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Strange Attractors in Nonlinear Time-delay Neural Systems

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Abstract—**In this paper, we provide a geometric construction of a transversal homoclinic orbit for a nonlinear model neuron with time delays. The existence of chaos for time-delay systems in a high-dimensional space is theoretically confirmed with several conditions. These units may be used as basic elements for networks with higher-order information processing capabilities.**

Keywords: Nonlinear Time-delay Systems, neural networks, Marotto's chaos, transversal homoclinic orbits

1 Introduction

Recently, there has been an increasing interest in the delayed equations: both differential and difference. Typically, for non-delayed discrete dynamical systems, it is assumed that all causes have an instantaneous effect. So far, the hypothesis is provided for a great number of systems which may be described by the differential/difference equations. However, it should be possible that a meaningful time-scale characteristic of a system is smaller than a time delay so that a simply model cannot characterize all information or changes. Hence, a model with time delay not only possesses more reasonable and extensible meaning for a living system but also offers new modeling possibilities such as optics [1], and biological systems [2], etc. It means that time delays can be viewed as intrinsic properties of the nervous system. Consequentially, the study of neural systems with time delays is very meaningful.

In 1998, Rabinovich and Abarbanel [3] argued that chaos itself is not in charge of the function of neural assemblies, but the resting state is at the edge of instability or beyond it. They thought that neural systems were transfered/forced to different working states by chaotic oscillations. That is, a behavior (a stable state) always starts from a resting state (a chaotic state). For example, the self-organizing phenomenon [4] can be observed on a sensorimotor coordination task. Several years ago, a Nagumo-Sato-based neural network with chaotic behaviors was initiated by Aihara et. al. [5] in order to characterize a number of sophisticated dynamics observed in a biological neural system. Their behaviors include stable fixed points, periodic oscillations, and chaos which are different from behaviors of static neural network models. In 1995, Chen and Aihara [6] proposed that *transiently chaotic neural network* (TCNN) with the increase of self-coupling from a sufficiently large negative to zero can mimic adaptable behaviors and successfully applied to optimization problems. The features of TCNN simultaneously contain two parts: stability and chaos [7, 8, 9, 10]. Therefore, TCNN can be exploited to support the view of Rabinovich and Abarbanel. Recently, Wu and Zhang [11] initiated a system of two difference equations coupled through an excitatory feedback with an integer delay and confirmed the system possesses larger capacity for associative memory than one with no delay item. It mainly deals with the coexistence of multiple stable patterns such as multiple equilibria and periodic orbits, lying at the basis of the mechanism for associative content-addressable memory storage and retrieval in neural networks, population models, and ecological models, etc. It has been shown that time delay provides an effective mechanism for a network to store and retrieve periodic patterns, and biologically these delays arise due to axonal conduction time, distances of interneurons and the finite switching speeds of amplifiers. For the time delay system, a chaotic behavior near the origin was confirmed by Huang and Zou [12]. By these two results, it is possible to prove that neural networks with time delays exist both stability and chaos.

Here, we review some developments and modifications about discrete-type chaos. Initially, a chaotic phenomenon was numerically found in the Lorenz's research [13] on weather prediction in 1963. Until 1975, the mathematical definition of chaos was initiated by Li and Yorke [14] for one-dimensional continuous maps, and the criterion of existence of chaos is "period three imply chaos" as follows. Let $f : I \to I$ be a continuous map of the compact interval I of the real line $\mathbb{R}$ into itself. If f has a periodic point of period three, then f exhibits chaotic behavior. After three years, the above result is generalized by Marotto [15]. He proposed the definition of the "snap-back repeller" and proved that "snap-back repellers imply chaos" in a multi-dimensional space. The type is specially called "Marotto's chaos" or chaos in the

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sense of Marotto. Later on, Marotto [16] gave some comments about changes, corrections and updates from many researchers. As an extension of the Li and York theorem on the one-dimensional maps, Marotto [15] established a significant theorem in confirming the chaotic dynamics for multi-dimensional systems. With presence of the so-called "snap-back repeller", the phase space possesses a topological structure which includes infinitely many periodic points and a scrambled set. Very erratic behaviors of the system then occur, including the lack of global stability for solutions, and the existence of an uncountable collection of orbits which do not eventually approach any periodic points. Up to now, the theorem is the best one for analyzing chaos in multi-dimensional maps. The detailed description of Marotto's chaos is presented in Appendix.

In the paper, we propose a simple, modified model as the following coupled system with two neurons and time delays $\tau_1,\ \tau_2$:

$$\begin{aligned} x(n) =& \mu_1 x(n-1) + \mu_2 x(n-\tau_1) + \omega_{11} g_\varepsilon(x(n-\tau_1)) \\ &+ \omega_{12} g_\varepsilon(y(n-\tau_2)) + \nu_1, \qquad (1) \\ y(n) =& \mu_1 y(n-1) + \mu_2 y(n-\tau_2) + \omega_{22} g_\varepsilon(y(n-\tau_2)) \\ &+ \omega_{21} g_\varepsilon(x(n-\tau_1)) + \nu_2, \qquad (2) \end{aligned}$$

where n ($\in \mathbb{N}$) is the iteration time; τ_i ($\in \mathbb{Z}$) are time-delayed parameters larger than one; μ_i between 0 and 1 are the damping factors of the nerve membrane ; ω_{ii} is the self-feedback connection weight; ω_{ij} is the connection weight from neuron i to neuron j; ε ($\varepsilon > 0$) is the steepness parameter of the neuronal activation function; ν_i is an input bias of neuron i; ρ ($-1 < \rho < 0$) is a negative parameter. Herein, $i = 1,\ 2$. A saturated output function $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

$$g_\varepsilon(x) = \left(2 + \left|\frac{x}{\varepsilon} + 1\right| - \left|\frac{x}{\varepsilon} - 1\right|\right) \Big/ 4. \qquad (3)$$

A traditional neural network is extended to one with delay times. In case of $\tau_1 = \tau_2 = 1$, the system (1)-(2) with Eq. (3) can be viewed as a cellular neural network [17]. We aim to provide the parameter conditions for the existence of chaos in the time delay system.

The remainder of the paper is organized as follows. A discrete neural network with one neuron and a time delay is investigated in Section 2. Herein, some complicated conditions for the existence of chaos in the sense of Marotto are presented. In Section 3, we proof the existence of chaos for a delayed discrete neural network with two neurons. The generic conditions for the existence of high-dimensional chaotic phenomena will be described without constrain of parameters as the system (1)-(2). Finally, we will draw conclusions in Section 4.

2 A delayed discrete neural network with one neuron

Let us consider the following simple model of a *delayed discrete neural network* (DDNN) with one neuron and a time delay τ:

$$x(n) = \mu_1 x(n-1) + \mu_2 x(n-\tau) + \omega\Big[g_\varepsilon(x(n-\tau)) + \rho\Big] + \nu, \quad (4)$$

where $|\nu| \le h$, $-1 < \rho < 0$, and a saturated output function g_ε is defined as Eq. (3) whose graph is shown in Figure 1.

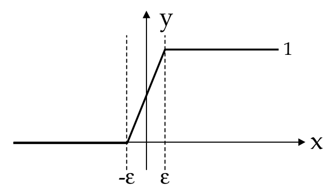

Figure 1: A saturated output function g_ε for $\varepsilon > 0$.

For a fixed $\varepsilon > 0$, the real line $\mathbb{R}$ can be represented as the union of three disjointed regions $\Omega_\ell = \{x \in \mathbb{R} | x < -\varepsilon\}$, $\Omega_{\rm m} = \{x \in \mathbb{R} | |x| \le \varepsilon\}$ and $\Omega_{\rm r} = \{x \in \mathbb{R} | x > \varepsilon\}$ based on the piecewise feature of the saturation function g_ε. Notably, the neural network (4) is smooth on the three disjointed intervals. To provide sufficient conditions of fixed points for DDNN (4), the following conditions are given.

(H1a) $\frac{1-\mu_1-\mu_2}{\omega}\varepsilon > 0$, $|\frac{1}{2\varepsilon}| > |\frac{1-\mu_1-\mu_2}{\omega}|$, $\frac{1-\mu_1-\mu_2}{\omega}\varepsilon + |\frac{h}{\omega}| - \rho < 1$, $\frac{1-\mu_1-\mu_2}{\omega}(-\varepsilon) - |\frac{h}{\omega}| - \rho > 0$.
(H1b) $\frac{1-\mu_1-\mu_2}{\omega}\varepsilon < 0$, $\frac{1-\mu_1-\mu_2}{\omega}\varepsilon + |\frac{h}{\omega}| - \rho < 1$, $\frac{1-\mu_1-\mu_2}{\omega}(-\varepsilon) - |\frac{h}{\omega}| - \rho > 0$.

Condition (H1a) confirms the existence of 3 fixed points for the family of one-dimensional DDNNs (4). Condition (H1b) assures that there exists at least one fixed point for the family of one-dimensional DTDNNs (4). The reasons to give these conditions are shown in the following lemmas.

Lemma 2.1 *Suppose that* (H1a) *holds. Then, for each* $|\nu| < h$, *there exist three points* $\phi^{0,\ell} \in \Omega_\ell$, $\phi^{0,{\rm m}} \in \Omega_{\rm m}$ *and* $\phi^{0,{\rm r}} \in \Omega_{\rm r}$ *such that* $(1-\mu_1-\mu_2)\phi^{0,\star} = \omega\Big[g_\varepsilon(\phi^{0,\star}) + \rho\Big] + \nu$, *where* $\star =$ "ℓ", "m", *or* "r".

proof: Assume that there is a fixed point $\overline{x}$ for a neural network (4). The equation

$$\overline{x} = \mu_1 \overline{x} + \mu_2 \overline{x} + \omega\Big[g_\varepsilon(\overline{x}) + \rho\Big] + \nu \qquad (5)$$

is hold. Therefore, we can reformulate Eq. (5) as a couple of equations

$$\begin{aligned} y &= [(1-\mu_1-\mu_2)x - \nu]/\omega - \rho, \qquad (6) \\ y &= g_\varepsilon(x). \end{aligned}$$

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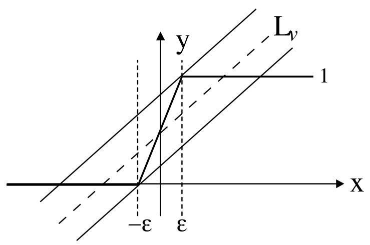

Figure 2: Diagram of finding a solution of a couple of equations. Let $L_\nu(x) = [(1-\mu_1-\mu_2)x-\nu]/\omega-\rho$ with the positive slope $\frac{1-\mu_1-\mu_2}{\omega}$. A solution exists if the intersection between the dotted line and g_ε is nonempty.

Obviously, Eq. (6) is a linear function of x. These two equations can be illustrated in the same x-y plane as Figure 2.

Let $\varepsilon > 0$. Consider $\omega > 0$. If $h \geq |\nu|$, $[(1-\mu_1-\mu_2)\varepsilon + h]/\omega-\rho < 1$ and $[-(1-\mu_1-\mu_2)\varepsilon-h]/\omega-\rho > 0$, then $[(1-\mu_1-\mu_2)\varepsilon-\nu]/\omega-\rho < 1$ and $[-(1-\mu_1-\mu_2)\varepsilon-\nu]/\omega-\rho > 0$. Hence, there exist three fixed points. Consider $\omega < 0$. If $h > |\nu|$, $[(1-\mu_1-\mu_2)\varepsilon-h]/\omega-\rho < 1$ and $[-(1-\mu_1-\mu_2)\varepsilon+h]/\omega-\rho > 0$, then $[(1-\mu_1-\mu_2)\varepsilon-\nu]/\omega-\rho < 1$ and $[-(1-\mu_1-\mu_2)\varepsilon-\nu]/\omega-\rho > 0$. Hence, there exist three fixed points. The argument can be supported by Figure 2 (a). Consequently, all conditions in (H1a) imply the existence of three fixed points.

Lemma 2.1 shows that, for each different parameter ν with $|\nu| < h$, these three fixed points $\phi^{0,\ell}(\nu)$, $\phi^{0,\mathrm{m}}(\nu)$ and $\phi^{0,\mathrm{r}}(\nu)$ in the interior areas of the corresponding regions Ω_ℓ, Ω_m and Ω_r can be thought as the functions of ν if six parameters μ_1, μ_2, ω, ρ, ε and h are compatible with Condition (H1a). With the property, we can extend the existence of fixed points for 1-d neural networks to one for neural networks in high dimensional space in the next section. Note that the first [resp. second] superscript in the symbol $\phi^{0,\star}$ represents a fixed point [resp. the position of a fixed point at Ω_r, Ω_m or Ω_ℓ].

Lemma 2.2 *Suppose that Condition* (H1b) *holds. Then, for each $|\nu| < h$, there exists at least one point $\phi^{0,\mathrm{m}} \in \Omega_\mathrm{m}$ such that* $(1-\mu_1-\mu_2)\phi^{0,\mathrm{m}} = \omega\Big[g_\varepsilon(\phi^{0,\mathrm{m}})+\rho\Big]+\nu$.

Proof: The proof resembles one in Lemma 2.1.

More illustrations for Lemma 2.2 are as follows. Assume $\mu_2+\frac{\omega}{2\varepsilon} < -1$ and $\mu_1 = 0$. If $0 < \mu_2 < 1$, there exists only one fixed point $\phi^{0,\mathrm{m}} \in \Omega_\mathrm{m}$. If $\mu_2 > 1$, there exists three fixed points $\phi^{0,\ell} \in \Omega_\ell$, $\phi^{0,\mathrm{m}} \in \Omega_\mathrm{m}$ and $\phi^{0,\mathrm{r}} \in \Omega_\mathrm{r}$. The reason of these two conditions is very clear by Figure 3.

Next, to construct a trajectory for a pre-image of one-dimensional DDNN (4), the following conditions are given.

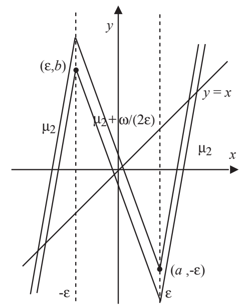

Figure 3: Illustration for Lemma (2.2). Assume that $\mu_2+\omega/(2\varepsilon) < -1$ and there exist a fixed point in Ω_m. If $0 < u_2 < 1$, there is no interaction in Ω_ℓ and Ω_r. If $\mu_2 > 1$, there exist a fixed point for every region.

(H2a1) $\varepsilon > 0$, $\mu_2+\frac{\omega}{2\varepsilon} > 1$, $\mu_2 < -1$, $\mu_2\varepsilon+\omega(1+\rho)-h > [-\varepsilon-\omega(1+\rho)-h]/\mu_2$.
(H2a2) $\varepsilon > 0$, $\mu_2+\frac{\omega}{2\varepsilon} > 1$, $\mu_2 < -1$, $\mu_2(-\varepsilon)+\omega\rho+h < (\varepsilon-\omega\rho+h)/\mu_2$.
(H2b1) $\varepsilon > 0$, $\mu_2+\frac{\omega}{2\varepsilon} < -1$, $\mu_2 > 0$, $(-\varepsilon-\omega\rho-h)/\mu_2 > \mu_2\varepsilon+\omega(1+\rho)+h$, $\mu_2(-\varepsilon)+\omega\rho-h > \varepsilon$.
(H2b2) $\varepsilon > 0$, $\mu_2+\frac{\omega}{2\varepsilon} < -1$, $\mu_2 > 0$, $-\mu_2\varepsilon+\omega\rho-h > [\varepsilon-\omega(1+\rho)+h]/\mu_2$, $-\varepsilon > \mu_2\varepsilon+\omega(1+\rho)+h$.

All of conditions (H2a1)-(H2a2) [or briefly (H2a)], (H2b1)-(H2b2) [or briefly (H2b)] assure the existence of the pre-image of some types of points for the neural networks (4) for a suitably small value μ_1, respectively. Furthermore, if there exists a fixed point, then the conditions can assure the existence of its pre-image point.

For precise explanation for the existence of a snap-back repeller of 1-d neural network with delayed time 3, we rewrite the 1-d model as the following equivalent discrete dynamical system on $\mathbb{R}^3$:

$$\Phi(n+1) = \mathbf{T}(\Phi(n)), \quad (7)$$

where $\Phi(n) = (\phi_3(n), \phi_2(n), \phi_1(n))$, and $\mathbf{T} : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$\mathbf{T}\begin{pmatrix} \phi_3(n) \\ \phi_2(n) \\ \phi_1(n) \end{pmatrix} = \begin{pmatrix} \phi_2(n) \\ \phi_1(n) \\ \mu_1\phi_1(n)+\mu_2\phi_3(n)+\omega\Big(g_\varepsilon(\phi_3(n))+\rho\Big)+\nu \end{pmatrix}.$$

Note that the subscript i for elements ϕ_i $(i = 1,2,3)$ of Φ can be thought as time delays.

For the simplification of notation, we define a map h from $\mathbb{N}\cup\{0\}$ to $\{\mathrm{m}, \mathrm{r}\}$ as follows:

$$h(z) = \begin{cases} \mathrm{r} & \text{if } z \in \{4,5,6\}, \\ \mathrm{m} & \text{otherwise.} \end{cases} \quad (8)$$

The map is exploited to describe the chosen location of the pre-image of the neural network (4). The domain of h means the pre-image number and its range describes its corresponding and specified position. Notably, the number of the preimage for the neural network (4) may

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be larger than one. Therefore, a notation $\phi^{n,h(n)}$ means that the point is the nth preimage of $\phi^{0,h(0)}$ under the action of the neural network (4) and belongs to the set of interior points in $\Omega_{h(n)}$.

To provide the expanding feature, we give the following conditions.

(H3a) $\mu_2 + \frac{\omega}{2\varepsilon} > 1 + |\mu_1|$.
(H3b) $\mu_2 + \frac{\omega}{2\varepsilon} < -(1 + |\mu_1|)$.

Absolute values of eigenvalues of linear part for the 3-d dynamical system (7) are larger than one if either Condition (H3a) or Condition (H3b) holds.

Lemma 2.3 *Suppose either* (H1a)-(H3a) *or* (H1b)-(H3b) *hold. There exists a sequence* $\{\phi^{n,h(n)} \in \mathbb{R} | n \in \mathbb{N} \cup \{0\}\}$ *which approaches to a fixed point* $\phi^{0,h(0)}$ *as* $n \to \infty$*. Hence, for neural network* (7)*, the constructed orbit is a transversal homoclinic orbit and the fixed point* $\phi^{0,h(0)}$ *is a snap-back repeller.*

proof: Let us consider the neural network (7). Since (H1a) holds, it follows from Lemma (2.1) that there exists a fixed point $\Phi^0 = (\phi^{0,h(0)}, \phi^{0,h(0)}, \phi^{0,h(0)})$ for Eq. (7), where $\phi^{0,h(0)} \in \Omega^0_{h(0)}$. Therefore, there exists $\widetilde{\mu}^1_1 > 0$ such that $\phi^{0,h(0)} - \mu_1\phi^{0,h(0)} \in \Omega^0_{h(0)}$ for all $\mu_1 \in \widetilde{U}_1$ where $\widetilde{U}_1 = [-\widetilde{\mu}^1_1, \widetilde{\mu}^1_1]$. By (H2a), there exists a point $\phi^{1,h(1)} \in \Omega^0_{h(1)}$ which satisfies $\phi^{0,h(0)} - \mu_1\phi^{0,h(0)} = \mu_2\phi^{1,h(1)} + \omega[g_\varepsilon(\phi^{1,h(1)}) + \rho] + \nu$. Let $\phi^{2,h(2)} = \phi^{1,h(1)}$, $\Phi^{-1} = (\phi^{1,h(1)}, \phi^{0,h(0)}, \phi^{0,h(0)})$ and $\Phi^{-2} = (\phi^{2,h(2)}, \phi^{1,h(1)}, \phi^{0,h(0)})$. Note that $\mathbf{T}(\Phi^{-1}) = \Phi^0$ and $\mathbf{T}(\Phi^{-2}) = \Phi^{-1}$. For an integer $n > 2$, we define

$$\Phi^{-n} = (\phi^{n,h(n)}, \phi^{n-1,h(n-1)}, \phi^{n-2,h(n-2)}) \tag{9}$$

whose components satisfy the following equation:

$$\phi^{n-3,h(n-3)} - \mu_1\phi^{n-2,h(n-2)} = \mu_2\phi^{n,h(n)} + \omega[g_\varepsilon(\phi^{n,h(n)}) + \rho] + \nu.$$

For $k = 1, \cdots, n-1$, assume a positive value $\widetilde{\mu}^k_1$ and $\phi^{k,h(k)} \in \Omega^0_{h(k)}$ exists with $\mu_1 \in \cap^{n-1}_{k=1}\widetilde{U}_k$ where $\widetilde{U}_k = [-\mu^k_1, \mu^k_1]$. By induction, there exists a suitable $\widetilde{\mu}^n_1 > 0$ such that $\phi^{n,h(n)} \in \Omega^0_{h(n)}$ exists if $\mu_1 \in \cap^n_{k=1}\widetilde{U}_k$ where $\widetilde{U}_k = [-\widetilde{\mu}^k_1, \widetilde{\mu}^k_1]$. The linear part $D\mathbf{T}$ of $\mathbf{T}$ at $\Phi = (\phi_1, \phi_2, \phi_3)$ is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \mu_2 + \omega g'(\phi_3) & 0 & \mu_1 \end{pmatrix}.$$

Note that $\phi^{k,h(k)} \in \Omega^0_{\mathrm{m}}$ as $k \geq 6$. Hence, $\mu_2 + \omega g'(\phi^{k,h(k)} = \mu_2 + \frac{\omega}{2\varepsilon}$. By (H3a) and Lemma (4.1), absolute values of eigenvalues of $D\mathbf{T}$ are larger than one as $\phi_3 \in \Omega_{\mathrm{m}}$. By Chen [18], there exists a norm $|\cdot|$ in $\mathbb{R}^3$ such that $|D\mathbf{T}(\Phi)\omega| \geq s|\omega|$ for all $\omega \in \mathbb{R}^3$ and $\phi_3 \in \Omega_{\mathrm{m}}$ for some $s = 1+\rho > 1$. Hence, there exists a sufficiently small radius $d > 0$ such that if Φ' belongs to a d-ball neighborhood of Φ^0, then $|\mathbf{T}^{-1}(\Phi^0) - \mathbf{T}^{-1}(\Phi')| < \frac{1}{s}|\Phi^0 - \Phi'|$. That is, the fixed point Φ^0 is stable and attractive for the map $\mathbf{T}^{-1}$. That is, if a point $\phi^{N,h(N)}$ locates in the basin of Φ^0 for the map $\mathbf{T}^{-1}$, then $\phi^{n,h(n)}$ approaches $\phi^{0,h(0)}$ for $n \geq N$. Actually, the sequence $\{\phi^{n,h(n)} \in \mathbb{R} | n \in \mathbb{N} \cup \{0\}\}$ is a transversal homoclinic orbit of $\phi^{0,h(0)}$ for the neural networks, Eq. (4).

3 A delayed discrete neural network with two neurons

Let us start with a DDNN with two neurons as follows:

$$\begin{aligned} x_1(n) &= \mu_1 x_1(n-1) + \mu_2 x_1(n-\tau_1) + \omega_{11}[g_\varepsilon(x_1(n-\tau_1)) - a_1] \\ &\quad + \omega_{12} g_\varepsilon(x_2(n-\tau_2)) + \nu_1, \qquad (10) \\ x_2(n) &= \mu_1 x_2(n-1) + \mu_2 x_2(n-\tau_3) + \omega_{22}[g_\varepsilon(x_2(n-\tau_3)) - a_2] \\ &\quad + \omega_{21} g_\varepsilon(x_1(n-\tau_4)) + \nu_2, \qquad (11) \end{aligned}$$

where, for simplification, we let $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \tau \in \mathbb{N}$, $a_1 = a_2 = -\rho$, $\omega_{11} = \omega_{22} = \omega$, $n \in \mathbb{N}$, and $g_\varepsilon : \mathbb{R} \to \mathbb{R}$ is a saturated output function as Eq. (3). Besides, the neural network (10)-(11) can be transformed as the following equivalent discrete map on $\mathbb{R}^{2\tau}$:

$$\Phi(n+1) = \mathbf{T}(\Phi(n)), \tag{12}$$

where $\Phi = (\Phi_1, \Phi_2)$ with $\Phi_1 = (\phi^1_\tau, \cdots, \phi^1_1)$, $\Phi_2 = (\phi^2_\tau, \cdots, \phi^2_1)$, and $\mathbf{T} : \mathbb{R}^{2\tau} \to \mathbb{R}^{2\tau}$ is given by

$$\mathbf{T}\begin{pmatrix} \phi^1_\tau \\ \vdots \\ \phi^1_2 \\ \phi^1_1 \\ \phi^2_\tau \\ \vdots \\ \phi^2_2 \\ \phi^2_1 \end{pmatrix} = \begin{pmatrix} \phi^1_{\tau-1} \\ \vdots \\ \phi^1_1 \\ \mu_1\phi^1_1 + \mu_2\phi^1_\tau + \omega[g_\varepsilon(\phi^1_\tau) + \rho] + \omega_{12} g_\varepsilon(\phi^2_\tau) + \nu_1 \\ \phi^2_{\tau-1} \\ \vdots \\ \phi^2_1 \\ \mu_1\phi^2_1 + \mu_2\phi^2_\tau + \omega[g_\varepsilon(\phi^2_\tau) + \rho] + \omega_{21} g_\varepsilon(\phi^1_\tau) + \nu_2 \end{pmatrix},$$

where $\mathbf{T} = (T^1_\tau, \cdots, T^1_1, T^2_\tau, \cdots, T^2_1)$.

Theorem 3.1 *If the parameters satisfy* (H1a) [*resp.* (H1b)]*, then there exists one fixed point in* $\Omega_{j_1\cdots j_n}$ [*resp.* $\Omega_{\mathrm{m}\cdots\mathrm{m}}$] *for the neural network* (10)-(11)*.*

Proof: Consider a fixed region $\Omega_{j_1\cdots j_n}$ for certain j_i = "ℓ" or "m" or "r" $(i = 1, \cdots, n)$. Let $(\widetilde{\xi}_1, \cdots, \widetilde{\xi}_n)$ be any point in $\mathbb{R}^n$. Then, by Lemma 2.1, there exist $\xi^\ell_i \in \Omega_\ell$, $\xi^{\mathrm{m}}_i \in \Omega_{\mathrm{m}}$, $\xi^{\mathrm{r}}_i \in \Omega_{\mathrm{r}}$ such that

$$\xi^*_i = \mu_1\xi^*_i + \mu_2\xi^*_i + \omega[g_\varepsilon(\xi^*_i) + \rho] + \sum^n_{j=1, j\neq i} \omega_{ij} g_\varepsilon(\widetilde{\xi}_j) + \nu_i, \tag{13}$$

where " $*$ " ="ℓ", "m", "r" for each i. Restated, each ξ^*_i is a fixed point of the one-dimensional map $\xi_i \mapsto \mu_1\xi_i + \mu_2\xi_i + \omega[g_\varepsilon(\xi_i) + \rho] + \sum^n_{j=1, j\neq i} \omega_{ij} g_\varepsilon(\widetilde{\xi}_j) + \nu_i$. Let $H : \mathbf{\Omega}_{j_1\cdots j_n} \mapsto \mathbf{\Omega}_{j_1\cdots j_n}$ be defined by $H(\widetilde{\xi}_1, \cdots, \widetilde{\xi}_n) =$

$(\xi_1^{j_1}, \cdots, \xi_n^{j_n})$. We want to show that there exists a fixed point for H. Define $\mathbf{G} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$G_i(\widetilde{\mathbf{x}}, \mathbf{x}) = x_i - \mu_1 x_i - \mu_2 x_i - \omega[g_\varepsilon(x_i) + \rho] - \sum_{j=1, j\neq i}^{n} \omega_{ij} g_\varepsilon(\widetilde{x}_j) - \nu_i,$$

where $\mathbf{G} = (G_1, \cdots, G_n)$, $\widetilde{\mathbf{x}} = (\widetilde{x}_1, \cdots, \widetilde{x}_n)$, $i = 1, \cdots, n$. Notably, $\mathbf{G}(\widetilde{\mathbf{x}}, H(\widetilde{\mathbf{x}})) = 0$ for every $\widetilde{\mathbf{x}} \in \Omega_{j_1 \cdots j_n}$, by Eq. (13). Now,

$$\frac{\partial \mathbf{G}}{\partial \mathbf{x}} = \mathrm{diag}[\chi_1, \cdots, \chi_n],$$

where $\chi_i = 1 - \mu_1 - \mu_2 - \omega g_\varepsilon^{'}(x_i)$, $i = 1, \cdots, n$. Note that $g_\varepsilon^{'} = 0$ in regions Ω_{r} and Ω_ℓ, $g_\varepsilon^{'} = \frac{1}{2\varepsilon}$ in the interior region Ω_{m}. For $\mathbf{x} = (x_1, \cdots, x_n) \in \Omega_{j_1 \cdots j_n}$, we have $\chi_i(x_i) = 1 - \mu_1 - \mu_2 - \frac{\omega}{2\varepsilon}$ or $1 - \mu_1 - \mu_2$ for each i whose values are nonzero due to the condition $0 < (1 - \mu_1 - \mu_2)/\omega < 1/(2\varepsilon)$. Hence, the Jacobian of $\mathbf{G}$ with respect to $\mathbf{x}$ is nonzero. It follows from the implicit function theorem that H is a C^1 function on the region $\Omega_{j_1 \cdots j_n}$. Furthermore, there exists one fixed point $\overline{\mathbf{x}}$ of H in $\Omega_{j_1 \cdots j_n}$, which is also a fixed point of $\mathbf{T}$. In fact, there exists only one fixed point in each $\Omega_{j_1 \cdots j_n}$. Consequently, there are 3^n fixed points of $\mathbf{T}$ in $\mathbb{R}^n$.

Theorem 3.2 *If the parameters satisfy either* (H1a-H3a) *or* (H1b-H3b)*, then there exists a snap-back repeller in the interior* $\Omega_{\mathrm{m} \cdots \mathrm{m}}$ *for the neural network* (10)-(11).

Proof: For simplification, $\tau = 3$ is given. Neural networks (10)-(11) becomes

$$\begin{aligned} x_1(n) &= \mu_1 x_1(n-1) + \mu_2 x_1(n-3) + \omega[g_\varepsilon(x_1(n-3)) + \rho] \\ &\quad + \omega_{12} g_\varepsilon(x_2(n-3)) + \nu_1, \qquad (14) \\ x_2(n) &= \mu_1 x_2(n-1) + \mu_2 x_2(n-3) + \omega[g_\varepsilon(x_2(n-3)) + \rho] \\ &\quad + \omega_{21} g_\varepsilon(x_1(n-3)) + \nu_2. \qquad (15) \end{aligned}$$

By Theorem 3.1, we know that there exists a fixed point $(\phi_1^{0,\mathrm{m}}, \phi_2^{0,\mathrm{m}}) \in \Omega_{\mathrm{mm}}$ for the full couple of neural networks (14) and (15). Let $(\xi_1^{'}, \xi_2^{'})$ be any point in Ω_{rr}. If these parameters $(\mu_1, \mu_2, \omega, \varepsilon, \rho, \nu)$, $i = 1, 2$, satisfy Condition (H2a), there exists a point $(\xi_1, \xi_2) \in \Omega_{\mathrm{rr}}$ such that

$$\mu_2 \xi_1 + \omega[g_\varepsilon(\xi_1) - a_{01}] + \omega_{12} g_\varepsilon(\xi_2^{'}) + a_1 = (1 - \mu_1)\phi_1^{0,\mathrm{m}},$$
$$\mu_2 \xi_2 + \omega[g_\varepsilon(\xi_2) - a_{02}] + \omega_{21} g_\varepsilon(\xi_1^{'}) + a_2 = (1 - \mu_1)\phi_2^{0,\mathrm{m}}.$$

Let $H : \mathbf{\Omega}_{j_1 j_2} \mapsto \mathbf{\Omega}_{j_1 j_2}$ be defined by $H(\widetilde{\xi}_1, \widetilde{\xi}_2) = (\xi_1^{j_1}, \xi_n^{j_2})$. We want to show that there exists a fixed point for H. Define a map $\mathbf{G} : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2$ with $\mathbf{G} = (G_1, G_2)$ by

$$G_1(\mathbf{x}^{'}, \mathbf{x}) = (1 - \mu_1)\phi_1^{0,\mathrm{m}} - \mu_2 x_1 - \omega[g_\varepsilon(x_1) - a_{01}] - \omega_{12} g_\varepsilon(x_2^{'}) - a_1,$$
$$G_2(\mathbf{x}^{'}, \mathbf{x}) = (1 - \mu_1)\phi_2^{0,\mathrm{m}} - \mu_2 x_2 - \omega[g_\varepsilon(x_2) - a_{02}] - \omega_{21} g_\varepsilon(x_1^{'}) - a_2,$$

where $\mathbf{x} = (x_1, x_2)$ and $\mathbf{x}^{'} = (x_1^{'}, x_2^{'})$. Then

$$(\partial \mathbf{G}/\partial \mathbf{x})(\mathbf{x}^{'}, \mathbf{x}) = \begin{pmatrix} -\mu_2 - \omega/2\varepsilon & 0 \\ 0 & -\mu_2 - \omega/2\varepsilon \end{pmatrix}.$$

The Jacobian of $\mathbf{G}$ with respect to $\mathbf{x}$ is nonzero, provided that Condition (H3a) holds. It follows from the implicit function theorem that H is a C^1 function on the region $\Omega_{j_1 j_2}$. Furthermore, there exists one fixed point $\widetilde{\mathbf{x}}$ of H in $\Omega_{j_1 j_2}$ by Brouwer's fixed point theorem. By the implicit function theorem, there exists a point $\widetilde{\mathbf{x}} = (\phi_1^{1,h(1)}, \phi_2^{1,h(1)}) \in \Omega_{\mathrm{rr}}$ such that $\mathbf{G}(\widetilde{\mathbf{x}}, \widetilde{\mathbf{x}}) = \mathbf{0}$. Denoting $\Phi^{-1} = (\Phi_1^{-1}, \Phi_2^{-1})$ with $\Phi_1^{-1} = (\phi_1^{1,h(1)}, \phi_1^{0,h(0)}, \phi_1^{0,h(0)})$ and $\Phi_2^{-1} = (\phi_2^{1,h(1)}, \phi_2^{0,h(0)}, \phi_2^{0,h(0)})$, it follows that $\mathbf{T}(\Phi^{-1}) = \Phi^0$. By the same certification, it is easy to construct a series $\{\Phi^{-i} | i = 2, 3, 4, \cdots\}$ where $\Phi_1^{-k} = \left(\phi_1^{k,h(k)}, \phi_1^{k-1,h(k-1)}, \phi_1^{k-2,h(k-2)}\right)$ and $\Phi_2^{-k} = \left(\phi_2^{k,h(k)}, \phi_2^{k-1,h(k-1)}, \phi_2^{k-2,h(k-2)}\right)$ with an integer $k \geq 2$. Recall that h is defined as Eq. (8). Furthermore, the Jacobian matrix of $\mathbf{T}^{-1}$ in the region $\Omega_{\mathrm{m},\cdots,\mathrm{m}}$ is the inverse of the matrix

$$D\mathbf{T}(\Phi) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \mu_2 + \omega g_\varepsilon^{'}(\phi_3^1) & 0 & \mu_1 & \beta_2 g_\varepsilon^{'}(\phi_3^2) & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ \beta_2 g_\varepsilon^{'}(\phi_3^1) & 0 & 0 & \mu_2 + \omega g_\varepsilon^{'}(\phi_3^2) & 0 & \mu_1 \end{pmatrix}.$$

The absolute values of all of its eigenvalues are larger than one by Gerschgorin's theorem for sufficiently small β_2. Therefore, the series approaches the fixed point in the region $\Omega_{\mathrm{m},\cdots,\mathrm{m}}$ under the action of $\mathbf{T}$. In other words, a homoclinic orbit for the neural network (10)-(11) is constructed. This fixed point Φ^0 is a snap-back repeller.

4 Conclusion

In the paper, we successfully develop a constructive method to explore the existence of a transversal homoclinic orbit for the system (1)-(2) with Eq. (3). The theorems based on the constructive method are theoretically supported by applying Marotto's theorem and have given sufficient conditions for the existence of both fixed points and their homoclinic orbits. The simple system with a delay item can demonstrate very complicated behavior near the origin. The analysis has indicated that, as multiple fixed points coexist, their homoclinic orbits position themselves in a tangle. In addition, the number of fixed points can grow exponentially in the number of neurons. This scenario has revealed the complication of the dynamics for the system. It is believed that more dynamical features other than snap-back repellers can be explored along this line of investigation. It is expected that the present work has great potential in many applications.

Appendix A: Marotto's Theorem

Consider a dynamical system: $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, $\mathbf{x}_k \in \mathbb{R}^n$ and $\mathbf{F}$ is in $C^1(\mathbb{R}^n, \mathbb{R}^n)$ or piecewise C^1. Suppose $\overline{\mathbf{x}}$ is a fixed point of $\mathbf{F}$ with all eigenvalues of $D\mathbf{F}(\overline{\mathbf{x}})$ exceeding 1 in magnitude, and suppose there exists a point $\mathbf{x}_0 \neq \overline{\mathbf{x}}$ in a repelling neighborhood of $\overline{\mathbf{x}}$, such that $\mathbf{F}^m(\mathbf{x}_0) = \overline{\mathbf{x}}$ and $\det(D\mathbf{F}^m(\mathbf{x}_0)) \neq 0$, for some $1 < m \in \mathbb{N}$. Then $\overline{\mathbf{x}}$ is called a *snap-back repeller* [16] of $\mathbf{F}$. If $\mathbf{F}$ has a snap-back repeller, then the system of $\mathbf{F}$ is chaotic in the following sense: (i) There exists a positive integer m_0 such that for each integer $p \geq m_0$, $\mathbf{F}$ has p-periodic points. (ii) There exists a scrambled set, that is, an uncountable set L containing no periodic points such that the following pertains: (a) $\mathbf{F}(L) \subset L$; (b) for every $\mathbf{y} \in L$ and any periodic point $\mathbf{x}$ of $\mathbf{F}$,

$$\limsup_{m\to\infty} ||\mathbf{F}^m(\mathbf{y}) - \mathbf{F}^m(\mathbf{x})|| > 0;$$

(c) for every $\mathbf{x}$, $\mathbf{y} \in L$ with $\mathbf{x} \neq \mathbf{y}$,

$$\limsup_{m\to\infty} ||\mathbf{F}^m(\mathbf{y}) - \mathbf{F}^m(\mathbf{x})|| > 0;$$

(3) There exists an uncountable subset L_0 of L such that for every $\mathbf{x}$, $\mathbf{y} \in L_0$,

$$\liminf_{m\to\infty} ||\mathbf{F}^m(\mathbf{y}) - \mathbf{F}^m(\mathbf{x})|| = 0.$$

Appendix B

Let $A = (a_{ij})$ be a $k \times k$ matrix where $a_{j,j+1} = 1$, $a_{k,1} = a$ and $a_{k,k} = b$; otherwise, $a_{i,j} = 0$. The characteristic polynomial of A is $\lambda^k - b\lambda^{k-1} - a$.

Lemma 4.1 *Consider*

$$\lambda^k - b\lambda^{k-1} - a = 0, \tag{16}$$

where a, b are real numbers and $k \geq 2$ is an integer. If (a) $a > 1$, $|b| < a - 1$, or (b) $a < -1$, $|b| < -a - 1$, then all of λ's solutions have absolute values larger than 1.

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