Financial Modelling with Ornstein–Uhlenbeck Processes Driven by Lévy Process

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***Abstract*—In this study we deal with aspects of the modeling of the asset prices by means Ornstein-Uhlenbech process driven by Lévy process. Barndorff-Nielsen and Shephard stochastic volatility model allows the volatility parameter to be a self-decomposable distribution. BNS models allow flexible modeling. For this reason we use as a model IG-Ornstein-Uhlenbeck process. We calibrate moments of Lévy process and OU process. Finally we fit the model some real data series. We present a simulation study.**

***Index Terms*—, Barndorff-Nielsen and Shephard model, Financial market, IG-Ornstein-Uhlenbeck process, Lévy processes**

I. INTRODUCTION

Empirical studies have also shown that the volatility is not constant as postulated by famous Black-Scholes model. In reality the logarithmic return distribution has fatter than the normal distribution implies. The characteristic properties of logarithmic returns are high kurtosis and negative skewness. These facts can not explain assumption of a constant volatility. Volatility has a stochastic structure. Therefore a mean-reverting dynamics can be suitable candidate for the modeling of volatility. The stock market prices evolve freely but other a lot of real asset, price spreads are observed in the short time, but in the long time, the demand of product is adjusted and the prices move towards around the level of production cost of asset. The stochastic volatility models are driven by Lévy processes is introduced by[8], [9] The Bates model is simpler but in this model jumps and stochastic volatility are independent. BNS model denotes a connection of jumps and stochastic volatility.

A Brownian motion may be a good model for a particle movement. After a hit the particle does not stop after changing position, but it moves continuously with decreasing speed. The Brownian motion is not differentiable anywhere . Ornstein -Uhlenbeck process was proposed by Uhlenbeck and Ornstein (1930) to improvement the model.

The paper is organized as follows. Section 2 reviews well known properties of Lévy process. In section 3 we set up OU-processes. We explain estimators. In section 4 we fit the model real data. Finally, the section 5 include conclusions.

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II. LÉVY PROCESSES

Definition 2.1 (Lévy process) A Lévy processes a continuous time stochastic processes $(L_t)_{t\geq 0}$ with

1) $L_0 = 0$
2) *Stationary increments*
For all $> 0, t > 0$, $L_{t+s} - L_t$ has the same distribution as L_s
3) *Independent increments*
For all $0 \leq t_0 < t_1 < \cdots < t_n$, $L_{t_i} - L_{t_{i-1}}$ $(i = 0,1,2,\ldots,n)$ are independent.
4) *Cadlağ paths*
The sample paths of a Lévy process are right continuous and left limits.

*Remark***:** In a Lévy processes discontinuous occurs at random times.

Brownian motion $B_t \sim N(\mu t, \sigma^2 t)$ and Poisson process $N_t \sim Pois(\lambda t)$ for some density $\lambda \in (0,\infty)$ are levy process.

The jumps of Lévy process $\Delta L_t = L_t - L_{t^-}$ are very important to understand structure levy processes.

Levy measure v is a measure satisfying $v\{0\} = 0$ and

$$\int_{-\infty}^{\infty} (|x|^2 \wedge 1)\, v(dx) < \infty \qquad (2.1)$$

For any Borel subset B of $\mathcal{R}\backslash\{0\}$,

$$v(B) = E[\#\{t \in [0,1] : \Delta L_t \neq 0 , \Delta L_t \in B\}] \qquad (2.2)$$

$v(B)$ is the expected number ,per unit time of jumps whose size belong to B (Tankov temperit stable,p.4).
Thus $v(dx)$ is intensity of jumps of size x.

Let $\Psi(u)$ is the characteristic function of a random variable L, If for every positive integer n , there exist a random variable $L^{(1/n)}$ such that,

$$\Psi_L(u) = \left(\Psi_{L^{(1/n)}}(u)\right)^n \qquad (2.3)$$

We say that the distribution of L is infinitely divisible. Anyone can define for any infinitely divisible distribution a stochastic process, $L = (L_t)_{t\geq 0}$ called Lévy process [2,p.44], [7,p.8].

Theorem (Lévy Khintchine representation). Let $(L_t)_{t\geq 0}$ be a Lévy process. The characteristic function $(L_t)_{t\geq 0}$ is of the form ,

$$E[e^{iuL_t}] = e^{t\Psi(u)}$$

Where $\Psi(u)$ is cumulant of L_1 given by the Lévy-Khintchine formula,

$$\Psi(u) = ibu - \frac{\sigma^2 u^2}{2} + \int_{-\infty}^{\infty} \left(e^{iux} - 1 - iux1_{\{|x|\leq 1\}}\right) v(dx)$$

(2.4) (b, σ^2, v) is called Lévy triplet.

The Lévy – Ito decomposition reveals the path structure of a Lévy process.

Theorem **(Levy-Ito Decomposition).** Let $(L_t)_{t\geq 0}$ be a Lévy process and v its Lévy measure and verifies,

$\int_{|x\leq 1|} |x|^2 \, v(dx) < \infty$ and $\int_{|x|\geq 1} v(dx) < \infty$

$$L_t = bt + \sigma B_t + \lim_{\varepsilon \downarrow 0}\{L_t^\varepsilon\} \qquad (2.5)$$

$$L_t^\varepsilon = \sum_{s\leq t} \Delta L_s 1_{\{|\Delta L_s > \varepsilon|\}} - t\int_{\varepsilon < |x| \leq 1} x \, v(dx) \qquad (2.6)$$

The *subordinators* are special case of Lévy process. All subordinators are pure upward jumping process. It has non decreasing sample paths(i.e Poisson and IG Lévy processes are subordinators)

Definition (Self-decomposability). Let $\Psi(u)$ be the characteristic function of random variable . We call X self-decomposable if

$$\Psi(u) = \Psi(cu)\Psi_c(u) \qquad (2.7)$$

For all $u \in R$ and all $c \in (0,1)$ and for some family of characteristic functions$\{\Psi_c : c \in (0,1)\}$[2,p.47].

Let $v(dx)$ denote the a Lévy measure of infinitely divisible measure P. The form of $v(dx)$ is $v(dx) = u(x)dx$ the such $|x|u(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$. [2,p.48]. Let $W(dx)$ denotes the Lévy measure of L_1 . If the Lévy density u of the self-decomposable law D is differentiable , then the Lévy measure W has a density w and u and w are related by

$$w(x) = -u(x) - x\frac{du(x)}{dx} \qquad (2.8)$$

Theorem : For any Lévy process $L = (L_t)_{t\geq 0}$ and for a function f ,satisfying regularity conditions,

$$ln\left[E\left(exp\{iu\int_{R+} f(s)dL_t\}\right)\right] = \int_{R+} ln[E(exp\{iuf(s)L_1\})]ds \qquad (2.9)$$

For proof, you can look [9] , [1].

Theorem : A random variable X is self-decomposable if and only if it there exist a Lévy process $L = (L_t)_{t\geq 0}$ such that X has representation of the form,

$$X = \int_0^\infty e^{-s} \, dL_s \qquad (2.10)$$

$v(dx)$ and $\mu(dx)$ are Lévy measures of respectively X and L. They are related by [1,p.31].

$$v(dx) = \int_0^\infty \mu(e^s dx)ds \qquad (2.11)$$

III. ORNSTEIN-UHLENBECK PROCESSES

Ornstein-Uhlenbeck process was proposed by Uhlenbeck and Ornstein (1930) as an alternative to Brownian motion.This process was driven by a Brownian motion with drift that is a Lévy process.

OU Process driven Brownian Motion

A one dimensional Gaussian OU process $X = (X_t)_{t\geq 0}$ can be defined as the solution to the stochastic differential equation,

$$dY_t = -\lambda Y_t dt + \sigma dW_t \qquad (3.1)$$

If X_t is the interest rate at time t and m is a reference value for the rate,

$$dX_t = -\lambda(X_t - m)dt + \sigma dW_t\,, \quad X_0 = x_0 \qquad (3.2)$$

with $\sigma > 0$ and $\lambda > 0$. Let $Y_t = X_t - m$. We get

$$dY_t = dX_t = -\lambda Y_t dt + \sigma dW_t \qquad (3.3)$$

So , $e^{\lambda t}dY_t + \lambda e^{\lambda t}Y_t dt = \sigma e^{\lambda t}dW_t$ consequently, $d(e^{\lambda t}Y_t) = \sigma e^{\lambda t}dW_t$. Let $Z_t = e^{\lambda t}Y_t$ $(Z_0 = x_0 - m)$

We obtain $Z_t = (x_0 - m) + \int_0^t \sigma e^{\lambda s}dW_s$ so

$$\begin{aligned} X_t &= Y_t + m = e^{-\lambda t}Z_t + m \\ &= e^{-\lambda t}\left((x_0 - m) + \int_0^t \sigma e^{\lambda s}dW_s\right) + m \\ &= m + e^{-\lambda t}(x_0 - m) + \sigma e^{-\lambda t}\int_0^t e^{\lambda s}dW_s \qquad (3.4) \end{aligned}$$

X_t is unique strong markov solution of (3.2) [19,p.298]. Finally we obtain that

$$X_t \sim N\left(m + e^{-\lambda t}(x_0 - m)\,,\ \frac{\sigma^2}{2\lambda}\left(1 - e^{2\lambda t}\right)\right) \qquad (3.5)$$

This distribution as $t \to \infty$ to the stationary distribution $N\left(m, \frac{\sigma^2}{2\lambda}\right)$. The probability distribution of X_t approach an equilibrium probability distribution called the stationary distribution. This stationary distribution has a stationary density function. For a time changed Brownian motion , another representation is here ,

$$X_t = m + e^{-\lambda t}(x_0 - m) + \sigma e^{-\lambda t}W_{(e^{2\lambda t} - 1)/2\lambda} \qquad (3.6)$$

$$E(X_t) = e^{-\lambda t} + m\left(1 - e^{-\lambda t}\right) \qquad (3.7)$$

$$Var(X_t) = \frac{\sigma^2}{2\lambda}\left(1 - e^{2\lambda t}\right) \qquad (3.8)$$

Theorem : The correlation function of X_t is

$$Corr[X_t, X_{t+k}] = \frac{e^{-\lambda k}\left(1 - e^{-2\lambda t}\right)}{\sqrt{\left(1 - e^{-2\lambda t}\right)\left(1 - e^{-2\lambda(t+k)}\right)}} \qquad (3.9)$$

When $t \to \infty$ the correlation of X_t tents to,

$$\lim_{t\to\infty} Corr[X_t, X_{t+k}] = e^{-\lambda k} \qquad (3.10)$$

OU Process driven General Lévy Processes

Let $L = (L_t)_{t\geq 0}$ is a time homogeneous Lévy process, for $\lambda > 0$, Ornstein-Uhlenbeck (OU) type process has

$$X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)}dL_s$$
$$= e^{-\lambda t}X_0 + e^{-\lambda t}\int_0^t e^{\lambda s}\, dL_s \qquad (3.11)$$

It is unique strong solution below SDE,

$$dX_t = -\lambda\, X_t dt + \sigma dL_t \;, X_0 = x_0 \qquad (3.12)$$

Where λ denotes the rate of decay. The λ enters the stationary solution of OU process. This leads difficulties solution of SDE. We can remove this difficulties by a simple change of time in the stochastic integrals [8,p.75]. We can rewrite OU process as follows

$$X_t = e^{-\lambda t}X_0 + \int_0^t e^{-\lambda(t-s)}dL_{\lambda s} \qquad (3.13)$$

If $Y = (Y_t)$ is an OU process with marginal law D , then we say that Y is a D-OU process. When given a one dimensional distribution D there exist a stationary OU process whose marginal law is D if and only if D is self-decomposable [16]. We have result that[8] ,

$$X_t = e^{-\lambda t}X_0 + e^{-\lambda t}\int_0^{\lambda t} e^{s}dL_s$$

Proposition: For any $t, h > 0$, [19] ,

$$e^{-\lambda t}\int_t^{t+h} e^{\lambda s}dL_{\lambda s} = \int_0^h e^{\lambda s}\, dL_{\lambda s}$$

Remark: $L^*(h) = \int_0^h e^{\lambda s}\, dL_{\lambda s} = \int_0^{\lambda h} L_s\, ds$

In the case of a D_OU process, process L_t denotes the background driving Lévy process(BDLP). We can write that relation between the characteristic functions of the BDLP Ψ_{L_1} and Ψ_X ,

$$ln\left[\Psi_{L_1}(u)\right] = u\, \frac{dln(\Psi_X(u))}{du} \qquad (3.14)$$

Let us denote by $k_D(u)$ the cumulant function of the self-decomposable law D and $k_L(u)$ the cumulant function of the BDLP at time $t = 1$ i.e $k_L(u) = \log E[e^{-uL_1}]$ Other words $k(u) = k_L(u) = \log E[e^{-uL_1}]$ is the cumulant function of L_1 [8] . We can say that the cumulant function of X_t can be directly found from the cumulant function of L_1.

If v_L denotes the Lévy measure of L_1, we will assume that

$$\int_R x^2 v_L(dx) \; < \infty$$

and we shall write $E(L_1) = \mu$ and $Var(L_1) = \sigma^2$ and we will assume that X_0 is independent of L and that [16] ,[10,p.3],

$$X_0 = \int_0^\infty e^{-s}dL_s$$

Parameter Estimation of Model

We will use moments estimation methods to estimate the model parameters (μ, σ^2, λ). We will take discrete spaced observations. Let $L = (L_t)_{t\geq 0}$ is a levy process, then $E(X_0) = \mu$ and $Var(X_0) = \sigma^2/2$ (for detail, [10,p.4]. In this section our aim is match the theoretical moments and empirical moments
Theoretical auto covariance and auto correlation of X_t is given by

$$a)\ \gamma(k) = Cov(X_{t+k}, X_t) = \frac{\sigma^2}{2}e^{-\lambda k}$$

$$b)\ \rho(k) = Corr(X_{t+k}, X_t) = e^{-\lambda k}$$

We can be write autocorrelation and empirical moments of time series X as follows,

$) \ \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$

$b)\ s^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$c)\ \widehat{\gamma_n} = (\hat{\gamma}(0), \hat{\gamma}(1), \dots, \hat{\gamma}(d)) \;, k \in \{0,1,\dots,d\}$

$$\hat{\gamma}(k) = \frac{1}{n}\sum_{i=1}^{n-k}(X_{i+k} - \bar{X})(X_i - \bar{X})$$

$d)\ \widehat{\rho_n} = (\hat{\rho}(0), \hat{\rho}(1), \dots, \hat{\gamma}(d)) \;, k \in \{0,1,\dots,d\}$

$$\hat{\rho}(k) = \frac{\hat{\gamma}(k)}{\hat{\rho}(0)}$$

Finally strongly consistent estimators of μ, σ^2, λ respectively [10:p.7],

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} X_i$$
$$\hat{\sigma}^2 = \frac{2}{n}\sum_{i=1}^{n}(X_i - \hat{\mu})^2$$
$$\hat{\lambda}_1 = -log(\hat{\rho}(1)) \qquad (3.15)$$
$$\hat{\lambda}_2 = \, min_\lambda \sum_{k=1}^{d}\left(\hat{\rho}(k) - e^{-\lambda k}\right)^2$$
$$\hat{\lambda}^* \;= min\{0, \hat{\lambda}_2\}$$

We can use the as an estimator of λ [10:p.7] .

Likelihood Estimation for a IG-OU Process

We can estimate parameters of IG process using observations $x_0, x_1, x_2, \dots, x_n$ sample $X_0, X_h, \dots, X_{nh}$ of X .

The initial estimates of a and b

- $Y_k = \int_{\lambda(k-1)h}^{\lambda kh} e^s dL_s = e^{\lambda h}X_{kh} - X_{(k-1)h}$, $k = 1,2,\dots,n$
- $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_k$ *and* $s_Y^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_k - \bar{Y})^2$

$$\hat{a} = \frac{\hat{b}\bar{Y}}{(e^{\lambda h}-1)} \quad and \quad \hat{b} = \frac{1}{s_Y}\sqrt{\frac{\bar{Y}(e^{2\lambda h}-1)}{(e^{\lambda h}-1)}} \qquad (3.16)$$

The Simulation of IG-OU Process

We simulate the paths of process by means of the Euler scheme,

$$X_{hk} = e^{-\lambda h}X_{(h(k-1)} + \sum_{k=1}^{100} e^{k\tau}L_\tau \qquad (3.17)$$

$L = \{L_t\}$ is the corresponding BDLP of X *and* $\tau = 0{,}0005$ [13:p.103].

Simulation of an IG process

We use the IG random number generator proposed by [2:p.111-112].

Generate a normal random number N

1) *Set* $Y = N^2$
2) *Set* $X_1 = (a/b) + (Y/2b^2) - \left[\sqrt{4abY + Y^2}/(2b^2)\right]$
3) *Generate a uniform random number U*
4) *If* $U \le a/(a + X_1 b)$ *then return* $X = X_1$, *else return* $X = a^2/(b^2X_1)$

Sample path of an IG process

We simulate a sample path of an IG process $L = \{L_t, t \ge 0\}$ the value of this process ay time points $\{nh; n =$ *0,1,2,... as follows,*

• Simulate n , i.i.d IG random variable I_n with parameter $IG(ah, b)$, $L_0 = 0$.

$$L_{nh} = L_{(n-1)h} + I_n \quad , \qquad n \ge 1$$

IV. The Modelling of Data

We describe the stock price process,

$$S_t = S_0 exp\left[\int_0^t \left(\mu + \left(\beta - \frac{1}{2}\right)\sigma_t^2\right) ds + \int_0^t \sigma_s dW_s\right] \qquad (4.1)$$

$$\int_{t-1}^t \sigma_s dW_s \cong \sigma_s\varepsilon \qquad ,\varepsilon \in N(0,1) \qquad (4.2)$$

Integrated volatility is difined as integral of the spot volatility

$$IV_t = \int_{t-1}^t \sigma_s^2 \; ds$$

A non parametric measure for integrated volatility is realized volatility. The realized volatility can be estimated by the sum of intra daily squared return

$$RV_t = \sum_{i=1}^{M} r_i^2 \quad , \qquad t = 1, \dots, T$$

M is the number of intra day observations.

If we use the discrete version of price process,

$$r_t = \mu + \beta\sigma_t^2 + \sigma_t\,\varepsilon_t \qquad (4.3)$$

Where μ , is the rate of return and β is the skewness parameter of the return. r_t denotes the return process. If σ^2 has an inverse Gaussian distribution then,

$$r = \mu + \beta\sigma^2 + \sigma\varepsilon$$

It has a normal inverse Gaussian distribution[18:p.280-281].

This r is an average of the continuous time volatility on one trading day.

Remarks: We assume that volatility process is a constant times the number of trades on each trading day.

$$\sigma_t^2 = \gamma\, v_t \qquad (4.5)$$

γ is a constant and %95 confidence interval for γ is $(2.065x10^{-7}, \; 2.414x10^{-7})$ We use $\gamma = 2.2395x10^{-7}$ constant value.

Application to Real Data

Our data set consist of General Motors stock prices from 1/2/1990 through 10/12/2007.The total number of observations is 4481.The parameters are calculated using with (3.15) formulas.

Table 1. Parameter estimation for close prices of GM

Parameter	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
Estimated value	47.81039	421.238	0.003005	0.012818

Table2. Parameter estimation for return of GM

Parameter	$\hat{\mu}$	$\hat{\sigma}^2$	$\hat{\lambda}_1$	$\hat{\lambda}_2$
Estimated value	0.000024	0.000906	3.985	0,022507

Figure1. Autocorrelation for close prices of GM

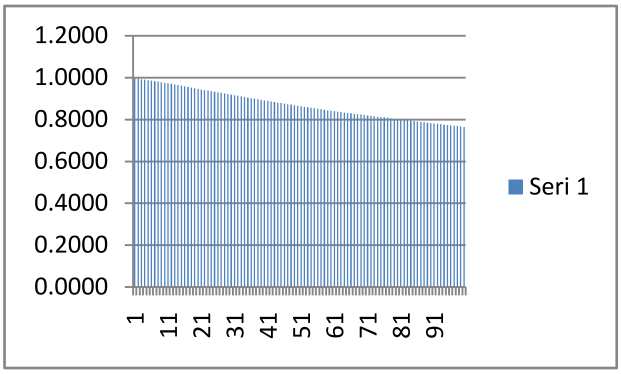

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Figure2 True autocorrelation and estimated autocorrelation

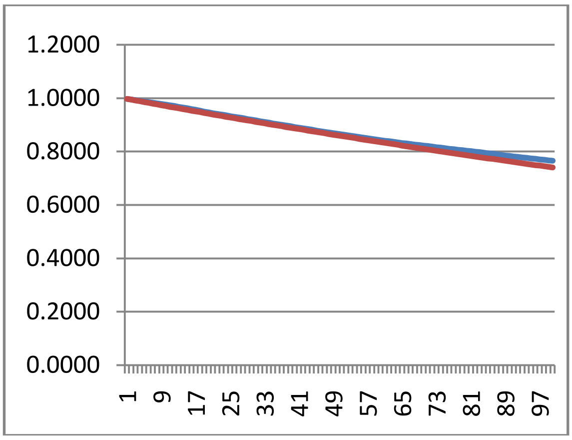


Figure3 Historical price data for GM

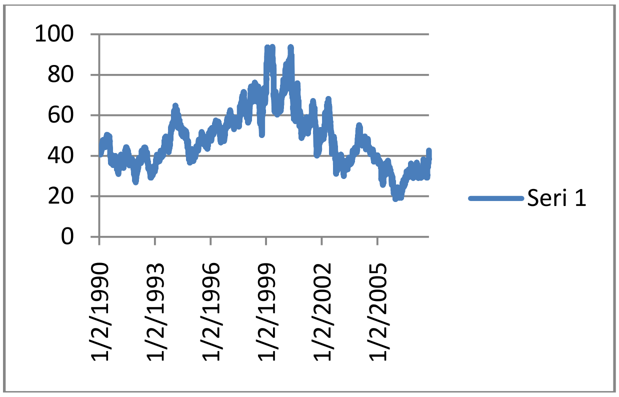
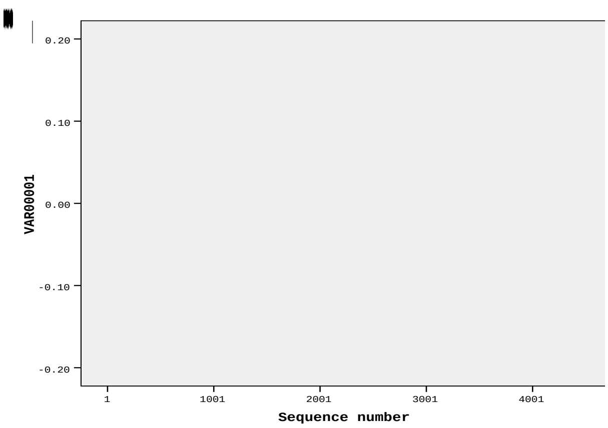


Figure 5: Return data for GM



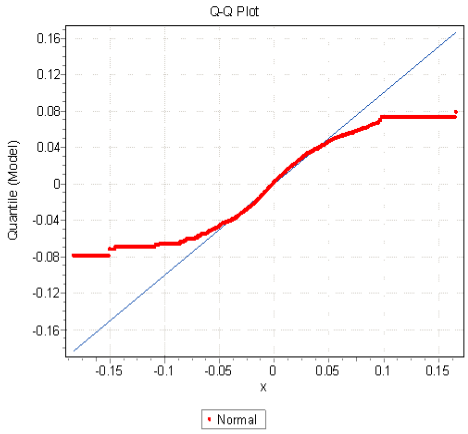


Table 1: Descriptive Statistics

Statistic	Value
Sample Size	4481
Range	0.3497
Mean	-9.5283E-6
Variance	4.5377E-4
Std. Deviation	0.0213
Coef. of Variation	-2235.6
Std. Error	3.1822E-4
Skewness	-0.03172
Excess Kurtosis	4.4648

Figure 6: GM return and Estimated return

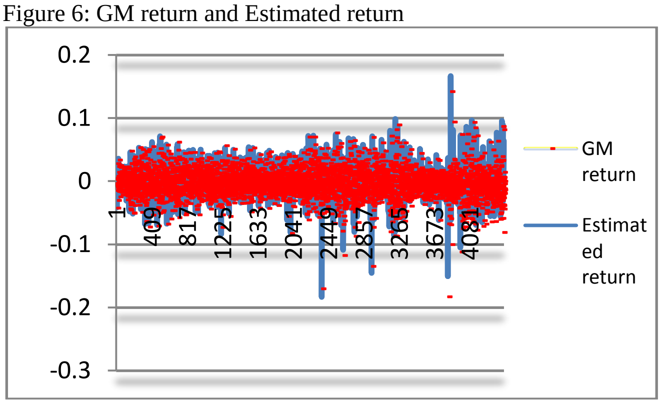


Figure 7: Volume data and INV.Gaussian pdf

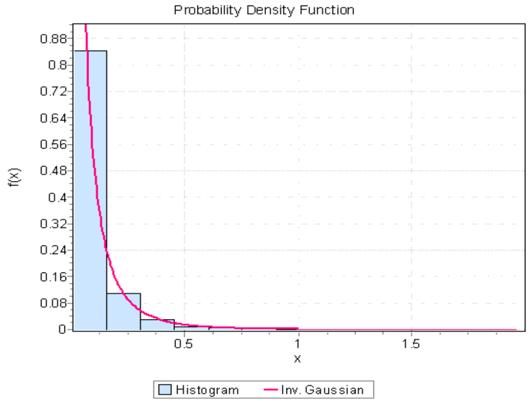


Figure 8: ACF of the squared residuals of GM actual return and estimated returns

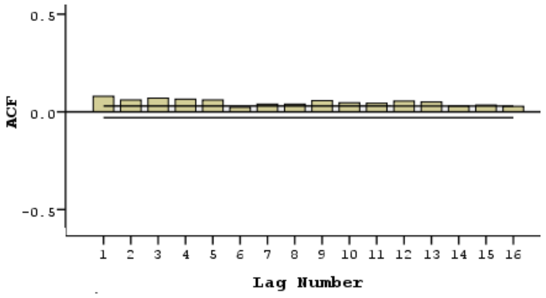
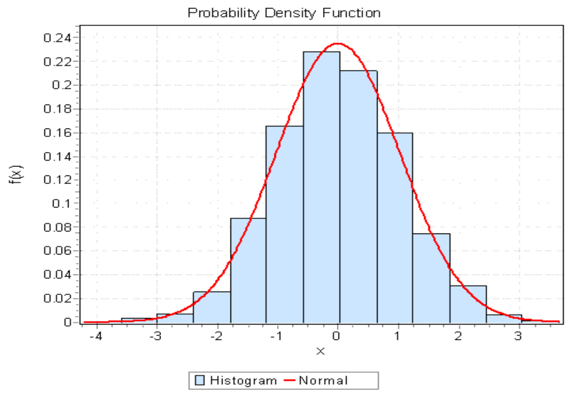


Figure 9: *Histgram* of the squared residuals of GM actual return and estimated returns



V. CONCLUSION

In this paper, we investigate an Ornstein-Uhlenbeck process driven by Lévy process for to model stock prices. We can be use the log return and stochastic volatility at the same time in a model. The autocorrelation function of an Ornstein-Uhlenbeck process is decreasing as exponential. Exponential autocorrelation function approximates well empirical autocorrelation function of General Motors stock. This result represent that Ornstein-Uhlenbeck process can be fit model for describe real data. Furthermore volume (trading intensity) can be use a model for the volatility. Accurate parameter estimates are important in mathematical finance and risk management.

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