolution equation, m-accretive operator, nonlocal Cauchy problem.

I. INTRODUCTION

We are concerned with the existence of solutions to the nonlocal Cauchy problem

$$u'(t) + A(t)u(t) \ni F(u)(t),\ t \in I = [0,T],$$
$$u(0) = g(u) \qquad (1)$$

in a real Banach space X. Here $\{A(t) : t \in I\}$ are m-accretive operators in X, while F, g are functionals defined on $C(I;X)$ with values in $L^1(I;X)$ and X, respectively. Such problems arise in physics and engineering, in particular in the mathematical modeling of heat or diffusion processes, or in the study of atomic reactors; see [1]-[3].

The study of abstract evolution equations with nonlocal initial conditions was initiated by Byszewski [4], who studied a problem of the form (1) where $A(t) = A$, linear and independent of time, $(Fu)(t) = f(t,u(t)),\quad f : I \times X \to X$, and

$$g(u) = \sum_{i=1}^{p} c_i u(t_i), \text{ with } 0 < t_1 < ... < t_p \le T$$

and $c_i \in R$.

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For some recent results on fully nonlinear nonlocal Cauchy problems, see [5]-[7]. In general, the existing results require a Lipschitz condition on F or g, or a compactness assumption on g. In [8], the authors remove these restrictions for a linear, autonomous version of (1). It is our goal to generalize their approach to the fully nonlinear, time-dependent case.

II. PRELIMINARIES

In this section, we collect some basic facts on m-accretive operators, evolution operators and non-autonomous evolution equations; see [9], [10] for details. Let $(X, \|.\|)$ be a real Banach space, of dual $(X^*, \|.\|_*)$. The duality mapping $J : X \to X^*$ is defined by

$$J(x) = \{x^* \in X^* : x^*(x) = \|x\|^2 = \|x^*\|_*^2\},\ \forall x \in X,$$

while the so-called upper semi-inner product is given by

$$< y, x >_+ = \sup\{x^*(y) : x^* \in J(x)\},\ \forall x, y \in X.$$

It can be shown that $<.,.>_+$ is upper semicontinuous on $X \times X$. Let A be a multivalued operator on X, of domain $D(A)$ and range $R(A)$. We say that A is *accretive* if $< y' - y, x' - x >_+ \ge 0$ for all $x, x' \in D(A)$ and all $y \in Ax, y' \in Ax'$. If also $R(Id + \lambda A) = X$ for all $\lambda > 0$, where Id denotes the identity on X, then A is called *m-accretive*.

Let $\{A(t) : t \in I\}$ be a family of m-accretive operators in X, of domains $D(A(t))$, with $\overline{D(A(t))} = D$ (independent of t), which satisfy the condition:

$(H_{A(t)})$ There exist two continuous functions $m_1 : I \to X, m_2 : R_+ \to R_+$ $(R_+ = [0,\infty))$ such that

$$< y_1 - y_2, x_1 - x_2 >_+ \geq -\|m_1(t) - m_2(s)\| \|x_1 - x_2\| \; m_2(\max\{\|x_1\|, \|x_2\|\}),$$

$\forall\, x_1 \in D(A(t)), y_1 \in A(t)x_1, x_2 \in D(A(s)),$
$y_2 \in A(s)x_2, 0 \leq s \leq t \leq T.$

If $(H_{A(t)})$ holds, then the family $\{A(t) : t \in I\}$ gives rise to an *evolution operator* $U(t,s)$ on D via the formula

$$U(t,s)x = \lim_{n\to\infty} \prod_{i=1}^{n} (Id + \frac{t-s}{n} A(s + i\frac{t-s}{n}))^{-1} x\,, \quad (2)$$

for all $x \in D, 0 \leq s \leq t \leq T.$

It follows that

$$U(t,t) = Id \text{ and } \|U(t,s)x - U(t,s)y\| \leq \|x - y\|,$$

for all $x, y \in D$, and all $0 \leq s \leq t \leq T$. The evolution operator U is said to be *compact* if $U(t,s)$ maps bounded subsets of D into relatively compact subsets of D for all $0 \leq s < t \leq T$. In the special case when $A(t) = A$ is a time-independent m-accretive operator, $U(t,0) = S(t)$ is the contraction semigroup generated by $-A$ on $\overline{D(A)}$.

Next, consider the Cauchy problem

$$\begin{aligned} &u'(t) + A(t)u(t) \ni f(t),\ t \in I, \\ &u(0) = u_0, \end{aligned} \quad (3)$$

where $\{A(t) : t \in I\}$ satisfy $(H_{A(t)})$, $f \in L^1(I;X)$, and $u_0 \in D.$

Definition 1. An *integral solution* of problem (3) is a function $u \in C(I;D)$ satisfying $u(0) = u_0$ and the inequality

$$\|u(t) - x\|^2 - \|u(s) - x\|^2 \leq 2\int_s^t [< f(\tau) - y, u(\tau) - x >_+ + M\|u(\tau) - x\| \|m_1(\tau) - m_1(\theta)\|]\, d\tau,$$

$\forall\, 0 \leq s \leq t \leq T, \theta \in [0,T], x \in D(A(\theta)), y \in A(\theta)x,$ and $M = m_2(\max\{\|x\|, \|u\|_{C(I;X)}\}).$

It is well-known that (3) has a unique integral solution for each $u_0 \in D$ and $f \in L^1(I;X)$, provided that $(H_{A(t)})$ is satisfied. In particular, $U(t,0)u_0$ is the integral solution of (3) when $f \equiv 0.$

Proposition 2. Let $(H_{A(t)})$ be satisfied, and let u, v be integral solutions of (3), corresponding to (u_0, f) and (v_0, g), respectively (with $u_0, v_0 \in D$ and $f, g \in L^1(I;X)$). Then

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\|\, d\tau,\ \forall 0 \leq s \leq t \leq T. \quad (4)$$

III. MAIN RESULTS

For a fixed finite $r > 0$, we set

$$B_r = \{x \in X : \|x\| \leq r\},$$
$$K_r = \{\phi \in C(I;X) : \phi(t) \in B_r, \forall t \in I\}.$$

We assume that:

(H_1) $\{A(t); t \in I\}$ satisfy $(H_{A(t)})$, and the corresponding evolution operator U (given by (2)) is compact;

(H_2) The operator $F : C(I;X) \to L^1(I;X)$ is continuous, and there exists $\alpha = \alpha_r \in L^1(I;R_+)$ such that

$$\|F(u)(t)\| \leq \alpha(t), \text{ a.e. on } I, \text{ for all } u \in K_r;$$

(H_3) The function $g : C(I;X) \to D$ is continuous and maps K_r into a bounded set;

(H_4) There exists $\delta \in (0,T)$ such that $F(u) = F(v),\ g(u) = g(v)$ for any $u, v \in K_r$ with $u(s) = v(s),\ s \in [\delta, T];$

(H_5) $\sup_{t\in I,\phi\in K_r}\|U(t,0)g(\phi)\| + \int_0^T \alpha(\tau)d\tau \le r.$

Definition 3. A function $u \in C(I;D)$ is called an integral solution of problem (1), if it is an integral solution, in the sense of Definition 1, of problem (3) with $F(u)(t)$ in place of $f(t)$, and $g(u)$ in place of u_0.

Theorem 4. Let $(H_1)-(H_5)$ be satisfied. Then problem (1) has at least one integral solution.

This result does not cover the case when $F(u)(t) = f(t,u(t))$ for a given function $f: I\times X \to X$, since (H_4) is not satisfied. We now replace (H_4) by the following condition:

$(H_4^{\#})$ $\lim_{\varepsilon\downarrow 0}\|g(\phi)-g(\phi^{\varepsilon})\| = 0$, uniformly for all $\phi \in K_r$, with $\phi^{\varepsilon}(t) = \begin{cases}\phi(\varepsilon), 0\le t\le \varepsilon,\\ \phi(t),\ \varepsilon \le t \le T.\end{cases}$

Theorem 5. Let $(H_1)-(H_3), (H_4^{\#})$ and (H_5) be satisfied. Then problem (1) has at least one integral solution.

If $A(t) = A$ (independent of time), then the evolution operator U is replaced by the contraction semigroup $S(t)$ generated by $-A$ (cf. Section II). The corresponding autonomous initial-value problem (3) has a unique integral solution (that is, a function $u \in C(I;\overline{D(A)})$ satisfying the inequality in Definition 1 with $M = 0$) for any $f \in L^1(I;X)$ and $u_0 \in \overline{D(A)}$. Theorem 5 now yields the following result:

Corollary 6. Let A be an m-accretive operator in X, such that $S(t)$, the semigroup generated by $-A$ on $\overline{D(A)}$, is compact for $t > 0$. If also $(H_2),(H_3)$ (with $D = \overline{D(A)}$), $(H_4^{\#})$ and (H_5) (with $S(t)$ in place of $U(t,0)$) are satisfied, then there exists an integral solution of the problem

$$u'(t) + Au(t) \ni F(u)(t),\ t\in I,\quad u(0) = g(u). \tag{5}$$

Remark 7. In the special case when

$$F(u)(t) = f(t,u(t)),\ f: I\times X \to X,$$

where f satisfies Caratheodory type conditions, Corollary 6 is comparable to Theorem 4.3 in [7], which was proved by a different method.

IV. PROOF OF THEOREM 4

We sketch the proof of Theorem 4, only. The proof of Theorem 5 can be carried out by using Theorem 4, and adapting the approximating procedure of [8]. The details will appear elsewhere.

Set

$$K_r(\delta) = \{u \in C([\delta,T];X): \|u(t)\| \le r, \forall t\in[\delta,T]\}.$$

For any $u \in K_r(\delta)$, let $\tilde{u} \in K_r$ be given by

$$\tilde{u}(t) = \begin{cases} u(\delta), 0 \le t \le \delta,\\ u(t),\ \delta \le t \le T.\end{cases}$$

Also, define

$$\tilde{F}(u)(t) = F(\tilde{u}(t)),\ t\in I;\ \ \tilde{g}(u) = g(\tilde{u}). \tag{6}$$

By $(H_2)-(H_4)$ and (6), it follows that $\tilde{F}$ and $\tilde{g}$ are continuous from $K_r(\delta)$ to $L^1(I;X)$ and D, respectively. In addition, we have

$$\|\tilde{F}(u)(t)\| \le \alpha(t), \text{ a.e. on } I,\ \forall u \in K_r(\delta), \tag{7}$$

$$\sup_{t\in I, u\in K_r(\delta)}\|U(t,0)\tilde{g}(u)\| = \sup_{t\in I, v\in K_r}\|U(t,0)g(v)\| < \infty. \tag{8}$$

Define the map $\Psi: K_r(\delta) \to C([\delta,T];X)$ by $\Psi(w)(t) = u_w(t),\ t\in[\delta,T],\ w\in K_r(\delta)$, where u_w is the unique integral solution of

$$\frac{d}{dt}u_w(t) + A(t)u_w(t) \ni \tilde{F}(w)(t),\ t\in I,\quad u_w(0) = \tilde{g}(w) \tag{9}$$

From (4) we infer that

$$\|u_w(t) - U(t,0)\tilde{g}(w)\| \le \int_0^t \|\tilde{F}(w)(s)\| ds.$$

This, (7), (8) and (H_5) lead to

$$\|u_w(t)\| \le \sup_{t\in I,\, v\in K_r} \|U(t,0)g(v)\| + \int_0^T \alpha(s)\, ds \le r,\ \forall t \in I, \tag{10}$$

so that Ψ maps $K_r(\delta)$ into itself. Moreover, on account of (4), (6), $(H_2),(H_3)$, Definition 1 and the upper semicontinuity of $<.,.>_+$, it is easily seen that Ψ is continuous. Next, employing (6), (7), $(H_1)-(H_3)$ and the theory of [11], we deduce that $\Psi(K_r(\delta))$ is relatively compact in $C([\delta,T];X)$. Applying Schauder's fixed point theorem, we conclude that Ψ has a fixed point $w^* \in K_r(\delta)$.

Let $u(t) = u_{w^*}(t), t \in I$ and remark that $w^*(t) = \Psi(w^*)(t) = u_{w^*}(t) = u(t), \forall t \in [\delta, T]$. This, together with (H_4) , (9) and (10), implies that u is an integral solution of (1), as desired. The proof is complete.

V. AN APPLICATION

For simplicity, we restrict ourselves to an example that illustrates Corollary 6. Consider the initial-boundary value problem

$$\begin{aligned} & u_t(t,x) - \Delta u(t,x) = h(t,u(t,x)) + \int_0^t k(t-s)\,u(s,x)\,ds,\ \ (t,x) \in I\times\Omega, \\ & -\frac{\partial u}{\partial n}(t,x) \in \beta(u(t,x)),\ (t,x)\in I\times\partial\Omega, \\ & u(0,x) = u_0(x) + \sum_{i=1}^p c_i u(t_i, x),\ x\in\Omega, \end{aligned} \tag{11}$$

where Ω is a bounded domain in R^N with a smooth boundary $\partial\Omega, \beta$ is an m-accretive operator on R with $0 \in \beta(0)$, $\frac{\partial}{\partial n}$ denotes the outward normal derivative, $0 < t_1 < ... < t_p \le T,\ c_i, i = 1,...p$ are given constants, $k \in L^1(I),\ h: I\times R \to R$, and $u_0 \in L^2(\Omega)$. This problem can be written in the form (1) in the space $X = L^2(\Omega)$, by setting

$$A = -\Delta,\ D(A) = \{u \in H^2(\Omega) : -\frac{\partial u}{\partial n} \in \beta(u),\ \text{a.e. on } \partial\Omega\},$$

$$F(u)(t)(x) = h(t,u(t,x)) + \int_0^t k(t-s)\,u(s,x)\,ds, \tag{12}$$

$$g(u)(x) = u_0(x) + \sum_{i=1}^p c_i u(t_i, x),\ (t,x) \in I\times\Omega.$$

It is well-known [10] that A is m-accretive in X with $\overline{D(A)} = X$, and that the semigroup $S(t)$ generated by $-A$ on X is compact (for $t > 0$), with $S(t)0 = 0, \forall t \ge 0.$ Assume that

(H_6) $t \to h(t,y)$ is measurable in t for all $y \in R$, and continuous in y for a.a. $t \in I$;

(H_7) There exist $a > 0, b \in L^1(I;R_+)$ such that

$$|h(t,y)| \le a|y| + b(t),$$

for almost all $t \in I$ and all $y \in R.$.

Then, it is easily verified that F and g, as given in (12), satisfy (H_2), and respectively (H_3) and $(H_4^\#)$.

Finally, in this set-up, condition (H_5) reduces to

$$(H_8) \qquad \|u_0\|_{L^2(\Omega)} + r(a + \sum_{i=1}^p |c_i| + T\|k\|_{L^1(I)}) + \|b\|_{L^1(I)}\,\mu(\Omega)^{1/2} \le r,$$

where μ denotes the Lebesgue measure.

Consequently, an application of Corollary 6 yields:

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Corollary 8. If $u_0 \in L^2(\Omega), k \in L^1(I)$ and $(H_6)-(H_8)$ hold, then problem (1) has at least one integral solution $u \in C(I; L^2(\Omega))$.

REFERENCES

[1] J. Chadam and H. M. Yin, Determination of an unknown function in a parabolic equation with an overspecified condition, *Math. Meth. Appl. Sci.* 13 (1990), 421-430.

[2] K. Deng, Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions, *J. Math. Anal. Appl.* 179 (1993), 630-637.

[3] Y. Lin, Analytical and numerical solutions for a class of nonlocal nonlinear parabolic differential equations, *SIAM J. Math. Anal.* 25 (1994), 1577-1596.

[4] L. Byszewski, Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.* 162 (1991), 494-505.

[5] S. Aizicovici and H. Lee, Existence results for nonautonomous evolution equations with nonlocal initial conditions, *Commun. Appl. Anal.* 11 (2007), 285-297.

[6] S. Aizicovici and V. Staicu, Multivalued evolution equations with nonlocal initial conditions in Banach spaces, *NoDEA, Nonlinear Differential Equations Appl.* 14 (2007), 361-376.

[7] X. Xue, Nonlinear differential equations with nonlocal conditions in Banach spaces, *Nonlinear Anal.* 63 (2005), 575-586.

[8] Q. Liu and R. Yuan, Existence of mild solutions for semilinear evolution equations with nonlocal initial conditions, *Nonlinear Anal.*, to be published.

[9] N. H. Pavel, *Nonlinear Evolution Operators and Semigroups*, Lecture Notes in Math., vol. 1260, Berlin: Springer-Verlag, 1987.

[10] I. I. Vrabie, *Compactness Methods for Nonlinear Evolutions*, Pitman Monographs Surveys Pure Appl. Math., vol. 75, Harlow: Longman Scientific & Technical, 1995.

[11] A. G. Kartsatos and K. Y. Shin, Solvability of functional evolution equations via compactness methods in general Banach spaces, *Nonlinear Anal*. 21 (1993), 517-535.