

Designing Optimal Flow Networks*

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Abstract—We investigate the problem of designing a minimum cost flow network interconnecting n sources and a single sink, each with known locations and flows. The network may contain other unspecified nodes, known as Steiner points. For concave increasing cost functions, a minimum cost network of this sort has a tree topology, and hence can be called a Minimum Gilbert Arborescence (MGA). We characterise the local topological structure of Steiner points in MGAs for linear cost functions. This problem has applications to the design of drains, gas pipelines and underground mine access.

Keywords: *optimisation, networks, network flows, Steiner trees*

1 Introduction

One of the most important advances in physical network design optimisation since the 1960's has been the development of theory for solving the *Steiner Minimum Tree (SMT) problem*. This problem asks for a shortest network spanning a given set of points, called *terminals*, in a given metric space. It differs from the minimum spanning tree problem in that additional points, called *Steiner points*, can be included to create a spanning network that is shorter than would otherwise be possible. This has numerous applications, including the design of telecommunications or transport networks for the problem in the Euclidean plane (the l_2 metric), and the design of microchips for the problem in the rectilinear plane (the l_1 metric) [5].

Gilbert [4] proposed a generalisation of the SMT problem whereby symmetric non-negative *flows* are assigned between each pair of terminals. The aim is to find a least cost network interconnecting the terminals, where each edge has an associated total such that the flow conditions between terminals are satisfied, and Steiner points satisfy Kirchhoff's rule (ie, the net incoming and outgoing flows at each Steiner point are equal). The cost of an edge is its length multiplied by a non-negative *weight*. The weight is determined by a given function of the total

flow being routed through that edge, where the function satisfies a number of conditions. The *Gilbert network problem* (GNP) asks for a minimum-cost network spanning a given set of terminals with given flow demands and a given weight function.

An important variation on this problem that we will show to be a special case of the GNP occurs when the terminals consist of n sources and a unique sink, and all flows not between a source and the sink are zero. This problem has applications to drainage networks [7], gas pipelines [1], and underground mining networks [2].

If the weight function is concave and increasing, the resulting minimum network has a tree topology, and provides a directed path from each source to the sink. Such a network can be called an *arborescence*, and we refer to this special case of the GNP as the *Gilbert arborescence problem* (GAP). Traditionally, the term 'arborescence' has been used to describe a rooted tree providing directed paths from the unique root (source) to a given set of sinks. Here we are interested in the case where the flow directions are reversed, i.e. flow is from n sources to a unique sink. It is clear, however, that the resulting weights for the two problems are equivalent, hence we will continue to use the term 'arborescence' for the latter case. Moreover, if we take the sum of these two cases, and rescale the flows (dividing flows in each direction by 2), then again the weights for the total flow on each edge are the same as in the previous two cases. This justifies our claim that the GAP can be treated as a special case of the GNP. It will be convenient, however, for the remainder of this paper to think of arborescences as networks with a unique sink.

A *minimum Gilbert arborescence* (MGA) is a (global) minimum-cost arborescence for a given set of terminals and flows, and a given cost function. All flows in the network are directed towards the unique sink. MGAs have been used to model drainage networks, such as plumbing networks in buildings [7], as well as gas pipeline networks [1]. In both cases heuristics were provided for obtaining low-cost solutions. A special complication for these networks is that the cost of an edge depends on the diameter of the pipe, and the diameter of the pipe depends on its length. This complication will not be considered in this work.

More recently, MGAs have been used to model under-

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ground mining networks [2]. Given a set of underground locations, called *draw points*, and their estimated ore tonnages, the development and haulage costs associated with an underground mine can be minimised by finding an MGA interconnecting the underground points with a fixed breakout point at the surface. Although the Euclidean metric is useful in some situations, more generally additional constraints must be imposed for truck navigability such as a gradient constraint, which can be handled by altering the metric [2].

In Section 2 we specify the nature of the weight function that we consider in this paper, and formally define minimum Gilbert networks and Gilbert arborescences in Minkowski spaces (which generalise Euclidean spaces). In Section 3 we give a general topological characterisation of Steiner points in such networks, for smooth Minkowski spaces. We then apply this characterisation, in Section 4, to the smooth Minkowski plane with a linear weight function to show that in this case all Steiner points have degree 3.

2 Preliminaries

The cost functions for the networks we consider in this paper make use of more general norms than simply the Euclidean norm. Hence, we introduce a generalisation of Euclidean spaces, namely finite-dimensional normed spaces or Minkowski spaces. See [12] for an introduction to Minkowski geometry.

A *Minkowski space* (or *finite-dimensional Banach space*) is \mathbb{R}^n endowed with a *norm* $\|\cdot\|$, which is a function $\|\cdot\| : \mathbb{R}^n \rightarrow \mathbb{R}$ that satisfies

- $\|x\| \geq 0$ for all $x \in \mathbb{R}^n$, $\|x\| = 0$ only if $x = 0$,
- $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in \mathbb{R}^n$, and
- $\|x + y\| \leq \|x\| + \|y\|$.

We now discuss some aspects of the SMT problem, since this is a special case of the GNP, where all flow are zero. Our terminology for the SMT problem is based on that used in [5]. Let T be a network interconnecting a set $N = \{p_1, \dots, p_n\}$ of points, called *terminals*, in a Minkowski space. A vertex in T which is not a terminal is called a *Steiner point*. Let $G(T)$ denote the *topology* of T , i.e. $G(T)$ represents the graph structure of T but not the embedding of the Steiner points. Then $G(T)$ for a shortest network T is necessarily a tree, since if a cycle exists, the length of T can be reduced by deleting an edge in the cycle. A network with a tree topology is called a *tree*, its links are called *edges*, and its nodes are called *vertices*. An edge connecting two vertices a, b in T is denoted by ab , and its length by $\|a - b\|$.

The *splitting* of a vertex is the operation of disconnecting two edges av, bv from a vertex v and connecting a, b, v

to a newly created Steiner point. Furthermore, though the positions of terminals are fixed, Steiner points can be subjected to arbitrarily small movements provided the resulting network is still connected. Such movements are called *perturbations*, and are useful for examining whether the length of a network is minimal.

A *Steiner tree* (ST) is a tree whose length cannot be shortened by a small perturbation of its Steiner points, even when splitting is allowed. By convexity, an ST is a minimum-length tree for its given topology. A *Steiner minimum tree* (SMT) is a shortest tree among all STs. For many Minkowski spaces bounds are known for the maximum possible degree of a Steiner point in an ST, giving useful restrictions on the possible topology of an SMT. For example, in Euclidean space of any dimension every Steiner point in an ST has degree three. Given a set N of terminals, the *Steiner problem* (or *Steiner Minimum Tree problem*) asks for an SMT spanning N .

Gilbert [4] proposed the following generalisation of the Steiner problem in Euclidean space, which we now extend to Minkowski space. Let T be a network interconnecting a set $N = \{p_1, \dots, p_n\}$ of n terminals in a Minkowski space. For each pair p_i, p_j , $i \neq j$ of terminals, a non-negative *flow* $t_{ij} = t_{ji}$ is given. The cost of an edge e in T is $w(t_e)l_e$, where l_e is the length of e , t_e is the total flow being routed through e , and weight $w(\cdot)$ is a unit cost function satisfying

$$w(t) \geq 0 \quad \text{and} \quad w(t) > 0 \text{ if } t > 0 \quad (1)$$

$$w(t_1 + t_2) \geq w(t_1) \quad \forall t_2 > 0 \quad (2)$$

$$w(t_1 + t_2) \leq w(t_1) + w(t_2) \quad \forall t_1, t_2 > 0 \quad (3)$$

$$w(\cdot) \text{ is concave} \quad (4)$$

A network satisfying Conditions (1) - (3) is called a *Gilbert network*. For a given edge e in T , $w(t_e)$ is called the *weight* of e , and is also denoted simply by w_e . The *total cost* of a Gilbert network T is the sum of all edge costs, i.e.

$$C(T) = \sum_{e \in E} w(t_e)l_e$$

where E is the set of all edges in T . A Gilbert network T is a *minimum Gilbert network* (MGN), if T has the minimum cost of all Gilbert networks spanning the same point set N , with the same flow demands t_{ij} and the same cost function $w(\cdot)$. By the arguments of [3], an MGN always exists in a Minkowski space when Conditions (1) - (4) are assumed for the weight function.

Conditions (1) - (3) above ensure that the weight function is non-negative, non-decreasing and triangular, respectively. These are natural conditions for most applications. Unfortunately, the first three conditions alone

do not guarantee that a minimum Gilbert network is a tree and an example of such a minimum network where the flow necessarily splits will be given in a later paper. (Note that in [3], Condition (3), which we call the *triangular condition*, was incorrectly interpreted as concavity of the cost function.)

The *Gilbert network problem* (GNP) is to find an MGN for a given terminal set N , flows t_{ij} and cost function $w(\cdot)$. Since its introduction in [4], various aspects of the GNP have been studied, although the emphasis has been on discovering geometric properties of MGNs (see [3], [11], [13], [14]). As in the Steiner problem, additional vertices can be added to create a Gilbert network whose cost is less than would otherwise be possible, and these additional points are again called *Steiner points*. A Steiner point s in T is called *locally minimal* if a perturbation of s does not reduce the cost of T . A Gilbert network is called *locally minimal* if no perturbation of the Steiner points reduces the cost of T .

The special case of the Gilbert model that is of interest in this work is when $N = \{p_1, \dots, p_n, q\}$ is a set of terminals in a Minkowski space, where p_1, \dots, p_n are *sources* with respective positive flows t_1, \dots, t_n , and q is the *sink*. All flows are between the sources and the sink; there are no flows between sources. It has been shown in [11] that concavity of the weight function implies that an MGN of this sort is a tree. Hence we refer to an MGN with this flow structure as a *minimum Gilbert arborescence* (MGA), and, as mentioned in the introduction, we refer to the problem of constructing such an MGA as the *Gilbert arborescence problem* (GAP).

If v_1 and v_2 are two adjacent vertices in a Gilbert arborescence, and flow is from v_1 to v_2 then we denote the edge connecting the two vertices by v_1v_2 .

3 Characterisation of Steiner Points

In this section, we generalise a theorem of Lawlor and Morgan [6] to give a local characterisation of Steiner points in an MGA. The characterisation in [6] holds for SMTs, which correspond to the case of MGAs with a constant weight function. Their theorem is formulated for arbitrary Minkowski spaces with differentiable norm. Our proof is based on the proof of Lawlor and Morgan's theorem given in [8]. A generalisation to non-smooth norms is contained in [10] for SMTs and in [15] for MGAs. Such a generalisation is much more complicated and involves the use of the subdifferential calculus.

We first introduce some necessary definitions relating to Minkowski geometry, in particular with relation to dual spaces. For more details, see [12].

We denote the inner product of two vectors $x, y \in \mathbb{R}^n$ by $\langle x, y \rangle$. For any given norm $\|\cdot\|$, the dual norm $\|\cdot\|^*$ is

defined as follows:

$$\|z\|^* = \sup_{\|x\| \leq 1} \langle z, x \rangle.$$

We say that a Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is *smooth* if the norm is differentiable at any $x \neq o$, i.e., if

$$\lim_{t \rightarrow 0} \frac{\|x + th\| - \|x\|}{t} =: f_x(h)$$

exists for all $x, h \in \mathbb{R}^n$ with $x \neq o$. It follows easily that f_x is a linear operator $f_x : \mathbb{R}^n \rightarrow \mathbb{R}$ and so can be represented by a vector $x^* \in \mathbb{R}^n$, called the dual vector of x , such that $\langle x^*, y \rangle = f_x(y)$ for all $y \in \mathbb{R}^n$, and $\|x^*\|^* = 1$. In fact x^* is just the gradient of the norm at x , i.e., $x^* = \nabla\|x\|$.

More generally, even if the norm is not differentiable at x , a vector $x^* \in \mathbb{R}^n$ is a *dual vector* of x if x^* satisfies $\langle x^*, x \rangle = \|x\|$ and $\|x^*\|^* = 1$. By the Hahn-Banach separation theorem, each non-zero vector in a Minkowski space has at least one dual vector. A Minkowski space is then smooth if and only if each non-zero vector has a unique dual vector.

A norm is *strictly convex* if $\|x\| = \|y\| = 1$ and $x \neq y$ imply that $\|\frac{1}{2}(x + y)\| < 1$, or equivalently, that the *unit sphere*

$$S(\|\cdot\|) = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

does not contain any straight line segment. A norm $\|\cdot\|$ is smooth [strictly convex] if and only if the dual norm $\|\cdot\|^*$ is strictly convex [smooth, respectively].

Theorem 1. *Suppose a smooth Minkowski space $(\mathbb{R}^n, \|\cdot\|)$ is given together with a concave weight function w , sources $p_1, \dots, p_n \in \mathbb{R}^n$, and a single sink $q \in \mathbb{R}^n$, all different from the origin o . Let the flow associated with p_i be t_i . (See Figure 1.) For each p_i let p_i^* denote its*

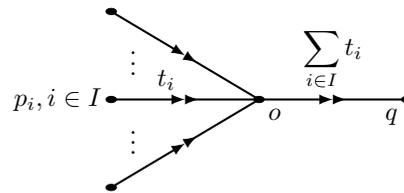


Figure 1: A Gilbert network with star topology.

dual vector, and let q^ denote the dual vector of q . Then the Gilbert arborescence with edges op_i , $i = 1, \dots, n$ and oq , where all flows are routed via the Steiner point o , is a minimal Gilbert arborescence if and only if*

$$\sum_{i=1}^n w(t_i)p_i^* + w\left(\sum_{i=1}^n t_i\right)q^* = o \tag{5}$$

and

$$\left\| \sum_{i \in I} w(t_i)p_i^* \right\|^* \leq w\left(\sum_{i \in I} t_i\right) \text{ for all } I \subseteq \{1, \dots, n\}. \tag{6}$$

Note: We think of Condition 5 as a flow-balancing condition at the Steiner point, and Condition 6 as a condition that ensures that the Steiner point does not split.

Proof. (\Rightarrow) We are given that the star is not more expensive than any other Gilbert network with the same sources, sink, flows and weight function.

In particular, o is the so-called weighted Fermat-Torricelli point of p_1, \dots, p_n, q with weights $t_1, \dots, t_n, \sum_{i=1}^n t_i$, respectively, which implies the balancing condition (5). We include a self-contained proof for completeness. If the Steiner point o is moved to $-te$, where $t \in \mathbb{R}$ and $e \in \mathbb{R}^n$ is a unit vector (in the norm), the resulting arborescence is not better, by the assumption of minimality. Therefore, the function

$$\begin{aligned} \varphi_e(t) &= \sum_{i=1}^n w(t_i)(\|p_i + te\| - \|p_i\|) \\ &\quad + w\left(\sum_{i=1}^n t_i\right)(\|q + te\| - \|q\|) \geq 0 \end{aligned}$$

attains its minimum at $t = 0$. For t in a sufficiently small neighbourhood of 0, $p_i + te \neq o$ and $q + te \neq o$, hence φ_e is differentiable. Therefore,

$$\begin{aligned} 0 &= \varphi'_e(0) = \lim_{t \rightarrow 0} \left(\sum_{i=1}^n w(t_i) \frac{\|p_i + te\| - \|p_i\|}{t} \right. \\ &\quad \left. + w\left(\sum_{i=1}^n t_i\right) \frac{\|q + te\| - \|q\|}{t} \right) \\ &= \sum_{i=1}^n w(t_i) \langle p_i^*, e \rangle + w\left(\sum_{i=1}^n t_i\right) \langle q^*, e \rangle \\ &= \left\langle \sum_{i=1}^n w(t_i) p_i^* + w\left(\sum_{i=1}^n t_i\right) q^*, e \right\rangle. \end{aligned}$$

Since this holds for all unit vectors e , (5) follows.

To show (6) for each $I \subseteq \{1, \dots, n\}$, we may assume without loss of generality that $I \neq \emptyset$ and $I \neq \{1, \dots, n\}$. Consider the Gilbert network obtained by splitting the Steiner point into two points o and $+te$ ($t \in \mathbb{R}$, e a unit vector) as follows. Each p_i , $i \notin I$, is still adjacent to o with flow t_i , and q is joined to o with flow $\sum_{i=1}^n t_i$, but now each p_i , $i \in I$, is adjacent to te with flow t_i , and te is adjacent to o with flow $\sum_{i \in I} t_i$, as shown in Figure 2.

Since the new network cannot be better than the original star, we obtain that for any unit vector e , the function

$$\psi_e(t) = \sum_{i \in I} w(t_i)(\|p_i - te\| - \|p_i\|) + w\left(\sum_{i=1}^n t_i\right)|t| \geq 0$$

attained its minimum at $t = 0$. Although ψ_e is not dif-

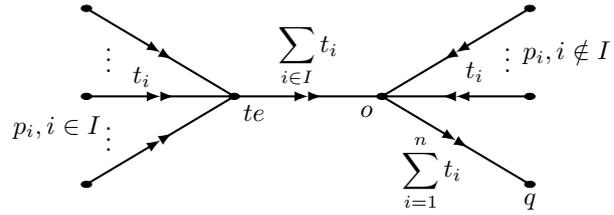


Figure 2: The Gilbert network obtained by splitting the Steiner point o .

ferentiable at 0, we can still calculate as follows:

$$\begin{aligned} 0 &\leq \lim_{t \rightarrow 0^+} \frac{\psi_e(t)}{t} \\ &= \lim_{t \rightarrow 0^+} \sum_{i \in I} w(t_i) \frac{\|p_i - te\| - \|p_i\|}{t} + w\left(\sum_{i=1}^n t_i\right) \\ &= \left\langle \sum_{i \in I} w(t_i) p_i^*, -e \right\rangle + w\left(\sum_{i=1}^n t_i\right). \end{aligned}$$

Therefore, $\langle \sum_{i \in I} w(t_i) p_i^*, e \rangle \leq w(\sum_{i=1}^n t_i)$ for all unit vectors e , and (6) follows from the definition of the dual norm.

(\Leftarrow) Now assume that p_1^*, p_n^*, q are dual unit vectors that satisfy (5) and (6). Consider an arbitrary Gilbert arborescence T for the given data. For each i , let P_i be the path in T from p_i to q , i.e., $P_i = x_1^{(i)} x_2^{(i)} \dots x_{k_i}^{(i)}$, where $x_1^{(i)} = p_i$, $x_{k_i}^{(i)} = q$, and $x_j^{(i)} x_{j+1}^{(i)}$ are distinct edges of T for $j = 1, \dots, k_i - 1$. For each edge e of T , let $S_e = \{i : e \text{ is on path } P_i\}$. Then the flow on e is $\sum_{i \in S_e} t_i$ and the total cost of T is

$$\sum_{\substack{e=xy \text{ is} \\ \text{an edge of } T}} w\left(\sum_{i \in S_e} t_i\right) \|x - y\|.$$

The cost of the star is

$$\begin{aligned} &\sum_{i=1}^n w(t_i) \|p_i\| + w\left(\sum_{i=1}^n t_i\right) \|q\| \\ &= \sum_{i=1}^n w(t_i) \langle p_i^*, p_i \rangle + w\left(\sum_{i=1}^n t_i\right) \langle q^*, q \rangle \\ &= \sum_{i=1}^n w(t_i) \langle p_i^*, p_i - q \rangle \quad \text{by (5)} \\ &= \sum_{i=1}^n w(t_i) \sum_{j=1}^{k_i-1} \langle p_i^*, x_j^{(i)} - x_{j+1}^{(i)} \rangle \end{aligned}$$

$\|x\| + \|y\|$ for any x, y such that the unit vectors $\frac{1}{\|x\|}x$ and $\frac{1}{\|y\|}y$ lie on this segment. In particular,

$$\|s_2 - s_1\| + \|s_1 - o\| = \|s_2 - o\|. \quad (8)$$

Now replace p_2o by the edges p_2s_2 and s_2s_1 , replace p_1o by p_1s_1 and s_1o , and add the flow t_2 to s_1o . The change in cost in the new Gilbert arborescence is

$$\begin{aligned} & (w(t_1)\|p_1 - s_1\| + w(t_1 + t_2)\|s_1 - o\| \\ & \quad + w(t_2)\|p_2 - s_2\| + w(t_2)\|s_2 - s_1\|) \\ & \quad - (w(t_1)\|p_1 - o\| - w(t_2)\|p_2 - o\|) \\ = & -w(t_1)\|s_1\| - w(t_2)\|s_2\| + w(t_1 + t_2)\|s_1\| \\ & \quad + w(t_2)(\|s_2\| - \|s_1\|) \quad \text{by (8)} \\ = & (w(t_1 + t_2) - w(t_1) - w(t_2))\|s_1\| \\ = & (d + h(t_1 + t_2) - (d + ht_1) - (d + ht_2))\|s_1\| \\ = & -d\|s_1\| < 0. \end{aligned}$$

We have shown that a Steiner point of degree at least 4 leads to a decrease in cost. In an MGA a Steiner point must then necessarily be of degree 3. \square

5 Conclusion

In this paper we have studied the problem of designing a minimum cost flow network interconnecting n sources and a single sink, each with known locations and flows, in general finite-dimensional normed spaces. The network may contain other unprescribed nodes, known as Steiner points. For concave increasing cost functions, a minimum cost network of this sort has a tree topology, and hence can be called a Minimum Gilbert Arborescence (MGA). We have characterised the local topological structure of Steiner points in MGAs for linear weight functions, specifically showing that Steiner points necessarily have degree 3.

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